Conclusion

The kind of thinking associated with category theory is geometrical and diagrammatic in essence, based as it is on formal relations of embedding and variation that hold between "figures" and "spaces" definable in very general terms. As we have seen in particular in chapter six, the role of spaces may be played for instance by ambient mathematical environments, and the corresponding figures are then the formal structures of quantum mechanics that are given as a heterogenous collection of data through the histories of experiment and theorization in modern physics. As these structural figures are embedded in the diverse mathematical spaces opened by category theory, topoi and partition logic, they become available to philosophical reflection as *diagrams* subject to experimental control and investigation. The concepts that in turn must be used to interpret these diagrams are then not given arbitrarily, but must instead be developed pragmatically through collective processes of experimentation and revision that are themselves scientific in character and are thereby available to critical scrutiny.

This way of thinking about scientific models opens up the possibility of a kind of realism with respect to processes of scientific modelization through mathematics themselves. This realist orientation with respect to epistemological method arises as a theoretical result, not a mere presupposition or posit. The result in question in any given case, however, is by no means a deductive certainty. Yet neither is it an inductive generalization or calculable probability based on given data. Instead, the reasoning at work here takes the shape of abductive inference. The "surprising event" initiating the abductive process may be, for example, the concrete history of twentieth and twenty-first century subatomic physics as both a constellation of real events and a sequence of epistemological forays, detours and extensions.

Much philosophy of science remains trapped in simplistically dualist conceptual frameworks, even and especially where a traditional Cartesian metaphysical dualism of extended and thinking substances is denied. The more insidious dualisms arise through the exportation of implicit meta-theoretical distinctions directly into meta-theoretical analysis and reflective discourse. It is obvious, for example, that the expression of most scientific theories is spontaneously "realist" in the straightfor-

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ward and naive sense that they are ostensibly theories "about" real objects and their various properties and relations. Some philosophers of science, however, seem to take this contingent fact as a fundamental ground for meta-theoretical reflection. Yet if this lifting of spontaneous meta-theory to philosophically reflective meta-theory is justifiable, it ought to be tested and justified explicitly. After all, we take it for granted in a modern scientific context that scientific theories may come to challenge our ordinary or "folk" theories about the natural world (Copernican astronomy, relativity, quantum mechanics, etc.). This is what scientific inquiry *essentially* does. Our "folk" conceptual frameworks regarding science need to be equally subject to critical inquiry and revision. No doubt when formulated in this way, nearly all philosophers of science would be inclined to agree. Such agreement, however, might be more realized in principle than in practice. Philosophers tend to forget that the iconic and diagrammatic models that characterize scientific thinking are just that: models. For instance, while formal languages often serve as useful models of scientific theories, scientific theories themselves simply *are not* sets of statements in formal languages. We may not know exactly what they are, any more than we know exactly what lightning is or gravity prior to scientific inquiry. But just because they are theoretical in character and employ models does not imply that they are equivalent to their own theoretical modelizations. They are, rather, always iconic expressions of their own epistemological trajectories through a partly obscure space of knowledge, ignorance and experience. Thus, not just the truth or falsity of scientific realism but the question of *what realism even means* ought to be a matter of sustained inquiry.

The earlier analyses of Peircean semiotics and logical grammar and Badiou's mathematical models of ontology and epistemology may be understood in the light of this notion of a scientific realism with respect to processes themselves of scientific investigation. For both of these philosophers, the formal models of knowledge and being that track the scientific investigation of nature and its essential use of mathematics are themselves expressions of ongoing abductive processes supported by complex cultural and historical dynamics at multiple levels. Such a view does not imply epistemological relativism, but rather emphasizes the context-bound reality of all actual movements of scientific inquiry. Abductive reasoning is a crucial aspect of all scientific modes of knowing, and it operates at both macroscopic and microscopic scales of inquiry. Continued investigation into the dynamics of abductive inference both as concretely evident in actual science and as formally guiding new procedures of optimized hypothesis construction remains one of the most important elements of philosophy of science and indeed of science itself.

Appendix A
Category Theory: A Primer

This appendix is meant only to fix some elementary notation and terminology for the uses made of category theory in the main text and to provide some motivating examples for those who might be unfamiliar with the material. Comprehensive surveys of this mathematical field must be sought elsewhere. For more complete treatments of elementary category theory see [1]. An excellent introduction for those with no background whatsoever in the subject is found in Lawvere [2].

A.1 Category Theory Axioms

A *category* is an abstract system of objects and relations. Categories are readily imagined by diagrams of dots and arrows. Objects are differentiated as distinct nodes in the system, whereas the relations link one individual to another in some determinate way. Between any two dots, there may be many arrows, one or none, and there may be arrows from any given object to itself (indeed every object must have at least one of these).

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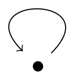
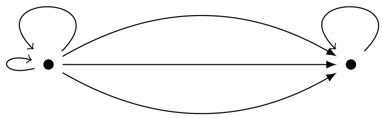




Arrows are then subject to the following three axioms, which all categories must satisfy and which are together sufficient to determine a category as such:

(A1) Axiom of Composition: Given any two arrows ordered such that the target of one is the source of the other, the composition of the first followed by the second exists in the category as a unique and definite arrow.

Formally, given two arrows

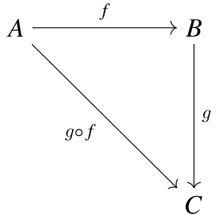
$$A \xrightarrow{\quad f \quad} B$$

$$B \xrightarrow{\quad g \quad} C$$

there exists one and only one arrow

$$A \xrightarrow{\quad g \circ f \quad} C$$

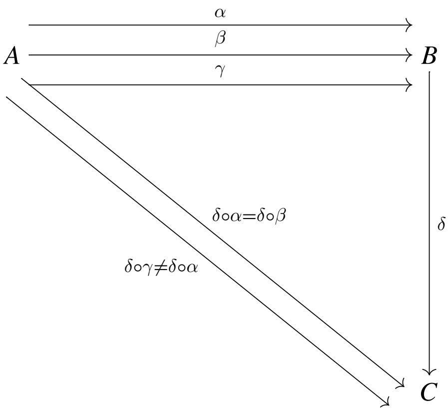
This arrow $g \circ f$ is called the *composite* of f and g. This "composition-relation" among arrows is intuitively captured by the following diagram:

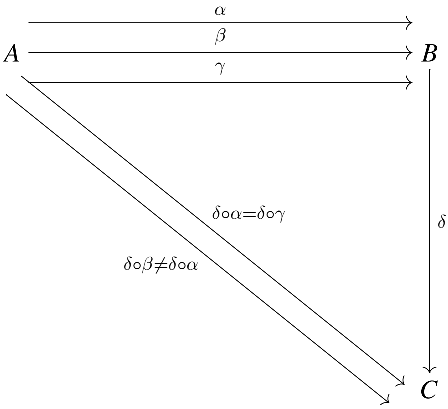


Remark 1 It is important to realize that the structure of a category is defined essentially in terms of how its arrows compose. A useful analogy here can be made with

group theory: typically, the structure of a group may be exhaustively expressed by the relations among its generators.

In order to highlight this fact, it is useful to look at the two pictures below. They both appear to have the same structure of objects and arrows, but the stipulated arrow-relations are different, therefore leading to two distinct mathematical entities.





(A2) Axiom of Associativity: Given any three arrows ordered such that the target of the first is the source of the second and the target of the second is the source of the third, the composite of the first two arrows composed with the third is the same as the first arrow composed with the composite of the second and third.

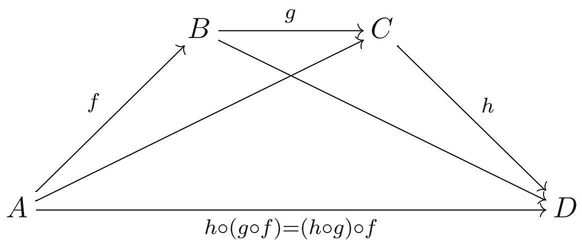
Formally, given three arrows

$$A \xrightarrow{f} B$$

$$B \xrightarrow{g} C$$

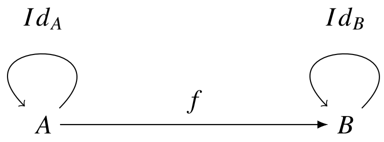
$$C \xrightarrow{h} D$$

the following diagram *commutes*, that is, all diagram arrow-paths with common sources and targets collapse to identical arrows (see below for a fuller explanation of this key diagrammatic notion):



In practice, this means that any path of arrows in a category composes to a unique arrow.

(A3) Axiom of Identity: Every object has an arrow called its *identity* (Id) which takes that object as both its source and its target and which composes "inertly" with all arrows having a composition defined with it. See the diagram and compositional equation immediately below:



$$f = f \circ Id_A = Id_B \circ f,$$

for all arrows f from A to B.

In practice, this means that identity-arrows may be arbitrarily added to or subtracted from paths of arrows without altering the composition of the path. In addition, since each identity arrow uniquely picks out the object to which it is attached and represents the static and inert iteration of that object's identity, from a categorical standpoint each object may typically be "identified" with its identity arrow.

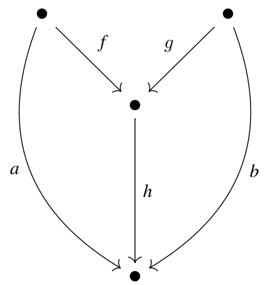
In fact, it is possible to define categories in an "arrows-only" fashion without reference to objects at all. As [3] in particular emphasizes (p. 261),

> the usual distinction between objects and morphisms of categories is formally dispensable: since with each object A of given category C is associated a unique identity morphism 1_A, one may formally identify objects of C with their corresponding identity morphisms and thus consider objects as morphisms of special sort. Thus, formally, a general category can be described as a class of things called morphisms provided with a (partial) binary associative operation called composition.

A.2 Diagrams

Informally, a diagram in a category is a collection of dots (or labels) representing objects connected by arrows representing arrows (note that in this way, arrows in a category theory diagram interestingly "represent themselves"). We may speak of the system of objects and arrows as *represented by* such a diagram as being that diagram *in* the category at issue, whereas the (same) diagram conceived as *representing* this system may be said to be a diagram *for* this same category.

A diagram is said to *commute* when all directed paths in the diagram with the same start and endpoints lead to the same result by composition. For instance, if we declare that the following diagram commutes:



we are really declaring that the arrow a *is and must be* the composition $h \circ f$ and the arrow b *is and must be* the composition $h \circ g$.

In this sense, when we say that a diagram is commutative we are in fact imposing certain relations on the variable components (arrows) of our structure (a category). This is similar to imposing relational identities (equations) on structures based on "classical" symbolic variables (think, for instance, of the equation $xy = yx$ which distinguishes abelian groups amongst all groups).

We want to highlight in this way that the notion of commutative diagram expresses the same "intuitionistic" problem intrinsic to the concept of equality in mathematics more generally: How can two ostensibly different terms, for instance in an algebraic expression, "come to be" conceived as representing in fact one and the same "thing"? By looking at the diagram above we see two arrow-paths $h \circ f$ and a, therefore two distinct objects, that are actually meant to represent (and are in fact forced to be) the same mathematical entity on the basis of the stipulated commutativity of the diagram taken as a whole.

Furthermore, when defining structures within a category, a variety of formal notational devices may be used to handle issues of quantification ("*for all arrows* f", "*there exists exactly one arrow* g", etc.). See, for instance, the innovative diagrammatic approach proposed in [4]. In practice, cases involving significant ambiguity seldom arise.

A.3 Isomorphisms

Two objects A and B are said to be *isomorphic* if there exist two arrows f and g as given below such that the following diagram commutes.

$$Id_A \circlearrowright A \underset{f}{\overset{g}{\rightleftarrows}} B \circlearrowleft Id_B$$

Informally, since this structure ensures that the objects A and B have identical systems of arrow-relations with all other objects in the ambient category, from a strictly category theoretical perspective they may be treated *in the context of this particular category* as being "the same".

A.4 Functors Between Categories

Categories may be related to one another via mappings called *functors*, which may be understood at a first approach on analogy with functions between sets.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor is in the first place a map F from objects of $\mathcal{C}$ into objects of $\mathcal{D}$ and from arrows of $\mathcal{C}$ into arrows of $\mathcal{D}$, that is, roughly, a function from the set of the objects of $\mathcal{C}$ to the set of the objects of $\mathcal{D}$ together with a function from the set of arrows of $\mathcal{C}$ to the set of arrows of $\mathcal{D}$.[1]

The condition that this map has to satisfy is that (a) relations linking arrows to their source and target objects are preserved and (b) composition relations between arrows are preserved across the mapping.

This amounts to saying that, if

$$A \xrightarrow{f} B$$

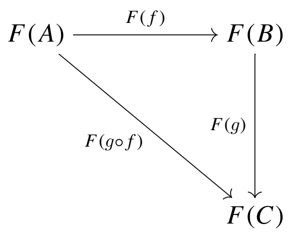
is an arrow in $\mathcal{C}$, then

$$F(A) \xrightarrow{F(f)} F(B)$$

is an arrow in $\mathcal{D}$.

Furthermore, the following diagram must commute in $\mathcal{D}$ for all suitable objects A, B and C and arrows f, g and $g \circ f$ in $\mathcal{C}$:

[1]This formulation only causes difficulties in the (not infrequent) cases when the objects and/or arrows of either $\mathcal{C}$ or $\mathcal{D}$ cannot be gathered into a set, for instance when one of these is the category **Sets** of sets and functions. The ensuing problems and the various strategies for resolving them are readily located in the standard literature on categories.



Finally, identity arrows must "track" with their objects across the mapping. Formally, for any object A in $\mathcal{C}$,

$$F(Id_A) = Id_{F(A)}$$

Such a mapping F is called a *covariant functor*. There is a dual notion of *contravariant functor*, which, essentially, instead of "preserving" arrows it "reverses" them. That is, $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be *contravariant* if, given two object A and B in $\mathcal{C}$ we have[2] that

$$(f : A \longrightarrow B) \Rightarrow F(f) : F(B) \longrightarrow F(A)^2$$

The reader familiar with abstract algebra will appreciate the striking similarity between the notion of functor and that of homomorphism in algebra. Indeed, if we think of category theory as an algebra whose elements are arrows and where defining equational structures are represented by commutative diagrams, the above diagram (for the covariant case) simply says that $F(g \circ f) = F(g) \circ F(f)$.

A.5 Diagrams as Functors

With the notion of functor in hand, it becomes possible to define categorical diagrams in a formal way.

Following the treatment given in [1], given two categories $\mathcal{J}$ and $\mathcal{C}$, we define a *diagram of type* $\mathcal{J}$ in $\mathcal{C}$ to be a functor

$$D : \mathcal{J} \to \mathcal{C}$$

The image of the functor D may then be said to be a diagram of $\mathcal{J}$ in $\mathcal{C}$. In particular, if we think of $\mathcal{J}$ as the index category and write its objects using the letters $i, j, \ldots$, we can then indicate the values of such functors with the symbols $D_i, D_j, \ldots$ and similarly with arrows.

The key idea here is to think of the diagram D as a "picture of $\mathcal{J}$ in $\mathcal{C}$."

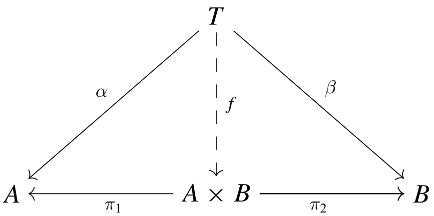
[2] Contravariant functors also reverse the direction of composition.

A.6 Limits

Among the diverse species of mathematical constructions possible in category theory, *limits* and *colimits* (dual to limits) are at once fundamental to and deeply iconic of the essential features of the categorical approach. The presentation given here follows in broad strokes that of [1], pp. 101–104.

Every basic introduction to category theory includes the definition of certain "special" objects that may or may not exist in a given category, such as "the product of two objects", "equalizer", "pullback", "exponential object", "subobject classifier", just to mention some of the most fundamental. These objects may be defined in terms of certain "universal properties" of types of diagrams in whatever category is in question (in the formal sense of diagram defined above).

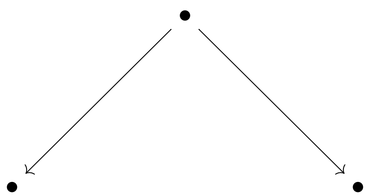
For instance, given two objects A and B of a category $\mathcal{C}$, we can interpret the commutative diagram below



as saying that the categorical product $A \times B$ is an object such that together with the two morphisms π_1 and π_2 from $A \times B$ into A and B, respectively, given *any* test object T in $\mathcal{C}$ and *any* two morphisms α and β from T to A and B respectively, there exists one and only one morphism from T to $A \times B$ (represented by the dashed arrow f) which makes the diagram as a whole commute. This construction determines *products* in the categorical sense as a particular type of *limit*.

Two main ingredients play a fundamental role in "unpacking" categorical diagrams as limits:

(1) a "basic shape", such as



and

(2) a *universal property* to be satisfied.

An elegant and very unifying approach to the understanding of categorical constructions is to formalize these two concepts (basic shape and universal property) through the notion of limits of diagrams, in the precise sense defined in the previous section.

Using the notation introduced above, we define a *cone* to a diagram D in $\mathcal{C}$ as consisting of an object $C \in \mathcal{C}$ and a family of arrows

$$c_j : C \longrightarrow D_j$$

one for each object $j \in \mathcal{J}$, such that for each arrow $\alpha : i \longrightarrow j$ in $\mathcal{J}$ the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{c_j} & D_j \\ {\scriptstyle c_i}\downarrow & \nearrow{\scriptstyle D_\alpha} & \\ D_i & & \end{array}$$

Quoting Awodey [1] “A cone to such a diagram D is then imagined as a many-sided pyramid over the ‘base’ D and a morphism of cones is an arrow between the apexes of such pyramids”.

The collection of all cones to D is denoted by **Cone**(D) and this collection can be shown to form a category in its own right once an opportune, natural notion of arrow, or morphism, between these objects is defined.[3]

We now define the straightforward notion of *terminal object* in a category and then use this construction in order to complete our general formulation of categorical limits as universal properties of types of diagrams.

Definition 10 In a category C an object A is a *terminal object* iff for any object T of C there is exactly one arrow $T \longrightarrow A$.

Definition 11 A *limit* for a diagram $D : \mathcal{J} \longrightarrow \mathcal{C}$ is a terminal object in **Cone**(D). A *finite limit* is a limit for a diagram on a finite index category $\mathcal{J}$.

These somewhat abstract definitions may be clarified through a simple example.

Example 11 (*Awodey*) Take $\mathcal{J} = \{1, 2\}$ the discrete category with two objects and no non-identity arrows. A diagram $\mathcal{D} : \mathcal{J} \longrightarrow \mathcal{C}$ is a pair of objects $D_1, D_2 \in \mathcal{C}$. A cone on D is an object of $\mathcal{C}$ equipped with arrows

$$D_1 \xleftarrow{c_1} C \xrightarrow{c_2} D_2$$

And a limit of D is a terminal such cone, that is, a product $D_1 \times D_2$ in $\mathcal{C}$ of D_1 and D_2.

[3]The intuitive notion of an arrow from one cone to another should be clear (essentially, it is a arrow from C to D in $\mathcal{C}$ such that “everything commutes”). The technical details may be found in [1], p. 102.

The reader should compare this last example with the diagram given at the beginning of the present section to understand the utility and economy of this formal approach.

A.7 Examples of Categories

We are especially interested in how the type/token relation is typically presented categorically. Many mathematical "types" naturally constitute categories in which the objects of the category are all of the tokens of the given type and the morphisms are maps between such objects that preserve their structure in some relevant way. Note that this effectively defines a given categorical "type" as a coherent system of relations. In many cases (such as in all of the examples provided below) this system of "external" relations among the tokens is sufficient to determine all of their relevant "internal" structure.

In addition, it is often the case that each of the tokens of a standard mathematical type may be represented by a category. This is the case for all the examples discussed below. Thus from this perspective, category theory may serve as a mathematical tool that in contexts like those illustrated by the following examples treats mathematical types and tokens from within a common framework.

- A *set* S is an unstructured collection of elements.
  - The category **Sets** has for objects sets $R, S, T \ldots$ and for arrows functions $f : R \longrightarrow S$ between sets, for all sets R and S. In this category isomorphisms are bijections, thus isomorphic objects correspond to sets of the same cardinality.
  - Every individual set S may also be conceived as a category $\mathcal{C}_S$. The objects of $\mathcal{C}_S$ are the elements $e, f, \ldots$ of S and the only arrows in $\mathcal{C}_S$ are the identity arrows. This is sometimes called the *discrete category* on S.

- A *group* G is a set endowed with an associative binary operation $\star$, such that: (a) there is an element $e \in G$ with the property that $e \star g = g = g \star e$ for all $g \in G$; (b) for any $g \in G$ there is an element g^{-1} such that $g \star g^{-1} = g^{-1} \star g = e$.
  - The category **Groups** has for objects groups and for arrows all group homomorphisms between groups.
  - Every individual group G may also be conceived as a category. The abstract structure of G is given by a category $\mathcal{C}_G$ with exactly one object and all of the arrows of which are invertible, that is, for every arrow f there is an arrow f^{-1} such that $f \circ f^{-1}$ is the identity arrow on the (only) object of $\mathcal{C}_G$. Arrow composition then corresponds to the group operation given by G. Functors from $\mathcal{C}_G$ into **Sets** select permutation groups on the elements of the individual sets selected by the mappings of the one object in $\mathcal{C}_G$.

- A *groupoid* is an algebraic generalization of a group in which the binary relation $\star$ is replaced by a partial function. For details see [5].

- The category **Groupoids** has groupoids for objects and groupoid homomorphisms for arrows.
- Each groupoid may be conceived straightforwardly as a category all of whose arrows are invertible. Intuitively, these are categories that are structured like groups but which may have more than one object.

• A *partial order* on a set P is a binary relation $\leq$ defined on P that is reflexive, antisymmetric and transitive.

- The category **ParOrders** has partial orders for objects and monotonic functions between partial orders (functions that preserve the order relation $\leq$) as arrows.
- Every individual partial order $P, \leq$ may also be conceived as a category $\mathcal{C}_{P,\leq}$. The objects of $\mathcal{C}_{P,\leq}$ are the elements of P and given any two elements P_p and P_q of P, an arrow $a : P_p \longrightarrow P_q$ exists in $\mathcal{C}_{P,\leq}$ if and only if $P_p \leq P_q$.[4]

• Partial orders may be generalized to *preorders* by relaxing the antisymmetry condition of partial orders. In other words, a preorder is a set P with a binary relation $\leq$ defined on P that is reflexive and transitive.

- The category **PreOrders** has preorders for objects and order-preserving functions between preorders as arrows.
- Each preorder $P, \leq$ determines a category defined in exactly the same way as a partial order above, although of course without the relation $\leq$ having to be antisymmetric. Alternately, preorder categories may be characterized simply as categories such that there is at most one arrow in either direction between any two objects. Note that this implies in particular that for such a category, A and B are necessarily isomorphic if arrows exist $f : A \longrightarrow B$ and $g : B \longrightarrow A$.

The reader should recognize how the generalization of groups to groupoids is strictly analogous to that of partial orders to preorders. Furthermore, the reader should note how the respective categorical characterizations make this analogy particularly evident.

• Finally, we introduce a somewhat unorthodox category $\mathcal{R}_W$ of dyadic relations defined on some set I of individuals in a world W. The objects of the category are these individuals themselves, that is, elements of I, and morphisms in the category are all dyadic relations instantiated in the world W between all pairs of such individuals. We presume that every individual has the relation "is identical to" to itself, and we take advantage of the fact that dyadic relations compose naturally in a canonical way, namely by concatenation. More precisely, given three individuals A, B and C and two dyadic relations $r_1 : A \longrightarrow B$ and $r_2 : B \longrightarrow C$, a dyadic relation $r_{1,2} : A \longrightarrow C$ is determined as a relation between A and C that concatenates r_2 following r_1. The concatenated relation is "generic" in the sense that the mediating object/individual is replaced by an indeterminate "something". For instance, if r_1 is the relation "is next to" and r_2 is the relation "is father of" then $r_{1,2}$ is the relation "is next to something that is the father of". The reader should

[4] This means in particular that every topological space induces a partial order category whose objects are the open sets and with morphisms corresponding to inclusion maps.

check how the categorical axioms of identity, composition and associativity are satisfied by this characterization. The logical and philosophical interest of this category rests in how the *incidence relations* of the arrows are conceived and stipulated for various semantic and metaphysical views of what relations are and how they work.

A.8 Functor Categories

A special and highly important categorical type consists of categories the objects of which are *functors* and the arrows of which are mappings between functors called *natural transformations*. More precisely, given two categories $\mathcal{C}$ and $\mathcal{D}$, the functor category $\mathcal{D}^{\mathcal{C}}$ is be defined as follows:

- The objects of $\mathcal{D}^{\mathcal{C}}$ are all *functors* $\mathcal{C} \to \mathcal{D}$.
- The arrows of $\mathcal{D}^{\mathcal{C}}$ are all *natural transformations* between functors $\mathcal{C} \to \mathcal{D}$.

Natural transformations are morphisms between functors: given two functors $F \in \mathcal{D}^{\mathcal{C}}$ and $G \in \mathcal{D}^{\mathcal{C}}$, a natural transformation between F and G is a family of morphisms η_O parametrized by the objects $O \in \mathcal{C}$ such that the following diagram commutes for any two objects A and B that are connected by a morphism f in $\mathcal{C}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

A.9 Yoneda's Lemma Proof

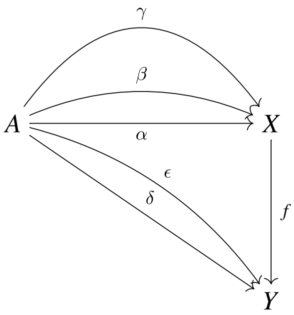
We follow here the presentation given in [6]. Let us restrict our previous constructions by setting $\mathcal{D} = \textbf{Sets}$ (with set functions as morphisms). In other words, let us consider the functor category $\textbf{Sets}^{\mathcal{C}}$ where objects are functors from an arbitrary category into the category of sets.[5]

We first define a canonical functor associated with every object of $\mathcal{C}$.

Definition 12 Given any object $A \in \mathcal{C}$, the *hom-functor based at* A is denoted by $Hom(A, -) : \mathcal{C} \longrightarrow \textbf{Sets}$ and takes any object $X \in \mathcal{C}$ into the set $Hom(A, X)$ of all the morphisms from A to X in $\mathcal{C}$. How arrows in $\mathcal{C}$ are treated by this functor is discussed below.

[5]For the rest of this section, unless otherwise specified, functors will be understood as belonging to $\textbf{Sets}^{\mathcal{C}}$.

Example 12



Let us assume the following structural equation for this diagram:

$$f \circ \alpha = \epsilon, f \circ \beta = \epsilon, f \circ \gamma = \delta$$

We then have

$$Hom(A, X) = \{\alpha, \beta, \gamma\}, Hom(A, Y) = \{\delta, \epsilon\}$$

Notice that so far we have only expressed the action of $Hom(A, -)$ on objects. It is an easy exercise to show that $Hom(A, -)$ is an actual functor by acting on morphisms by pre-composition.

In the example above, for instance, $Hom(A, f)$ is a map between the sets $Hom(A, X)$ and $Hom(A, Y)$ defined as follows:

$$Hom(A, f)(\alpha) = f \circ \alpha = \epsilon$$

$$Hom(A, f)(\beta) = f \circ \beta = \epsilon$$

$$Hom(A, f)(\gamma) = f \circ \gamma = \delta$$

Definition 13 A functor F for which there exists an object A such that $F(X) = Hom(A, X)$ for any object $X \in \mathcal{C}$ is said to be *representable*.

It is not quite true that all functors from $\mathcal{C}$ into **Sets** are representable. Yoneda's lemma, however, proves that all such functors can be obtained from hom-functors through natural transformations, and it explicitly parametrizes all such transformations:

Theorem 9 *There is a one-to-one correspondence between natural transformations from $Hom(A, -)$ to F and elements of $F(A)$.*

Proof Consider two objects A and B in $\mathcal{C}$ connected by a morphism f:

$$A \xrightarrow{f} B$$

The naturality square for pairs of functors F and $Hom(A, -)$ is the following:

$$\begin{CD} Hom(A, X) @>{Hom(A,f)}>> Hom(A, Y) \\ @V{\eta_X}VV @VV{\eta_Y}V \\ F(X) @>{F(f)}>> F(Y) \end{CD}$$

Consider now an element h of $Hom(A, X)$ and let us look at the equation expressed by the commutativity of the above diagram. By remembering that $Hom(A, f)$ acts by pre-compositions, we obtain

$$\eta_Y(f \circ h) = F(f)[\eta_X(h)]$$

By specializing this construction to the case $X = A$, we obtain the following naturality square:

$$\begin{CD} Hom(A, A) @>{Hom(A,f)}>> Hom(A, Y) \\ @V{\eta_A}VV @VV{\eta_Y}V \\ F(X) @>{F(f)}>> F(Y) \end{CD}$$

In this case h is a morphism from A into A. Since we know that id_A is for sure one of those, we have the following relation:

$$\eta_Y(f) = F(f)[\eta_A(id_A)]$$

Notice the implication of this result: the image of η_Y of any morphism $f \in Hom(A, X)$ is completely determined by assigning a value to η_A at the morphism id_A, that is, any choice of such value will determine a natural transformation.

Conversely, given any natural transformation η, this result tells us that we can evaluate η at id_A to obtain a point in $F(A)$. □

A.10 Further Developments

Higher-level categorical constructions based on Yoneda's lemma open up a vast and still largely uncharted domain of mathematics. In particular, the basic approach to mathematics as grounded in functor categories allows for a purely relational characterization of many established results that frequently provides new insights into their applicability in other mathematical areas. We refer the reader in particular to the intermediate-level text [7] for indications of possible directions for further exploration.

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Appendix B
Topos Theory: A Primer

Topos theory is a particular mathematical genre in which, among other things, the ontological and constructive aspects of abduction may be modeled. The core idea of a topos is that it is a type of category with sufficient resources to build up the objects and operations characterizing ordinary mathematics. Any topos is thus sufficiently rich in objects and relations to construct "meta-objects" (such as products $A \times B$) given any relevant constellation of objects within the topos (such as any pair of objects A and B). The most important point is that the topos is given as a whole: All the meta-objects by which it is structured are—from the internal perspective of the topos itself—simply additional objects. Thus the material out of which ordinary mathematics may be constructed within a topos is the fabric of structural relations that compose the topos itself as a unified and internally differentiated whole.

Topoi are essentially categories possessing a sufficiently rich internal structure of objects and relations to model many of the most important core mathematical and logical constructions needed in ordinary mathematical discourse. A particularly canonical topos and one that will help the uninitiated reader find his or her bearings throughout the following categorical characterization is the topos **Sets** of sets and functions. A topos may in many cases be thought of as a category that is set-like in certain determinate ways.

Topos theory constitutes a vast ocean of mathematical research and insight. Helpful introductions may be found in [1–3]. An aging but still highly relevant attempt to provide a comprehensive survey of topos theoretical results is found in [4]. What follows merely outlines the fundamental axioms of elementary topoi, with a special emphasis on explaining the precise sense in which topoi grasp the essential structure of the category of sets.

G. Caterina and R. Gangle, *Iconicity and Abduction*,
Studies in Applied Philosophy, Epistemology and Rational Ethics 29,
DOI 10.1007/978-3-319-44245-7

B.1 Structure of Sets

In this section we first recall some structural features of sets, using a set-theoretical language, with the goal of framing them in a categorical (hence relational) context. The important point in each of the following constructions is that a familiar property of sets and functions may be characterized in the category **Sets** solely in terms of relational properties of arrows. By then relying *only* on these categorical characterizations, a greatly enriched space of "set-like" categories emerges.

B.1.1 Injective Functions

A function f between two sets A and B is said to be *injective* if $(f(x) = f(y)) \Rightarrow x = y$.

The categorical translation of this property in **Sets** is given by the notion of monic arrow: Given

$$A \xrightarrow{\quad f \quad} B$$

f is said to be *monic* if, for any two arrows

$$Z \underset{g}{\overset{h}{\rightrightarrows}} A$$

such that the pair of compositions represented by the diagram

$$Z \underset{g}{\overset{h}{\rightrightarrows}} A \xrightarrow{\quad f \quad} B$$

are equal (that is, such that $f \circ g = f \circ h$), it must be the case that

$$h = g$$

It can be shown that, in the category **Sets**, every monic arrow corresponds to an injective function.

B.1.2 Pullbacks

Given two functions with the same codomain and possibly two different domains, say

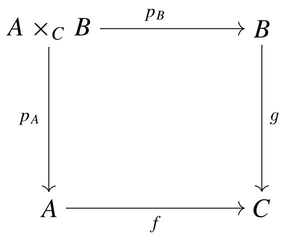
$$f : A \longrightarrow C$$

and

$$g : B \longrightarrow C,$$

we can form the set $A \times_C B$ constituted by the ordered pairs (a, b), with $a \in A$ and $b \in B$, such that $f(a) = g(b)$.

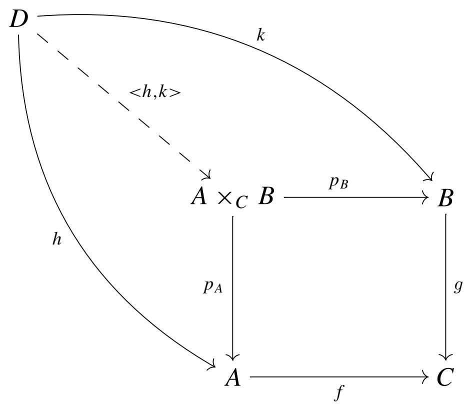
Let us now look at this construction within the categorical setting of **Sets**, by first considering the following diagram:



where p_A and p_B denote the projections over A and B, respectively.

As we learned in Appendix A, once a diagram is given in a category, a correspondent limit can be defined. In the case of the diagram above, taking the limit leads to the notion of *pullback* or *fibered product* of f and g. The reader should notice how the following definition refers only to objects and arrows in an otherwise undetermined category $\mathcal{C}$. In other words, the pullback or fibered product is here defined in a solely categorical manner (that generalizes the special notion of fibered product in set theory):

Definition 14 Let $f : A \longrightarrow C,\ g : B \longrightarrow C$ be a pair of morphisms in a category $\mathcal{C}$. A *pullback* (also called a *fibered product*) of f and g is an object $A \times_C B$ in $\mathcal{C}$ together with arrows $p_A : A \times_C B \longrightarrow A$ and $p_B : A \times_C B \longrightarrow B$, called projections, such that $f \circ p_A = g \circ p_B$, and for any object D in $\mathcal{C}$ and morphisms $h : D \longrightarrow A$ and $k : D \longrightarrow B$ such that $f \circ h = g \circ k$, there exists a unique morphism $< h, k >: D \longrightarrow A \times_C B$ such that the diagram



commutes.

We say that a category $\mathcal{C}$ has pullbacks iff every diagram

$$A \xrightarrow{f} C \xleftarrow{g} B$$

has a pullback.

One fundamental instance of pullbacks is given when g is a monic arrow, which, within a set-theoretical context, corresponds to declaring that—up to isomorphism of sets—$B \subseteq C$ (in the diagram above).[6] In this case, the pullback will return simply the pre-image of f as restricted to B:

$$A \times_C B \cong f^{-1}(C_{|B})$$

When this is generalized to categories, such a pullback along any monic arrow retains this feature of, roughly, the arrow as restricted to or fibered over the "sub-object" of C designated by the monic (see below for the formal characterization of subobjects).

B.1.3 Exponentials

Given two sets A and C, we can consider the collection C^A of all the functions from A to C:

$$C^A := \{f \mid f : A \longrightarrow C\}$$

and define the *evaluation* function

$$ev : A \times C^A \longrightarrow C$$

such that

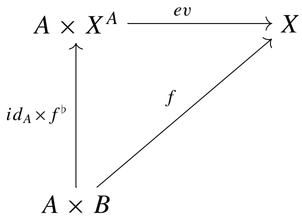
$$ev(x, f) = f(x)$$

This property of **Sets** can be generalized in strictly categorical terms as follows [5].

Definition 15 An object A of a category $\mathcal{C}$ is called *exponentiable* iff for every object X of $\mathcal{C}$ there exists an object X^A, called an *exponential*, and a morphism $ev : A \times X^A \longrightarrow X$ called an *evaluation*, such that for any $f : A \times B \rightarrow X$ there

[6]Throughout the present discussion we gloss over the distinction between subset and injective function (that is, subset equivalence up to isomorphism) for ease of exposition.

exists a unique $f^\flat : B \longrightarrow X^A$, called the *exponential transpose*, for which the diagram:



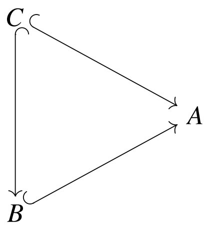
commutes. We say that category $\mathcal{C}$ has exponentials iff every object is exponentiable.

We leave as an exercise to the reader to show that the category **Sets** has exponentials.

Definition 16 A category $\mathcal{C}$ is called *cartesian closed* if and only if all objects of $\mathcal{C}$ are exponentiable (equivalently, $\mathcal{C}$ has exponentials) and $\mathcal{C}$ has finite products.

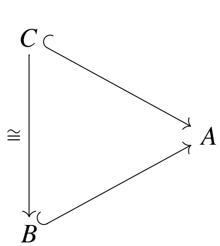
B.1.4 Subobjects

We can represent three nested sets A, B and C such that $C \subseteq B \subseteq A$ via the set-theoretical diagram



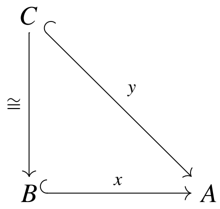
where the hooked arrow represents inclusion (an injection).

If we want to express the fact that C is (set-theoretically) equivalent to B, we need to replace the inclusion with a bijection, so that we get the following commutative diagram



Let us now lift this construction into a purely categorical framework.

Definition 17 Two monic arrows x and y in a category $\mathcal{C}$ which satisfy



are called *equivalent*, which is denoted as $x \sim y$. The equivalence class of x is denoted as $[x] = \{y \mid y \sim x\}$. A *subobject* of any object A in $\mathcal{C}$ is defined as an equivalence class of monic arrows into A. The class of subobjects of an object A is denoted as

$$Sub(A) := \{[f] \mid cod(f) = A \; and \; f \; is \; monic\}$$

B.1.5 Subobject Classifier

Armed with the notions of monic arrow, subobject and pullback, we can now derive the categorical analogue to the set-theoretical notion of *characteristic function* for sets. Recall that, given a set X and a subset $A \subseteq X$, the characteristic function $\chi_A : X \longrightarrow \{0, 1\}$ takes every element of A into 1 and every element in the complement of A into 0. In other words, the notion of B being a subset of A can be expressed, in a set-theoretical context, using the characteristic function $\chi_B : A \to \{0, 1\}$, which is defined as follows:

$$\chi_B(x) = \begin{cases} 1 & if \; x \in B \\ 0 & if \; x \notin B \end{cases}$$

Working in the category **Sets**, the same idea is expressed by treating the inclusion $1 \subseteq 2$ as a stipulated monic arrow from an arbitary one-element set into an arbitary two-element set. This "true" arrow $\top$ then works to characterize monic arrows (set-theoretic inclusions) in exactly in the same way. To show this, let us consider a set **2** (that is, any two-element set), a terminal object **1** (that is, any singleton set), an inclusion arrow (a monic) $f : B \longrightarrow A$ that represents the subset B as an injective function, and finally a function $g : A \longrightarrow \mathbf{2}$. Because **1** is a terminal object, by definition there is exactly one arrow from B into it. We thus have the following data, diagrammatically arranged:

$$\begin{array}{ccc} B & \overset{!}{\dashrightarrow} & \mathbf{1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \top} \\ A & \underset{g}{\longrightarrow} & \mathbf{2} \end{array}$$

Then the reader should see that only if that diagram is a pullback is it the case that g in fact properly represents the inclusion (monic) arrow from B to A, that is, $g = \chi_f$:

$$\begin{array}{ccc} B & \overset{!}{\dashrightarrow} & \mathbf{1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \top} \\ A & \underset{\chi_f}{\longrightarrow} & \mathbf{2} \end{array}$$

Notice the shift in notation from χ_B to χ_f. Thus the characteristic function in a categorical framework represents a monic arrow (a function), not strictly speaking an object. From a categorical perspective, a subset is a type of arrow, not an object.

Notice the heuristic of all of this: for any set B included in A as represented by an inclusion arrow f, the two-element set **2** is such that f is uniquely represented by a certain "special function" (χ_f) into **2**.

The generalization of this construction from **Sets** to a general category is immediate:

Definition 18 A *subobject classifier* or a generalised truth-value object is an object Ω in $\mathcal{C}$, together with an arrow $\top : \mathbf{1} \to \Omega$, called the true arrow, such that for each monic arrow $m : B \hookrightarrow A$ there is a unique arrow $\chi_m : A \longrightarrow \Omega$, called the characteristic arrow of m (or of B), such that the diagram

$$\begin{array}{ccc} B & \overset{!}{\dashrightarrow} & \mathbf{1} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \top} \\ A & \underset{\chi_m}{\longrightarrow} & \Omega \end{array}$$

is a pullback.

Therefore, the generalization of the role of **2** in **Sets** to the role of Ω in a category $\mathcal{C}$ allows for a more differentiated structure of the analogue of set-theoretical complementation. In short, there may be a richer array of truth-values (more than just two).

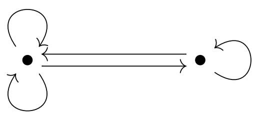
B.2 Topoi

It goes without saying that the environment of sets and functions is a useful one for making a great variety of mathematical constructions. What if precisely the features that make this environment so useful for doing mathematics could be abstracted from those residual features of the set-theoretical universe(s) that may be at times unwieldy or even obstructive to mathematical investigation and production? In effect, this is what the abstraction from the category **Sets** to the more general notion of a topos accomplishes. We saw above how certain fundamental constructions in sets may be characterized in a purely relational or "arrows-only" fashion and then lifted to a more general categorical context. We are now prepared to characterize the result of this process through the categorical notion of a topos.

Definition 19 A category $\mathcal{C}$ is a *topos* (sometimes called an elementary topos) iff it is a cartesian closed category with a subobject classifier.

It is remarkable that such a simple definition can give rise to such a varied and powerful mathematical framework. We conclude by merely indicating several important examples of topoi:

- The canonical topos is the category **Sets** of sets and functions. In the previous sections we have indeed seen that **Sets** satisfies the elementary topos axioms, and this provides a helpful intuition of how the axioms together determine the possibility of certain basic mathematical constructions, even in topoi that are much more highly-structured or pathologically underdetermined from the standpoint of standard set theory.
- The category **Graphs** of directed graphs and graph homomorphisms is a useful environment for modeling networks, dynamical systems and many other related phenomena. **Graphs** is also a topos. The reader will find it instructive to show as an exercise that this category is indeed cartesian closed and to derive the necessary structure of its subobject classifier. As a helpful guiding thread, we note that the subobject classifier Ω of the topos **Graphs** is in fact the directed graph pictured here:



- An important class of topoi is given by the following construction. Take any small category $\mathcal{C}$ and then generate the category $\mathbf{Sets}^{\mathcal{C}^{\mathbf{op}}}$ of contravariant functors from $\mathcal{C}$ into **Sets**. This category of functors (its objects are functors and its arrows are natural transformations) will always be a topos. As one example, take the simple category $\mathcal{D}$ consisting of just two objects A and B, the identity arrows Id_A and Id_B and two distinct arrows $f : A \longrightarrow B$ and $g : A \longrightarrow B$. Any contravariant functor F from $\mathcal{D}$ into **Sets** will correspond to a directed graph ($F(A)$ will be a set representing the graph's dots and $F(B)$ will be a set representing the graph's arrows; $F(f)$ and $F(g)$ determine *source* and *target* dots respectively for every arrow). A natural transformation from one such functor to another will correspond to a homomorphism of directed graphs. Thus the topos **Graphs** discussed immediately above may also be represented as a functor category $\mathbf{Sets}^{\mathcal{D}^{\mathbf{op}}}$. In this way, the reader may perhaps begin to grasp how the basic conceptual approach of categories (characterizing the "internal" structure of objects through "external" relations to other objects of its type) carries over to the rich terrain of topoi as well.

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