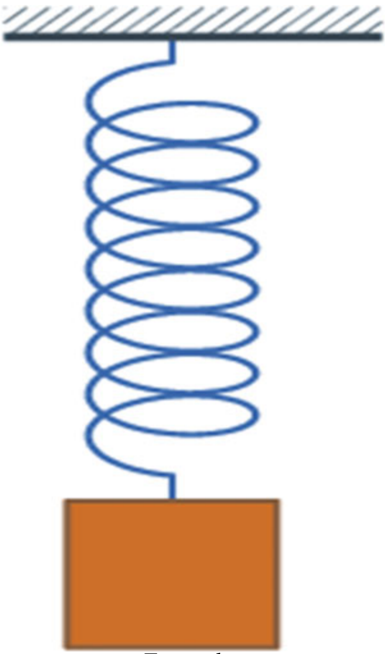
Appendix A: Simple Harmonic Motion

A.1 We Start with Hooke's Law

Harmonic oscillator is depicted here and Hooke's law defines the following equation:



$$F = -kx.$$

But using Newton's second law of motion, we can write

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$$F = ma,$$

where in both equations F is the force, x is displacement of mass m, and k is spring constant as well as the acceleration of the mass.

By equating both equations, we obtain the following:

$$ma = -kx$$

or

$$m\frac{d^2x}{dt^2} = -kx$$

and

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

Define $\omega_0^2 = \frac{k}{m}$ then we have

$$\frac{1}{\omega_0^2} = \frac{d^2x}{dt^2} + x = 0$$

or

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{A.1}$$

where ω is called angular frequency and can be defined as $\omega = 2\pi f = \frac{2\pi}{T}$ where f is frequency and T is period of oscillation.

Define

$$\dot{x} = \frac{dx}{dt}$$

then we have

$$\frac{d^2x}{dt^2} = \ddot{x} = \frac{d\dot{x}}{dt}\frac{dx}{dx} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{d\dot{x}}{dx}\dot{x}\,.$$

Substituting the above result in Eq. A.1, then we have

$$\frac{d\dot{x}}{dx}\dot{x} + \omega_0^2 x = 0,$$

$$\dot{x}\, d\dot{x} + \omega_0^2 x dx = 0.$$

Integrating over the differential equation, we have

$$\int \dot{x}\, d\dot{x} + \int \omega_0^2 x\, dx = 0,$$
$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2 = \text{cte},$$
$$\dot{x}^2 + \omega_0^2 x^2 = 2\text{cte} = K = (A\omega_0)^2,$$
$$\dot{x}^2 = A^2\omega_0^2 - \omega_0^2 x^2,$$
$$\frac{dx}{dt} = \dot{x} = \pm\omega_0\sqrt{A^2 - x^2},$$

Separating of variable gives the following results:

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega_0\, dt$$

or

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \int \omega_0\, dt. \tag{A.2}$$

Two possible solutions

$$\arcsin\frac{x}{A} = \omega_0 t + \phi,$$
$$\arccos\frac{x}{A} = \omega_0 t + \phi.$$

ϕ Integrating constant term

Note: To do left-hand side integral, we can do the following steps: Assume

$$x = A\ \sin y \Rightarrow dx = A \cos y dy$$

and

$$\sqrt{A^2 - x^2} = \sqrt{A^2 - A^2 \sin^2 y} = A\sqrt{1 - \sin^2 y} = A \cos y.$$

Therefore,

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \int \frac{A\cos y}{A\cos y} dy = \int dy = y.$$

Since $x = A\sin y$, we can conclude the following:

$$\sin y = \frac{x}{A} \Rightarrow y = \arcsin \frac{x}{A}.$$

Therefore, if we substitute the last step result in Eq. A.2, then we have

$$\arcsin \frac{x}{A} = \int \omega_0 dt = \omega_0 t + \phi.$$

Similarly we can have same results for the second solution:

$$\arccos \frac{x}{A} = \int \omega_0 dt = \omega_0 t + \phi.$$

So the general solution is written as follows:

$$x = A\cos(\omega_0 t + \phi).$$

However, remember we assumed that $\omega = 2\pi f = \frac{2\pi}{T}$ where $f = \frac{1}{T}$ and f is frequency while T is the period.

Appendix B: Pendulum Problem

B.1 Definition

A pendulum is a mass (or bob) on the end of a string of negligible mass that, when initially displaced, will swing back and forth under the influence of gravity over its central (lowest) point. The regular motion of a pendulum can be used for time-keeping; pendulums are used to regulate (Fig. B.1).

A simple is an idealization, working on the assumption that:

- The rod or cord on which the bob swings is massless, inextensible, and always remains taut.
- The motion occurs in a two-dimensional plane, i.e., the bob does not trace an ellipse.
- The motion does not lose energy to friction.

The differential equation, which represents the motion of the pendulum very similar to simple harmonic motion, is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \tag{B.1}$$

See Appendix A for Eq. B.1 derivation as well as the following pages.

In order to derive the simple pendulum equation and prove the dimensional analysis case about we show the following depiction (Fig. B.2):

Note: The path of the pendulum sweeps out an arc of a circle. The angle θ is measured in radians, and this is crucial for this formula. The blue arrow is the gravitational force acting on the bob, and the violet arrows are that same force resolved into components parallel and perpendicular to the bob's instantaneous motion. The direction of the bob's instantaneous velocity always points along the red axis, which is considered the tangential axis because its direction is always tangent to the circle. Consider Newton's second law:

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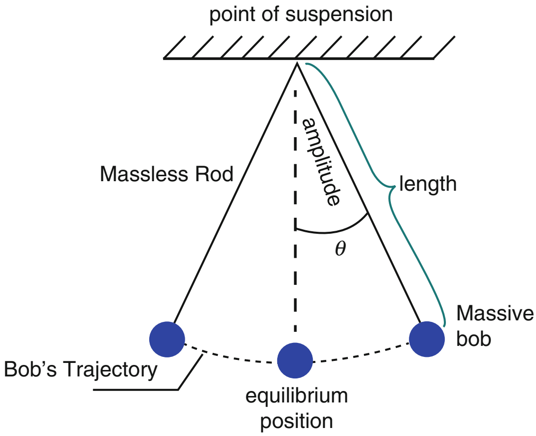

Fig. B.1 Simple gravity pendulum assumes no air resistance and no friction

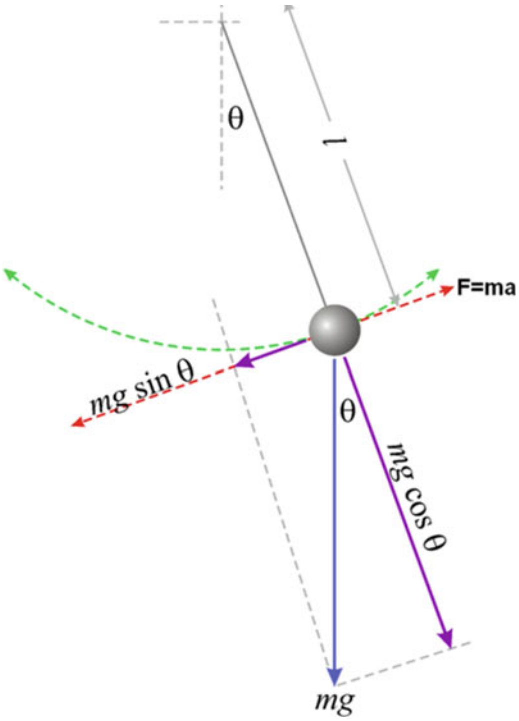

Fig. B.2 Force diagram of a simple gravity pendulum

$$F = ma,$$

where F is the sum of forces on the object, m is the mass, and a is the instantaneous acceleration. Because we are only concerned with changes in speed, and because the bob is forced to stay in a circular path, we apply Newton's equation to the tangential axis only. The short violet arrow represents the component of the gravitational force in the tangential axis, and trigonometry can be used to determine its magnitude. Thus,

$$F = -mg\sin\theta = ma$$
$$a = -g\sin\theta,$$

where g is the acceleration due to gravity near the surface of the earth. The negative sign on the right-hand side implies that θ and a always point in opposite directions. This makes sense because when a pendulum swings further to the left, we would expect it to accelerate back toward the right.

This linear acceleration a along the red axis can be related to the change in angle θ by the arc length formulas; s is arc length:

$$s = l\theta,$$
$$v = \frac{ds}{dt} = l\frac{d\theta}{dt},$$
$$a = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}.$$

Thus,

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\ = 0, \tag{B.2}$$

This is the differential equation which, when solved for $\theta(t)$, will yield the motion of the pendulum. It can also be obtained via the conservation of mechanical energy principle: any given object, which fell a vertical distance h, would have acquired kinetic energy equal to that which it lost to the fall. In other words, gravitational potential energy is converted into kinetic energy. Change in potential energy is given by

$$\Delta U = mgh$$

change in kinetic energy (body started from rest) is given by

$$\Delta K = \frac{1}{2}mv^2.$$

Since no energy is lost, those two must be equal:

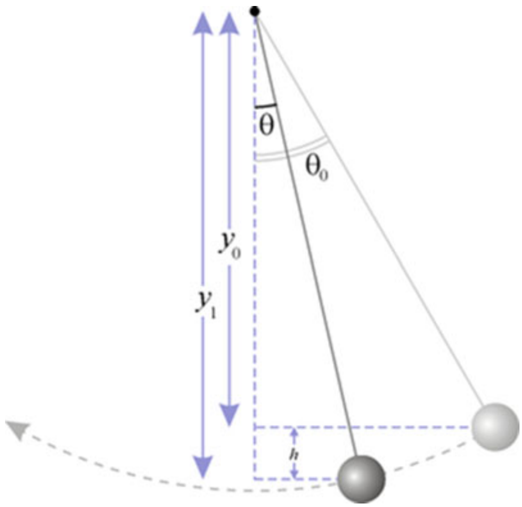

Fig. B.3 Trigonometry of a simple gravity pendulum

$$\frac{1}{2}mv^2 = mgh,$$
$$v = \sqrt{2gh}.$$

Using the arc length formula above, this equation can be rewritten in favor of $\frac{d\theta}{dt}$

$$\frac{d\theta}{dt} = \frac{1}{l}\sqrt{2gh},$$

where h is the vertical distance the pendulum fell. Consider Fig. B.3. If the pendulum starts its swing from some initial angle θ_0, then y_0, the vertical distance from the screw, is given by

$$y_0 = l\cos\theta_0$$

similarly, for y_1, we have

$$y_1 = l\cos\theta$$

then h is the difference of the two

$$h = l(\cos\theta - \cos\theta_0)$$

substituting this into the equation for $\frac{d\theta}{dt}$ gives

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}. \tag{B.3}$$

This equation is known as the *first integral of motion*; it gives the velocity in terms of the location and includes an integration constant related to the initial displacement (θ_0). We can differentiate, by applying the chain rule, with respect to time to get the acceleration:

$$\begin{aligned}
\frac{d}{dt}\frac{d\theta}{dt} &= \frac{d}{dt}\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \\
\frac{d^2\theta}{dt^2} &= \frac{1}{2}\frac{-(2g/l)\sin\theta}{\sqrt{(2g/l)(\cos\theta - \cos\theta_0)}}\frac{d\theta}{dt} \\
&= \frac{1}{2}\frac{-(2g/l)\sin\theta}{\sqrt{(2g/l)(\cos\theta - \cos\theta_0)}}\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} = -\frac{g}{l}\sin\theta \\
\frac{d^2\theta}{dt^2} &= -\frac{g}{l}\sin\theta,
\end{aligned}$$

which is the same result as obtained through force and dimensional analysis.

Appendix C: Similarity Solution Methods for Partial Differential Equations (PDEs)

Here we discuss briefly how to handle and solve a partial differential equation of high order by reducing to and ordinary differential equation using self-similar methods given by George W. Bluman and J. D. Cole.

C.1 Self-Similar Solutions by Dimensional Analysis

Consider the diffusion problem from the last section, with point-wise release (reference: *Similarity Methods for Differential Equations* (Applied Mathematical Sciences, Vol. 13)—Paperback (Dec. 2, 1974) by George W. Bluman and J.D. Cole (Sect. 2.3):

$$\begin{cases} \dfrac{\partial c}{\partial t} = D\dfrac{\partial^2 c}{\partial x^2} + Q_0\delta(x)\delta(t) \\ c(x,0) = 0, \quad c(\pm\infty, t) = 0. \end{cases}$$

Initial release within infinitely narrow neighborhood of $x = 0$, such that $\Pi(x)/d = \delta(x)$ and $L/d \to \infty$. Note Q_0 has different dimension as the previous Q because of the cross-sectional area S and time contained in $\delta(t)$.

1. *Dimensional analysis*
   $\{c\} = ML^{-3}, \{D\} = L^2T^{-1}, \{Q_0\} = ML^{-2}$ (mass release per unit cross-sectional area) $\{x\} = L$, $\{t\} = T$. Thus, we expect 2Pi groups:

$$\Pi_1 = \frac{\sqrt{Dt}}{Q_0}c, \quad \Pi_2 = \frac{x}{\sqrt{Dt}}$$

   and the solution to the PDE problem must be of the form $\Pi_1 = f(\Pi_2)$ or

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$$c = \frac{Q_0}{\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right).$$

Normally we expect dimensional analysis to reduce the number of variables and parameters. However, here we reduce the number of independent variables from 1 to 1!

2. *Transformation of PDE to ODE*
   Now we can plug this form back into the PDE. First, the partial derivatives:

$$\frac{\partial c}{\partial t} = -\frac{Q_0}{2t\sqrt{Dt}} f - \frac{Q_0 x}{2Dt^2} f', \quad \frac{\partial c}{\partial x} = \frac{Q_0}{Dt} f', \quad \frac{\partial^2 c}{\partial x^2} = \frac{Q_o}{(Dt)^{3/2}} f''.$$

For $t > 0$, there is no more injection: $\delta(t) = 0$. After inserting the above into the PDE:

$$-\frac{f}{2} - \frac{x}{2\sqrt{Dt}} f' = f'' \quad \text{or} \quad f'' + \frac{\xi}{2} f' + \frac{f}{2} = 0, \tag{C.1}$$

where $\xi = \frac{x}{\sqrt{Dt}}$ is our new independent variable. We have successfully transformed the PDE into an ODE. How about the initial and boundary conditions? Note that $t = 0$ and $x = \infty$ both correspond to $\xi = \infty$, so that the initial and boundary conditions can be rolled into one:

$$f(\pm\infty). \tag{C.2}$$

However, we need another condition on f, one that reflects the amount of initial injection. This is obtained by integrating the PDE over the following intervals:

$$\int_{0^-}^{t} dt \int_{-\infty}^{+\infty} [\text{PDE}] dx, \qquad \text{where } t = 0^- \text{ means } \text{"just before } t = 0\text{"}.$$

Now the left-hand side is

$$\int_{0^-}^{t} dt \int_{-\infty}^{+\infty} \frac{\partial^2 c}{\partial x^2} dx = \int_{-\infty}^{+\infty} dx \int_{0^-}^{t} \frac{\partial c}{\partial t} dt = \int_{-\infty}^{+\infty} [c(x,t) - c(x,0)] dx = \int_{-\infty}^{+\infty} c(x,t) dx.$$

Now we have $\int_{-\infty}^{+\infty} c(x,t) dx = Q_0$, which can be transformed, using the variable ξ, into

$$\int_{-\infty}^{+\infty} f(\xi) d\xi = 1. \tag{C.3}$$

ODE Eq. C.1, along with condition Eqs. C.2 and C.3, will uniquely determine $f(\xi)$, from which we get $c(x, t)$. We are not concerned with the actual solution of the new ODE problem. Rather, the interesting question is *how did we manage to turn a PDE to an ODE*.

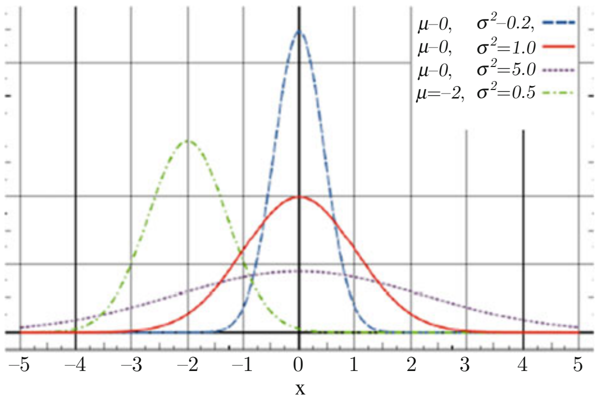
3. *Discussion*

(a) The problem admits a *self-similar solution:* if x is scaled by the diffusion length $(Dt)^{1/2}$, then the $c(x, t)$ profiles at different times can be collapsed onto each other if c is scaled by $Q_0/(Dt)^{1/2}$

(b) This means that x and t are not really two independent variables; as far as c is concerned, they can be rolled into one independent variable ξ.

(c) Similarity solutions are "happy coincidences" in physical process. Can we always find them for any PDEs? No. This problem is special in that there is no inherent length scale. Thus, we are not able to form dimensionless groups for each of the variables x, t and c; instead, we have to combine them and end up with only 2Pi groups. That is how we ended up with ODE. If we had the release length dS or the domain length L, the self-similar will be ruined.

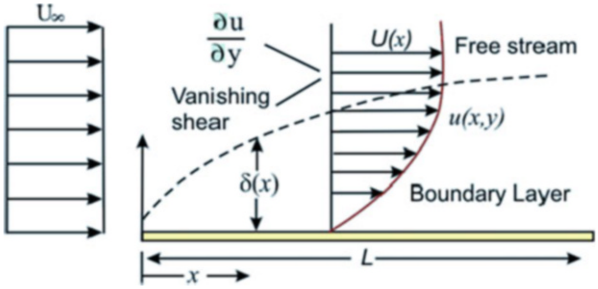
(d) Can we always find similarity solutions by dimensional analysis? No. However, we will study another example next and then introduce the general "stretching transformation" idea for detecting similarity solutions.



C.2 Similarity Solutions by Stretching Transformation

It is rare that similarity solutions can be obtained from dimensional analysis. In this section, we introduce the idea of stretching transformation which is a more general procedure for seeking out similarity in PDE problems. The materials are based on Barenblatt (Sect. 5.2) and Bluman and Cole (Sect. 2.5).

As a concrete example, we will take Prandtl's boundary-layer equation for flow over a flat semi-plane. After the boundary-layer approximation (that viscosity acts only within a thin layer, that the gradient in the flow direction (x) is much smaller than in the transverse direction (y), and that the pressure is constant in the y direction), the governing equations are



$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u(x,0) = 0, \quad v(x,0) = 0 \\ u(x,\infty) = U_\infty, \quad u(0,y) = U_\infty \end{cases}$$

where U_∞ is the free-stream velocity and ν is the kinematic viscosity. If you recall your fluid mechanics, this problem does have a similarity solution (Blasius's solution), and the PDE can be reduced to ODE. (Try to distinguish the velocity v from the viscosity ν. We could use different symbols but these are the conventional ones).

1. *Would dimensional analysis work?*

   Let us write out the dimensions of all the variables and parameters:

$$\{u\} = \{v\} = \{U_\infty\} = L/T, \quad \{\nu\} = L^2/T, \quad \{x\} = \{y\} = L$$

   There are two independent dimensions involved (L and T), and we can construct four

$$\Pi_1 = \frac{u}{U_\infty}, \quad \Pi_2 = \frac{v}{U_\infty}, \quad \Pi_3 = \frac{U_\infty x}{\nu}, \quad \Pi_4 = \frac{U_\infty y}{\nu}$$

   in addition, and we expect solutions such as

$$\Pi_1 = f(\Pi_3, \Pi_4), \quad \Pi_2 = g(\Pi_3, \Pi_4).$$

Plugging these back into the equations, we will see that we have **not** achieved a reduction of the number of independent variable. Dimensional analysis has failed to give us the similarity solution. Why? Even through the problem has no intrinsic time or length scales. There are only two indecent dimensions (L and T) instead of three. Thus, it is possible for x and y to form their own Pi groups; they do not have to be forced into a single one.
It turns out that in this particular example, a trivial manipulation can "cure" the above problem. This is not a general technique, but nevertheless, it is fun to illustrate here. We will take this little detour before marching into the general technique that is the focus of this section. Based on the physical insight that things happen at different scales along the x and y directions, which is the fundamental idea behind the boundary-layer approximation, we assign two different dimensions to x and y, L and H, and for the moment pretend that they are different dimensions. Now the list of variables and unknowns are scaled as such:

$$\{u\} = \left\{U\grave{A}\hat{U}\right\} = L/T, \quad \{v\} = H/T, \quad \{\pounds h\} = H2/T, \quad \{x\} = L, \quad \{y\} = H.$$

There are now three independent dimensions involved (L, H, and T), and we can construct only three dimensionless groups out of these:

$$\widetilde{\Pi}_1 = \frac{u}{U_\infty}, \quad \widetilde{\Pi}_2 = \sqrt{\frac{v}{vU_\infty/x}, \quad \widetilde{\Pi}_3 = \frac{y}{\sqrt{vx/V}} = \zeta.}$$

Now we expect a similarity solution in this form:

$$u = U_\infty f(\zeta), \quad v = \sqrt{\frac{vU_\infty}{x}} g(\zeta).$$

Plugging this into the original PDE will show that, indeed, we have reduced the PDE problem to a couple of ODEs, whose solution is detailed in *Fluid Mechanics* textbooks. For another example of such "ingenious¨ dimensional analysis, see the Rayleigh problem analyzed in the next section (see also Bluman and Cole, p. 195). We typically seek to increase the number of independent dimensions (as done above) or decrease the number of dimensional parameters (as done in Bluman and Cole's example).

2. *Stretching transformation*
The "ingenious" dimensional analysis method is specific to the problems. There is, however, a general scheme for seeking out possible similarity solutions. The scheme sometimes goes by the name of "renormalization groups" or "invariant transformation groups" and is based on rather formalistic mathematical manipulations. We will skip the proofs and focus on the technique itself.
Since the essence of similarity is that the solution is *invariant* after certain scaling of the independent and dependent variables, we consider the following

stretching transformation, and see if such transformations will leave the PDE and the boundary conditions invariant.
Consider:

$$\begin{cases} U = \alpha^a u, & V = \alpha^b v \\ X = \alpha^c x, & Y = \alpha^d y \end{cases},$$

where α is a positive number. Under this transformation, we have

$$\frac{\partial u}{\partial x} = \alpha^{c-a} \frac{\partial U}{\partial X}, \quad \frac{\partial u}{\partial y} = \alpha^{d-a} \frac{\partial U}{\partial Y}, \quad \frac{\partial v}{\partial y} = \alpha^{d-a} \frac{\partial V}{\partial Y}, \quad \frac{\partial^2 u}{\partial y^2} = \alpha^{2d-a} \frac{\partial^2 U}{\partial Y^2}.$$

Plugging these into the original PDE and boundary conditions, we will see what choices of a, b, c, and d may maintain the invariance of the problem. The continuity equation yields:

$$c - a = d - b.$$

The three terms of the momentum equation requires:

$$c - 2a = d - a - b = 2d - a.$$

Note that the first equation above is identical to the preceding equation, and thus the momentum equation adds only one additional constraint on the power indices. Finally the boundary conditions require

$$a = 0$$

because for the problem in the new variables to be invariant, the nonhomogeneous BC should remain as $U(X_\infty) = U_\infty$. Now we have three equations that constrain the four indices, and we rewrite the transformation as

$$\begin{cases} U = u, & V = \dfrac{v}{\varepsilon} \\ X = \varepsilon^2 x, & Y = \varepsilon y \end{cases}, \quad \text{where} \quad \varepsilon = \alpha^d.$$

This transformation will leave the problem the same as before, in the new "stretched" and scaled variables. The fact that this *one-parameter family* of transformations will maintain the invariance of the PDE problem reveals the intrinsic self-similarity of the problem. In other words, if we stretch the coordinate y by a factor ε, then we must stretch x by ε^2 and the velocity component ε^2 by $1/\varepsilon$ in order to collapse the velocity profiles. From this argument, we recognize that

$$u, \upsilon\sqrt{x}, \frac{y}{\sqrt{x}}$$

shall remain the same no matter how we stretch the coordinates. These are known as the invariants of the transformation, and immediately suggest the following similarity solution:

$$\begin{cases} u = f(\zeta) \\ \upsilon = \dfrac{1}{\sqrt{x}} g(\zeta) \end{cases}, \quad \text{with the similarity variable } \zeta = \frac{y}{\sqrt{x}}.$$

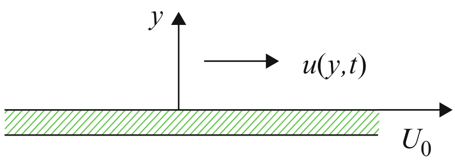
This is the same form as obtained from the "ingenious dimensional analysis," aside from a few constant factors. Note that we reached the conclusion here not through dimensional considerations, but through the idea of invariance under general stretching transformations.
Now it is a simple matter to plug these forms into the original PDE problem, and transform it into the following ODE problem:

$$\begin{cases} \upsilon f'' + f'\left(\dfrac{\zeta}{2} f - g\right) = 0 \\ \zeta f' - 2g' = 0 \\ f(\infty) = U_\infty, f(0) = 0, g(0) = 0 \end{cases}$$

the solution of which will not be of immediate interest to us here. Note that the two BCs at $x = 0$ and $y = \infty$ both project onto $\zeta = \infty$.

C.3 Similarity Solution for the Rayleigh Problem

The Rayleigh problem is another classical example with a self-similar solution. Consider the transient motion in a viscous fluid induced by a flat plate moving in its own plane. Initially both the plate and the fluid are at rest. Starting at, the plate moves with a constant velocity. The Nervier–Stokes equations, simplified for this problem, along with the initial and boundary conditions, can be written as



$$\begin{cases} \dfrac{\partial u}{\partial t} = v\dfrac{\partial^2 u}{\partial y^2} \\ u(y,0) = 0, \quad u(0,t) = U_0, \quad u(\infty,t) \end{cases}.$$

(a) *Dimensional analysis*
From the following dimensions,

$$\{u\} = \{U_0\} = L/T, \quad \{v\} = H^2/T, \quad \{t\} = T, \quad \{y\} = L,$$

where v is fluid viscosity.
We can make 3Pi groups, say u/U_0, U_0y/v, and U_0^2t/v, and there is no reduction to ODE. Again, we can play trick here, by either *increasing* the number of "independent dimension" or *decreasing* the number of parameters, so as to reduce the number of Pi groups.
Using the physical observation that viscous diffusion happens along the y direction, while the primary flow is in the x direction, we can introduce different length scales:

$$\{u\} = \{U_0\} = L/T, \quad \{v\} = H^2/T, \quad \{t\} = T, \quad \{y\} = H.$$

Now there are only 2Pi groups:

$$\Pi_1 = \frac{u}{U_0}, \quad \Pi_2 = \frac{y}{\sqrt{vt}}$$

in addition, we can try a similarity solution of the form

$$u(y,t) = U_0 f\left(\frac{y}{\sqrt{vt}}\right)$$

Alternatively, we can reduce the number of parameters by scaling u by U_0, and calling $\widetilde{u}(y,t) = u(x,t)/U_0$ the new dependent variable. Now the problem has one less parameter, and again only admits 2Pi groups. In the following, however, let us carry out the formal procedure of *stretching transformation* as an exercise.

(b) *Stretching transformation*
Consider

$$U = \alpha^a u, \quad Y = \alpha^b, \quad T = \alpha^c t,$$

where α is a positive number. Under this transformation, we have

$$\frac{\partial u}{\partial t} = \alpha^{c-a}\frac{\partial U}{\partial T}, \qquad \frac{\partial^2 u}{\partial y^2} = \alpha^{2b-a}\frac{\partial^2 U}{\partial Y^2}.$$

To maintain invariance of the PDE, we require

$$c - a = 2b - a \quad \text{or} \quad c = 2b.$$

The boundary condition $u(0,t) = U_0$ requires $a = 0$. Thus, the following transformation renders the problem invariant:

$$U = u, \quad Y = \varepsilon y, \quad T = \varepsilon^2 t, \quad \text{which} \quad \varepsilon = \alpha^b.$$

This transformation dictates that y and t be transformed in a coordinated way. Thus u and $\zeta = y/\sqrt{t}$ shall be our new variables that remain unchanged for any stretching α or ε:

$$u = f\left(\frac{y}{\sqrt{t}}\right) = f(\zeta)$$

This reduces the original PDE into the following ODE problem:

$$\begin{cases} 2\nu f'' + \zeta f' = 0 \\ f(0) = U_\infty, \quad f(\infty) = 0 \end{cases},$$

which can be integrated analytically to give:

$$f = c_1\int_0^\zeta \exp\left(-\frac{z^2}{4\nu}\right)dz + c_2.$$

Noting that $\int_0^\zeta \exp\left(-\frac{z^2}{4\nu}\right)dz = 2\sqrt{\nu}\int_0^\infty \exp\left(-\xi^2\right)d\xi = \sqrt{\pi\nu}$, the two constants of integration are determined:

$$\begin{aligned} c_1 &= -U_0/\sqrt{\pi\nu}. \\ c_2 &= U_0 \end{aligned}$$

Finally, the solution can be written in terms of the *complementary error function*:

$$f = U_0\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\nu}}\right) = U_0\mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)$$

with $\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-z^2\right)dz.$

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