Appendix A
Background Material in Linear Systems Theory

A.1 Quadratic Forms

In this section, a few fundamental facts about symmetric matrices and quadratic forms are reviewed.

Symmetric matrices. Let P be a $n \times n$ *symmetric* matrix of real numbers (that is, a matrix of real numbers satisfying $P = P^{\mathrm{T}}$). Then there exist an *orthogonal* matrix Q of real numbers[1] and a diagonal matrix Λ of real numbers

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

such that

$$Q^{-1}PQ = Q^{\mathrm{T}}PQ = \Lambda .$$

Indeed, the numbers $\lambda_1, \dots, \lambda_n$ are the eigenvalues of P. Thus, a symmetric matrix P of real numbers has real eigenvalues and a purely diagonal Jordan form.

Note that the previous identity can be rewritten as

$$PQ = Q\Lambda$$

from which it is seen that the ith column q_i of Q is an eigenvector of P, associated with the ith eigenvalue λ_i. If P is invertible, so is the matrix Λ, and

$$P^{-1}Q = Q\Lambda^{-1} ,$$

[1]That is, matrix Q of real numbers satisfying $QQ^{\mathrm{T}} = I$, or—what is the same—satisfying $Q^{-1} = Q^{\mathrm{T}}$.

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from which it is seen that Λ^{-1} is a Jordan form of P^{-1} and the ith column q_i of Q is also an eigenvector of P^{-1}, associated with the ith eigenvalue λ_i^{-1}.

Quadratic forms. Let P be a $n \times n$ matrix of real numbers and $x \in \mathbb{R}^n$. The expression

$$V(x) = x^{\mathrm{T}} P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

is called a *quadratic form* in x. Without loss of generality, in the expression above we may assume that P is *symmetric*. In fact,

$$V(x) = x^{\mathrm{T}} P x = \frac{1}{2}[x^{\mathrm{T}} P x + x^{\mathrm{T}} P x] = x^{\mathrm{T}} \left[\frac{1}{2}(P + P^{\mathrm{T}}) \right] x$$

and $\frac{1}{2}(P + P^{\mathrm{T}})$ is by construction symmetric.

Let P be symmetric, express it as $P = Q \Lambda Q^{\mathrm{T}}$ (see above) with eigenvalues sorted so that $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$\begin{aligned} x^{\mathrm{T}} P x &= x^{\mathrm{T}} (Q \Lambda Q^{\mathrm{T}}) x = (Q^{\mathrm{T}} x)^{\mathrm{T}} \Lambda (Q^{\mathrm{T}} x) = \textstyle\sum_{i=1}^{n} \lambda_i (Q^{\mathrm{T}} x)_i^2 \\ &\leq \lambda_1 \textstyle\sum_{i=1}^{n} (Q^{\mathrm{T}} x)_i^2 = \lambda_1 (Q^{\mathrm{T}} x)^{\mathrm{T}} Q^{\mathrm{T}} x = \lambda_1 x^{\mathrm{T}} Q Q^{\mathrm{T}} x = \lambda_1 x^{\mathrm{T}} x \\ &= \lambda_1 \|x\|^2. \end{aligned}$$

With a similar argument we can show that $x^{\mathrm{T}} P x \geq \lambda_n \|x\|^2$. Usually, λ_n is denoted as $\lambda_{\min}(P)$ and λ_1 is denoted as $\lambda_{\max}(P)$. In summary, we conclude that

$$\lambda_{\min}(P) \|x\|^2 \leq x^{\mathrm{T}} P x \leq \lambda_{\max}(P) \|x\|^2.$$

Note that the inequalities are tight (*hint*: pick, as x, the last and, respectively, the first column of Q).

Sign-definite symmetric matrices. Let P be symmetric. The matrix P is said to be *positive semidefinite* if

$$x^{\mathrm{T}} P x \geq 0 \qquad \text{for all } x.$$

The matrix P is said to be *positive definite* if

$$x^{\mathrm{T}} P x > 0 \qquad \text{for all } x \neq 0.$$

We see from the above that P is positive semidefinite if and only if $\lambda_{\min} \geq 0$ and is positive definite if and only if $\lambda_{\min} > 0$ (which in turn implies the nonsingularity of P).

There is another criterion for a matrix to be positive definite, that does not require the computation of the eigenvalues of P, known as Sylvester's criterion. For a *symmetric* matrix P, the n minors

$$D_1 = \det(p_{11}), \quad D_2 = \det\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad D_3 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \ldots$$

are called the *leading principal minors*. Note that $D_n = \det(P)$.

Lemma A.1 *A symmetric matrix is positive definite if and only if all leading principal minors are positive, i.e.,* $D_1 > 0,\ D_2 > 0, \ldots, D_n > 0$.

Another alternative criterion, suited a for block-partitioned matrix, is the criterion due to Schur.

Lemma A.2 *The symmetric matrix*

$$\begin{pmatrix} Q & S \\ S^{\mathrm{T}} & R \end{pmatrix} \tag{A.1}$$

is positive definite if and only if[2]

$$R > 0 \qquad \text{and} \qquad Q - SR^{-1}S^{\mathrm{T}} > 0. \tag{A.2}$$

Proof Observe that a necessary condition for (A.1) to be positive definite is $R > 0$. Hence R is nonsingular and (A.1) can be transformed, by congruence, as

$$\begin{pmatrix} I & 0 \\ -R^{-1}S^{\mathrm{T}} & I \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & S \\ S^{\mathrm{T}} & R \end{pmatrix} \begin{pmatrix} I & 0 \\ -R^{-1}S^{\mathrm{T}} & I \end{pmatrix} = \begin{pmatrix} Q - SR^{-1}S^{\mathrm{T}} & 0 \\ 0 & R \end{pmatrix}.$$

from which the condition (A.2) follows. ◁

A symmetric matrix P is said to be *negative semidefinite* (respectively, *negative definite*) if $-P$ is *positive semidefinite* (respectively, *positive definite*). Usually, to express the property (of a matrix P) of being positive definite (respectively, positive semidefinite) the notation $P > 0$ (respectively, $P \geq 0$) is used.[3] Likewise, the notation $P < 0$ (respectively $P \leq 0$) is used to express the property that P is negative definite (respectively, negative semidefinite). If P and R are symmetric matrices, the notations

$$P \geq R \qquad \text{and} \qquad P > R$$

stand for "*the matrix* $P - R$ *is positive semidefinite*" and, respectively, for "*the matrix* $P - R$ *is positive definite*".

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[2]The matrix $Q - SR^{-1}S^{\mathrm{T}}$ is called the *Schur's complement* of R in (A.1).

[3]Note that this *is not* the same as $p_{ij} > 0$ for all i, j. For example, the matrix

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

in which the off-diagonal elements are negative, is positive definite (use Sylvester's criterion above).

Any matrix P that can be written in the form $P = M^{\mathrm{T}}M$, in which M is a possibly non-square matrix, is positive semidefinite. In fact $x^{\mathrm{T}}Px = x^{\mathrm{T}}M^{\mathrm{T}}Mx = \|Mx\|^2 \geq 0$.

Conversely, any matrix P which is positive semidefinite can always be expressed as $P = M^{\mathrm{T}}M$. In fact, if P is positive semidefinite all its eigenvalues are nonnegative. Let r denote the number of nonzero eigenvalues and let the eigenvalues be sorted so that $\lambda_{r+1} = \cdots = \lambda_n = 0$. Then

$$P = Q\Lambda Q^{\mathrm{T}} = Q\begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^{\mathrm{T}}$$

in which Λ_1 is an $r \times r$ diagonal matrix, whose diagonal elements are all positive. For $i = 1, \ldots, r$, let σ_i denote the positive square root of λ_i, let Q_1 be the $n \times r$ matrix whose columns coincide with the first r columns of Q and set

$$M = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_r \end{pmatrix} Q_1^{\mathrm{T}}$$

Then, the previous identity yields

$$P = M^{\mathrm{T}}M$$

in which M is a $n \times r$ matrix of rank r.

Finally, recall that, if P is symmetric and invertible, the eigenvalues of P^{-1} are the inverse of the eigenvalues of P. Thus, in particular, if P is positive definite, so is also P^{-1}.

A.2 Linear Matrix Equations

In this section, we discuss the existence of solutions of two relevant linear matrix equations that arise in the analysis of linear systems. One of such equation is the so-called *Sylvester's equation*

$$AX - XS = R \tag{A.3}$$

in which $A \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{d\times d}$, in the unknown $X \in \mathbb{R}^{n\times d}$. An equation of this kind arises, for instance, when it is desired to transform a given block-triangular matrix into a (purely) block-diagonal one, by means of a similarity transformation, as in

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & R \\ 0 & S \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Another instance in which an equation of this kind arises is the analysis of the stability of a linear system, where this equation assumes the special form $AX + XA^{\mathrm{T}} = Q$, known as *Lyapunov's equation*.

Another relevant linear matrix equation is the so-called *regulator* or *Francis's equation*

$$\begin{aligned} \Pi S &= A\Pi + B\Psi + P \\ 0 &= C\Pi + Q \end{aligned} \tag{A.4}$$

in which $A \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, in the unknowns $\Pi \in \mathbb{R}^{n\times d}$ and $\Psi \in \mathbb{R}^{m\times d}$. This equation arises in the study of the problem of output regulation of linear systems.

The two equations considered above are special cases of an equation of the form

$$A_1Xq_1(S) + \cdots + A_kXq_k(S) = R \tag{A.5}$$

in which, for $i = 1, \ldots, k$, $A_i \in \mathbb{R}^{\bar{n}\times\bar{m}}$ and $q_i(\lambda)$ is a polynomial in the indeterminate λ, $S \in \mathbb{R}^{\bar{d}\times\bar{d}}$, $R \in \mathbb{R}^{\bar{n}\times\bar{d}}$, in the *unknown* $X \in \mathbb{R}^{\bar{m}\times\bar{d}}$. In fact, the Sylvester's equation corresponds to the case in which $k = 2$ and

$$A_1 = A, \quad q_1(\lambda) = 1, \quad A_2 = I, \quad q_2(\lambda) = -\lambda,$$

while the Francis' equation corresponds to the case in which $k = 2$ and

$$A_1 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad q_1(\lambda) = 1, \quad A_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad q_2(\lambda) = -\lambda, \quad R = \begin{pmatrix} -P \\ -Q \end{pmatrix}.$$

Equations of the form (A.5) are also known as *Hautus' equations*.[4] Noting that the left-hand side of (A.5) can be seen as a linear map

$$\begin{aligned} \mathscr{H} : \mathbb{R}^{\bar{m}\times\bar{d}} &\to \mathbb{R}^{\bar{n}\times\bar{d}} \\ : \quad X &\mapsto \mathscr{H}(X) := A_1Xq_1(S) + \cdots + A_kXq_k(S), \end{aligned}$$

to say that (A.5) has a solution is to say that $R \in \mathrm{Im}(\mathscr{H})$.

In what follows, we are interested in the case in which (A.5) has solutions for all R, i.e., in the case in which the map $\mathscr{H}$ is *surjective*.[5]

Theorem A.1 *The map $\mathscr{H}$ is surjective if and only if the $\bar{n}$ rows of the matrix*

$$A(\lambda) = A_1q_1(\lambda) + \cdots + A_kq_k(\lambda)$$

are linearly independent for each λ which is an eigenvalue of S. If this is the case and $\bar{n} = \bar{m}$, the solution X of (A.5) is unique.

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[4] See [3].

[5] Note that, if this is the case and $\bar{n} = \bar{m}$, the map is also *injective*, i.e., it is an invertible linear map. In this case the solution X of (A.5) is *unique*.

From this, it is immediate to deduce the following Corollaries.

Corollary A.1 *The Sylvester's equation (A.3) has a solution for each R if and only if* $\sigma(A) \cap \sigma(S) = \emptyset$*. If this is the case, the solution X is unique.*

Corollary A.2 *The Francis' equation (A.4) has a solution for each pair* (P, Q) *if and only if the rows of the matrix*

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$$

are linearly independent for each λ *which is an eigenvalue of S. If this is the case and* $m = p$*, the solution pair* (Π, Ψ) *is unique.*

A.3 The Theorems of Lyapunov for Linear Systems

In this section we describe a powerful criterion useful to determine when a $n \times n$ matrix of real numbers has all eigenvalues with negative real part.[6]

Theorem A.2 (Direct Theorem) *Let* $A \in \mathbb{R}^{n \times n}$ *be a matrix of real numbers. Let* $P \in \mathbb{R}^{n \times n}$ *be a symmetric positive-definite matrix of real numbers and suppose that the matrix*

$$PA + A^{\mathrm{T}}P$$

is negative definite. Then, all eigenvalues of the matrix A have negative real part.

Proof Let λ be an eigenvalue of A and x an associated eigenvector. Let x_R and x_I denote the real and, respectively, imaginary part of x, i.e., set $x = x_R + jx_I$ and let $x^* = x_R^{\mathrm{T}} - jx_I^{\mathrm{T}}$. Then

$$x^*Px = (x_R)^{\mathrm{T}}Px_R + (x_I)^{\mathrm{T}}Px_I.$$

Since P is positive definite and x_R and x_I cannot be both zero (because $x \neq 0$), we deduce that

$$x^*Px > 0. \tag{A.6}$$

With similar arguments, since $A^{\mathrm{T}}P + PA$ is negative definite, we deduce that

$$x^*(A^{\mathrm{T}}P + PA)x < 0. \tag{A.7}$$

Using the definition of x and λ (i.e., $Ax = x\lambda$, that implies $x^*A^{\mathrm{T}} = \lambda^*x^*$), obtain

$$x^*(A^{\mathrm{T}}P + PA)x = \lambda^*x^*Px + x^*Px\lambda = (\lambda + \lambda^*)x^*Px$$

[6] For further reading, see e.g., [1].

from which, using (A.6) and (A.7) we conclude

$$\lambda + \lambda^* = 2\mathrm{Re}[\lambda] < 0. \quad \triangleleft$$

Remark A.1 The criterion described in the previous theorem provides a sufficient condition under which all the eigenvalues of a matrix A have negative real part. In the analysis of linear systems, this criterion is used as a sufficient condition to determine whether the equilibrium $x = 0$ of the autonomous system

$$\dot{x} = Ax \tag{A.8}$$

is (globally) asymptotically stable. In this context, the previous proof—which only uses algebraic arguments—can be replaced by the linear version of the proof of Theorem B.1, which can be summarized as follows. Let $V(x) = x^{\mathrm{T}}Px$ denote the positive-definite quadratic function associated with P, let $x(t)$ denote a generic trajectory of system (A.8) and consider the composite function $V(x(t)) = x^{\mathrm{T}}(t)Px(t)$. Observe that

$$\frac{\partial V}{\partial x} = 2x^{\mathrm{T}}P.$$

Therefore, using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \left.\frac{\partial V}{\partial x}\right|_{x=x(t)} \frac{\mathrm{d}x}{\mathrm{d}t} = 2x^{\mathrm{T}}(t)PAx(t) = x^{\mathrm{T}}(t)(PA + A^{\mathrm{T}}P)x(t).$$

If P is positive definite and $PA+A^{\mathrm{T}}P$ is negative definite, there exist positive numbers a_1, a_2, a_3 such that

$$a_1\|x\|^2 \le V(x) \le a_2\|x\|^2 \quad \text{and} \quad x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x \le -a_3\|x\|^2.$$

From this, it is seen that $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le -\lambda V(x(t))$$

with $\lambda = a_3/a_2 > 0$ and therefore

$$a_1\|x(t)\|^2 \le V(x(t)) \le e^{-\lambda t}V(x(0)) \le e^{-\lambda t}a_2\|x(0)\|^2.$$

Thus, for any initial condition $x(0)$, $\lim_{t\to\infty} x(t) = 0$. This proves that all eigenvalues of A have negative real part. $\triangleleft$

Theorem A.3 (Converse Theorem) *Let $A \in \mathbb{R}^{n\times n}$ be a matrix of real numbers. Suppose all eigenvalues of A have negative real part. Then, for any choice of a symmetric positive-definite matrix Q, there exists a unique symmetric positive-definite matrix P such that*

$$PA + A^{\mathrm{T}}P = -Q. \tag{A.9}$$

Proof Consider (A.9). This is a Sylvester equation, and—since the spectra of A and $-A^{\mathrm{T}}$ are disjoint—a unique solution P exists. We compute it explicitly. Define

$$M(t) = e^{A^{\mathrm{T}}t}Qe^{At}$$

and observe that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At} + e^{A^{\mathrm{T}}t}Qe^{At}A = A^{\mathrm{T}}M(t) + M(t)A.$$

Integrating over $[0, T]$ yields

$$M(T) - M(0) = A^{\mathrm{T}}\int_0^T M(t)dt + \int_0^T M(t)dt\, A.$$

Since the eigenvalues of A have negative real part,

$$\lim_{T\to\infty} M(T) = 0$$

and

$$P := \lim_{T\to\infty}\int_0^T M(t)dt < \infty.$$

We have shown in this way that P satisfies (A.9). It is the unique solution of this equation.

To complete the proof it remains to show that P is positive definite, if so is Q. By contradiction, suppose is not. Then there exists $x_0 \neq 0$ such that

$$x_0^{\mathrm{T}}Px_0 \leq 0\,,$$

which, in view of the expression found for P, yields

$$\int_0^\infty x_0^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At}x_0dt \leq 0.$$

Setting

$$x(t) = e^{At}x_0$$

this is equivalent to

$$\int_0^\infty x^{\mathrm{T}}(t)Qx(t)dt \leq 0\,,$$

which, using the estimate $x^{\mathrm{T}}Qx \geq \lambda_{\min}(Q)\|x\|^2$ in turn yields

$$\int_0^\infty \|x(t)\|^2 dt \leq 0$$

which then yields

$$x(t) = 0, \qquad \text{for all } t \in [0, \infty).$$

Bearing in mind the expression of $x(t)$, this implies $x_0 = 0$ and completes the proof.

A.4 Stabilizability, Detectability and Separation Principle

In this section, a few fundamental facts about the stabilization of linear systems are reviewed.[7] Consider a linear system modeled by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{A.10}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$ and we summarize the properties that determine the existence of a (dynamic) output feedback controller of the form

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y, \end{aligned} \tag{A.11}$$

with state $x_c \in \mathbb{R}^{n_c}$, that stabilizes the resulting closed-loop system

$$\begin{aligned} \dot{x} &= (A + BD_c C)x + BC_c x_c \\ \dot{x}_c &= B_c C x + A_c x_c. \end{aligned} \tag{A.12}$$

Definition A.1 The pair (A, B) is *stabilizable* if there exists a matrix F such that $(A + BF)$ has all the eigenvalues in $\mathbb{C}^-$.

Definition A.2 The pair (A, C) is *detectable* if there exists a matrix G such that $(A - GC)$ has all the eigenvalues in $\mathbb{C}^-$.

Noting that the closed-loop system (A.12) can be written as $\dot{x}_{c\ell} = A_{c\ell} x_{c\ell}$, with $x_{c\ell} = \text{col}(x, x_c)$ and

$$A_{c\ell} = \begin{pmatrix} (A + BD_c C) & BC_c \\ B_c C & A_c \end{pmatrix}, \tag{A.13}$$

we have the following fundamental result.

Theorem A.4 *There exist matrices A_c, B_c, C_c, D_c such that (A.13) has all the eigenvalues in $\mathbb{C}^-$ if and only if the pair (A, B) is stabilizable* and *pair (A, C) is detectable.*

[7] For further reading, see e.g., [1].

Proof (Necessity) Suppose all eigenvalues of (A.13) have negative real part. Then, by the converse Lyapunov's Theorem, there exists a unique, symmetric, and positive-definite solution $P_{c\ell}$ of the matrix equation

$$P_{c\ell}A_{c\ell} + A_{c\ell}^{T}P_{c\ell} = -I. \tag{A.14}$$

Let $P_{c\ell}$ be partitioned as in

$$P_{c\ell} = \begin{pmatrix} P & S \\ S^{T} & P_{c} \end{pmatrix}$$

consistently with the partition of $A_{c\ell}$ (note, in this respect, that the two diagonal blocks may have different dimensions n and n_c). Note also that P and P_c are necessarily positive definite (and hence also nonsingular) because so is $P_{c\ell}$. Consider the matrix

$$T = \begin{pmatrix} I & 0 \\ -P_{c}^{-1}S^{T} & I \end{pmatrix}.$$

Define $\tilde{P} := T^{T}P_{c\ell}T$ and note that

$$\tilde{P} = \begin{pmatrix} P - SP_{c}^{-1}S^{T} & 0 \\ 0 & P_{c} \end{pmatrix}. \tag{A.15}$$

Define $\tilde{A} := T^{-1}A_{c\ell}T$ and, by means of a simple computation, observe that

$$\tilde{A} = \begin{pmatrix} A + B(D_{c}C - C_{c}P_{c}^{-1}S^{T}) & * \\ * & * \end{pmatrix}$$

in which we have denoted by an asterisk blocks whose expression is not relevant in the sequel.

From (A.14) it is seen that

$$\begin{aligned} \tilde{P}\tilde{A} + \tilde{A}^{T}\tilde{P} &= (T^{T}P_{c\ell}T)(T^{-1}A_{c\ell}T) + (T^{T}A_{c\ell}^{T}(T^{-1})^{T})(T^{T}P_{c\ell}T) \\ &= T^{T}(P_{c\ell}A_{c\ell} + A_{c\ell}^{T}P_{c\ell})T = -T^{T}T. \end{aligned} \tag{A.16}$$

This shows that the matrix $\tilde{P}\tilde{A} + \tilde{A}^{T}\tilde{P}$ is negative definite (because T is nonsingular) and hence so is its upper-left block. The latter, if we set

$$P_0 = P - SP_{c}^{-1}S^{T}, \qquad F = D_{c}C - C_{c}P_{c}^{-1}S^{T}$$

can be written in the form

$$P_0(A + BF) + (A + BF)^{T}P_0. \tag{A.17}$$

The matrix P_0 is positive definite, because it is the upper-left block of the positive-definite matrix (A.15). The matrix (A.17) is negative definite, because it is the upper-left block of the negative-definite matrix (A.16). Thus, by the direct criterion of Lyapunov, it follows that the eigenvalues of $A + BF$ have negative real part. This completes the proof that, if $A_{c\ell}$ has all eigenvalues in $\mathbb{C}^-$, there exists a matrix F such that $A + BF$ has all eigenvalues in $\mathbb{C}^-$, i.e., the pair (A, B) is stabilizable. In a similar way it is proven that the pair (A, C) is detectable.

(Sufficiency) Assuming that (A, B) is stabilizable and that (A, C) is detectable, pick F and G so that $(A+BF)$ has all eigenvalues in $\mathbb{C}^-$ and $(A-GC)$ has all eigenvalues in $\mathbb{C}^-$. Consider the controller

$$\begin{aligned} \dot{x}_c &= (A + BF - GC)x_c + Gy \\ u &= Fx_c, \end{aligned} \tag{A.18}$$

i.e., set

$$A_c = A + BF - GC, \quad B_c = G, \quad C_c = F, \quad D_c = 0.$$

This yields a closed-loop system

$$\begin{aligned} \dot{x} &= Ax + BFx_c \\ \dot{x}_c &= GCx + (A + BF - GC)x_c. \end{aligned}$$

The change of variables $z = x - x_c$ changes the latter into the system

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A + BF & -BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

This system is in block-triangular form and both diagonal blocks have all eigenvalues in $\mathbb{C}^-$. Thus the controller (A.18) guarantees that the matrix $A_{c\ell}$ has all eigenvalues in $\mathbb{C}^-$.[8]

To check whether the two fundamental properties in question hold, the following tests are useful.

Lemma A.3 *The pair* (A, B) *is stabilizable if and only if*

$$\operatorname{rank} \begin{pmatrix} A - \lambda I & B \end{pmatrix} = n \tag{A.19}$$

for all $\lambda \in \sigma(A)$ *having nonnegative real part.*

[8] Observe that the choices of F and G are *independent* of each other, i.e., F is only required to place the eigenvalues of $(A + BF)$ in $\mathbb{C}^-$ and G is only required to place the eigenvalues of $(A - GC)$ in $\mathbb{C}^-$. For this reason, the controller (A.18) is said to be a controller inspired by a *separation principle*.

Lemma A.4 *The pair* (A, C) *is detectable if and only if*

$$\text{rank}\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} = n \tag{A.20}$$

for all $\lambda \in \sigma(A)$ *having nonnegative real part.*

Remark A.2 For the sake of completeness, we recall how the two properties of stabilizability and detectability invoked above compare with the properties of reachability and observability. To this end, we recall that the linear system (A.25) is *reachable* if and only if

$$\text{rank}\begin{pmatrix} B \; AB \; \dots \; A^{n-1}B \end{pmatrix} = n \tag{A.21}$$

or, what is the same, if and only if the condition (A.19) holds *for all* $\lambda \in \sigma(A)$ (and not just for all such λ's having nonnegative real part). The linear system (A.25) is *observable* if and only if

$$\text{rank}\begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} = n \tag{A.22}$$

or, what is the same, if and only if the condition (A.20) holds *for all* $\lambda \in \sigma(A)$ (and not just for all such λ's having nonnegative real part).

It is seen from this that, in general, reachability is a property stronger than stabilizability and observability is a property stronger than detectability. The two (pairs of) properties coincide when all eigenvalues of A have nonnegative real part. If the rank of the matrix on the left-hand side of (A.21) is $n_1 < n$, the system is *not* reachable and there exists a nonsingular matrix T such that[9]

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad TB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

in which $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ and the pair (A_{11}, B_1) is reachable. This being the case, it is easy to check that the pair (A, B) is stabilizable if and only if all eigenvalues of A_{22} have negative real part. A similar criterion determines the relation between detectability and observability. If the rank of the matrix on the left-hand side of (A.22) is $n_1 < n$, the system is *not* observable and there exists a nonsingular matrix T such that

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad CT^{-1} = \begin{pmatrix} 0 & C_2 \end{pmatrix}$$

in which $A_{22} \in \mathbb{R}^{n_1 \times n_1}$ and the pair (A_{22}, C_2) is observable. This being the case, it is easy to check that the pair (A, C) is detectable if and only if all eigenvalues of A_{11} have negative real part.

[9]This is the well-know *Kalman's* decomposition of a system into reachable/unreachable parts.

A.5 Steady-State Response to Harmonic Inputs

Invariant subspaces. Let A be a fixed $n \times n$ matrix. A subspace $\mathscr{V}$ of $\mathbb{R}^n$ is *invariant* under A if

$$v \in \mathscr{V} \quad \Rightarrow \quad Av \in \mathscr{V}.$$

Let $d < n$ denote the dimension of $\mathscr{V}$ and let $\{v_1, v_2, \ldots, v_d\}$ be a basis of $\mathscr{V}$, that is a set of d linearly independent vectors $v_i \in \mathbb{R}^n$ such that

$$\mathscr{V} = \mathrm{Im}(V)$$

where V is the $n \times d$ matrix

$$V = \begin{pmatrix} v_1 \; v_2 \; \ldots \; v_d \end{pmatrix}.$$

Then, it is an easy matter to check that $\mathscr{V}$ is invariant under A if and only if there exists a $d \times d$ matrix $A_{\mathscr{V}}$ such that

$$AV = VA_{\mathscr{V}}.$$

The map $z \mapsto A_{\mathscr{V}}z$ characterizes the *restriction* to $\mathscr{V}$ of the map $x \mapsto Ax$. This being the case, observe that if (λ_0, z_0) is a pair eigenvalue-eigenvector for $A_{\mathscr{V}}$ (i.e., a pair satisfying $A_{\mathscr{V}}z_0 = \lambda_0 z_0$), then (λ_0, Vz_0) is a pair eigenvalue-eigenvector for A.

Let the matrix A have n_s eigenvalues in $\mathbb{C}^-$, n_a eigenvalues in $\mathbb{C}^+$ and n_c eigenvalues in $\mathbb{C}^0 = \{\lambda \in \mathbb{C} : \mathrm{Re}[\lambda] = 0\}$, with (obviously) $n_\mathrm{s} + n_\mathrm{a} + n_\mathrm{c} = n$. Then (passing for instance through the Jordan form of A) it is easy to check that there exist three invariant subspaces of A, denoted $\mathscr{V}_\mathrm{s}$, $\mathscr{V}_\mathrm{a}$, $\mathscr{V}_\mathrm{c}$, of dimension $n_\mathrm{s}, n_\mathrm{a}, n_\mathrm{c}$ that are complementary in $\mathbb{R}^n$, i.e., satisfy

$$\mathscr{V}_\mathrm{s} \oplus \mathscr{V}_\mathrm{a} \oplus \mathscr{V}_\mathrm{c} = \mathbb{R}^n, \tag{A.23}$$

with the property that the restriction of A to $\mathscr{V}_\mathrm{s}$ is characterized by a $n_\mathrm{s} \times n_\mathrm{s}$ matrix A_s whose eigenvalues are precisely the n_s eigenvalues of A that are in $\mathbb{C}^-$, the restriction of A to $\mathscr{V}_\mathrm{a}$ is characterized by a $n_\mathrm{a} \times n_\mathrm{a}$ matrix A_a whose eigenvalues are precisely the n_a eigenvalues of A that are in $\mathbb{C}^+$ and the restriction of A to $\mathscr{V}_\mathrm{c}$ is characterized by a $n_\mathrm{c} \times n_\mathrm{c}$ matrix A_c whose eigenvalues are precisely the n_c eigenvalues of A that are in $\mathbb{C}^0$. These three subspaces are called the *stable eigenspace*, the *antistable eigenspace* and the *center eigenspace*.

Consider now the autonomous linear system

$$\dot{x} = Ax \tag{A.24}$$

with $x \in \mathbb{R}^n$. It is easy to check that if subspace $\mathscr{V}$ is invariant under A, then for any $x^\circ \in \mathscr{V}$, the integral curve $x(t)$ of (A.24) passing through x° at time $t = 0$ is such that $x(t) \in \mathscr{V}$ for all $t \in \mathbb{R}$.[10] Because of (A.23), any trajectory $x(t)$ of (A.24) can be uniquely decomposed as

$$x(t) = x_{\rm s}(t) + x_{\rm a}(t) + x_{\rm c}(t)$$

with $x_{\rm s}(t) \in \mathscr{V}_{\rm s}$, $x_{\rm a}(t) \in \mathscr{V}_{\rm a}$, $x_{\rm c}(t) \in \mathscr{V}_{\rm c}$. Moreover, since the restriction of A to $\mathscr{V}_{\rm s}$ is characterized by a matrix $A_{\rm s}$ whose eigenvalues are all in $\mathbb{C}^-$,

$$\lim_{t\to\infty} x_{\rm s}(t) = 0$$

and, since the restriction of A to $\mathscr{V}_{\rm a}$ is characterized by a matrix $A_{\rm a}$ whose eigenvalues are all in $\mathbb{C}^+$,

$$\lim_{t\to-\infty} x_{\rm a}(t) = 0.$$

A geometric characterization of the steady-state response. It is well known that a stable linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

subject to a (harmonic) input of the form

$$u(t) = u_0 \cos(\omega_0 t) \tag{A.25}$$

exhibits a well-defined *steady-state response*, which is itself a harmonic function of time. The response in question can be easily characterized by means of a simple geometric construction. Observe that the input defined above can be viewed as generated by an autonomous system of the form

$$\begin{aligned} \dot{w} &= Sw \\ u &= Qw \end{aligned} \tag{A.26}$$

in which $w \in \mathbb{R}^2$ and

$$S = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}, \qquad Q = u_0 \begin{pmatrix} 1 & 0 \end{pmatrix},$$

[10] Note that also the converse of such implication holds. If $\mathscr{V}$ is a subspace with the property that, for any $x^\circ \in \mathscr{V}$, the integral curve $x(t)$ of (A.24) passing through x° at time $t = 0$ satisfies $x(t) \in \mathscr{V}$ for all $t \in \mathbb{R}$, then $\mathscr{V}$ is invariant under A. The property in question is sometimes referred to as the *integral version* of the notion of invariance, while the property indicated in the text above is referred to as the *infinitesimal version* on the notion of invariance.

set in the initial state[11]

$$w(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{A.27}$$

In this way, the *forced* response of the given linear system, from any initial state $x(0)$, to the input (A.25) can be identified with the *free* response of the composite system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} \tag{A.28}$$

from the initial state $(x(0), w(0))$ with $w(0)$ given by (A.27).

Since A has all eigenvalues with negative real part and S has eigenvalues on the imaginary axis, the Sylvester equation

$$\Pi S = A\Pi + BQ \tag{A.29}$$

has a unique solution Π. The composite system (A.28) possesses two complementary invariant eigenspaces: a *stable eigenspace* and a *center eigenspace*, which can be respectively expressed as

$$\mathscr{V}^{\mathrm{s}} = \mathrm{span} \begin{pmatrix} 0 \\ I \end{pmatrix}, \qquad \mathscr{V}^{\mathrm{c}} = \mathrm{span} \begin{pmatrix} I \\ \Pi \end{pmatrix}.$$

The latter, in particular, shows that the center eigenspace is the set of all pairs (w, x) such that $x = \Pi w$.

Consider now the change of variables $\tilde{x} = x - \Pi w$ which, after a simple calculation which uses (A.29), yields

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{x}} &= A\tilde{x}. \end{aligned}$$

Since the matrix A is Hurwitz, for any initial condition

$$\lim_{t \to \infty} \tilde{x}(t) = 0\,,$$

which shows that the (unique) projection of the trajectory along the stable eigenspace asymptotically tends to zero. In the original coordinates, this reads as

$$\lim_{t \to \infty} [x(t) - \Pi w(t)] = 0\,,$$

[11] To check that this is the case, simply bear in mind that the solution $w(t)$ of (A.26) is given by

$$w(t) = e^{St} w(0) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$$

.

from which we see that the *steady-state response* of the system to any input generated by (A.26) can be expressed as

$$x_{ss}(t) = \Pi w(t). \tag{A.30}$$

It is worth observing that the steady-state response $x_{ss}(t)$ thus defined can also be identified with an *actual* forced response of the system to the input (A.25), provided that the initial state $x(0)$ is appropriately chosen. In fact, since the center eigenspace is invariant for the composite system (A.28), if the initial condition of the latter is taken on $\mathscr{V}^c$, i.e., if $x(0) = \Pi w(0)$, the motion of such system remains confined to $\mathscr{V}^c$ for all t, i.e., $x(t) = \Pi w(t)$ for all t. Thus, if $x(0) = \Pi w(0)$, the actual forced response $x(t)$ of the system to the input (A.25) coincides with the steady-state response $x_{ss}(t)$. Note that, in view of the definition of $w(0)$ given by (A.27), this initial state $x(0)$ is nothing else than the first column of the matrix Π.

The calculation of the solution Π of the Sylvester equation (A.29) is straightforward. Set

$$\Pi = \begin{pmatrix}\Pi_1 & \Pi_2\end{pmatrix}$$

and observe that the equation in question reduces to

$$\Pi \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} = A\Pi + Bu_0 \begin{pmatrix}1 & 0\end{pmatrix}.$$

An elementary calculation (multiply first both sides on the right by the vector $\begin{pmatrix}1 & j\end{pmatrix}^{\mathrm{T}}$) yields

$$\Pi_1 + j\Pi_2 = (j\omega_0 I - A)^{-1}Bu_0,$$

i.e.,

$$\Pi = \begin{pmatrix}\mathrm{Re}[(j\omega_0 I - A)^{-1}B]u_0 & \mathrm{Im}[(j\omega_0 I - A)^{-1}B]u_0\end{pmatrix}.$$

As shown above, the steady-state response has the form (A.30). Hence, in particular, the periodic input

$$u(t) = u_0 \cos(\omega_0 t)$$

produces the periodic state response

$$x_{ss}(t) = \Pi w(t) = \Pi_1 \cos(\omega_0 t) - \Pi_2 \sin(\omega_0 t), \tag{A.31}$$

and the periodic output response

$$\begin{aligned} y_{ss}(t) &= Cx_{ss}(t) + Du_0 \cos(\omega_0 t) \\ &= \mathrm{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \mathrm{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t), \end{aligned} \tag{A.32}$$

in which

$$T(j\omega) = C(j\omega I - A)^{-1}B + D.$$

A.6 Hamiltonian Matrices and Algebraic Riccati Equations

In this section, a few fundamental facts about algebraic Riccati equations are reviewed.[12] An algebraic Riccati equation is an equation of the form

$$A^{\mathrm{T}}X + XA + Q + XRX = 0 \tag{A.33}$$

in which all matrices involved are $n \times n$ matrices and R, Q are *symmetric* matrices. Such equation can also be rewritten the equivalent form as

$$\begin{pmatrix} X & -I \end{pmatrix} \begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = 0.$$

From either one of these expressions, it is easy to deduce the following identity

$$\begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = \begin{pmatrix} I \\ X \end{pmatrix} (A + RX) \tag{A.34}$$

and to conclude that X is a solution of the Riccati equation (A.33) if and only if the subspace

$$\mathscr{V} = \mathrm{Im} \begin{pmatrix} I \\ X \end{pmatrix} \tag{A.35}$$

is an (n-dimensional) invariant subspace of the matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{pmatrix}. \tag{A.36}$$

In particular, (A.34) also shows that if X is a solution of (A.33), the matrix $A + RX$ characterizes the restriction of H to its invariant subspace (A.35). A matrix of the form (A.36), with real entries and in which R and Q are symmetric matrices, is called a Hamiltonian matrix. Some relevant features of the Hamiltonian matrix (A.36) and their relationships with the Riccati equation (A.33) are reviewed in what follows.

Lemma A.5 *The spectrum of the Hamiltonian matrix (A.36) is symmetric with respect to the imaginary axis.*

[12] For further reading, see e.g., [2].

Proof Set

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and note that

$$J^{-1}HJ = \begin{pmatrix} -A^{\mathrm{T}} & Q \\ -R & A \end{pmatrix} = -H^{\mathrm{T}}.$$

Hence H and $-H^{\mathrm{T}}$ are similar. As a consequence, if λ is an eigenvalue of H so is also $-\lambda$. Since the entries of H are real numbers, and therefore the spectrum of this matrix is symmetric with respect to the real axis, the result follows. ◁

Suppose now that the matrix (A.36) has no eigenvalues on the imaginary axis. Then, the matrix in question has exactly n eigenvalues in $\mathbb{C}^-$ and n eigenvalues in $\mathbb{C}^+$. As a consequence, there exist two complementary n-dimensional invariant subspaces of H: a subspace $\mathscr{V}^{\mathrm{s}}$ characterized by property that restriction of H to $\mathscr{V}^{\mathrm{s}}$ has all eigenvalues in $\mathbb{C}^-$, the *stable eigenspace*, and a subspace $\mathscr{V}^{\mathrm{a}}$ characterized by property that restriction of H to $\mathscr{V}^{\mathrm{a}}$ has all eigenvalues in $\mathbb{C}^+$, the *antistable eigenspace*. A situation of special interest in the subsequent analysis is the one in which the stable eigenspace (respectively, the antistable eigenspace) of the matrix (A.36) can be expressed in the form (A.35); in this case in fact, as observed before, it is possible to associate with this subspace a particular solution of the Riccati equation (A.33).

If there exists a matrix X^- such that

$$\mathscr{V}^{\mathrm{s}} = \mathrm{Im}\begin{pmatrix} I \\ X^- \end{pmatrix},$$

this matrix satisfies

$$A^{\mathrm{T}}X^- + X^-A + Q + X^-RX^- = 0$$

and the matrix $A+RX^-$ has all eigenvalues in $\mathbb{C}^-$. This matrix is the *unique*[13] solution of the Riccati equation (A.33) having the property that $A + RX$ has all eigenvalues in $\mathbb{C}^-$ and for this reason is called *the stabilizing* solution of the Riccati equation (A.33).

Similarly, if there exists a matrix X^+ such that

$$\mathscr{V}^{\mathrm{a}} = \mathrm{Im}\begin{pmatrix} I \\ X^+ \end{pmatrix},$$

this matrix satisfies

$$A^{\mathrm{T}}X^+ + X^-A + Q + X^+RX^+ = 0$$

---

[13]If $\mathscr{V}$ is an n-dimensional subspace of $\mathbb{R}^{2n}$, and $\mathscr{V}$ can be expressed in the form (A.35), the matrix X is necessarily unique.

and the matrix $A + RX^+$ has all eigenvalues in $\mathbb{C}^+$. This matrix is the unique solution of the Riccati equation (A.33) having the property that $A + RX$ has all eigenvalues in $\mathbb{C}^+$ and for this reason is called *the antistabilizing* solution of the Riccati equation (A.33).

The existence of such matrices X^- and X^+ is discussed in the following statement.

Proposition A.1 *Suppose the Hamiltonian matrix (A.36) has no eigenvalues on the imaginary axis and R is a (either positive or negative) semidefinite matrix.*

If the pair (A, R) is stabilizable, the stable eigenspace $\mathscr{V}^{\mathrm{s}}$ of (A.36) can be expressed in the form

$$\mathscr{V}^{\mathrm{s}} = \mathrm{Im}\begin{pmatrix} I \\ X^- \end{pmatrix}$$

in which X^- is a symmetric matrix, the (unique) stabilizing solution of the Riccati equation (A.33).

If the pair (A, R) is antistabilizable, the antistable eigenspace $\mathscr{V}^{\mathrm{a}}$ of (A.36) can be expressed in the form

$$\mathscr{V}^{\mathrm{a}} = \mathrm{Im}\begin{pmatrix} I \\ X^+ \end{pmatrix}$$

in which X^- is a symmetric matrix, the (unique) antistabilizing solution of the Riccati equation (A.33).

The following proposition describes the relation between solutions of the algebraic Riccati equation (A.33) and of the algebraic Riccati *inequality*

$$A^{\mathrm{T}}X + XA + Q + XRX > 0. \tag{A.37}$$

Proposition A.2 *Suppose R is negative semidefinite. Let X^- (respectively X^+) be a solution of the Riccati equation (A.33) having the property that $\sigma(A + RX^-) \in \mathbb{C}^-$ (respectively, $\sigma(A + RX^+) \in \mathbb{C}^+$). Then, the set of solutions of*

$$A^{\mathrm{T}}X + XA + Q + XRX > 0\,,$$

is not empty and any X in this set satisfies $X < X^-$ (respectively, $X > X^+$).

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Appendix B
Stability and Asymptotic Behavior of Nonlinear Systems

B.1 The Theorems of Lyapunov for Nonlinear Systems

We assume in what follows that the reader is familiar with basic concepts concerning the stability of equilibrium in a nonlinear system. In this section we provide a sketchy summary of some fundamental results, mainly to the purpose of introducing notations and results that are currently used throughout the book.[14]

Comparison functions. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathscr{K}_\infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathscr{KL}$ if, for each fixed t, the function

$$\begin{aligned} \alpha : [0, a) &\to [0, \infty) \\ r &\mapsto \beta(r, t) \end{aligned}$$

belongs to class $\mathscr{K}$ and, for each fixed r, the function

$$\begin{aligned} \varphi : [0, \infty) &\to [0, \infty) \\ t &\mapsto \beta(r, t) \end{aligned}$$

is decreasing and $\lim_{t\to\infty} \varphi(t) = 0$.

The composition of two class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, denoted $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathscr{K}$ (respectively, class $\mathscr{K}_\infty$) function. If $\alpha(\cdot)$ is a class $\mathscr{K}$ function, defined on $[0, a)$ and $b = \lim_{r\to a} \alpha(r)$, there exists a unique function, $\alpha^{-1} : [0, b) \to [0, a)$, such that

$$\begin{aligned} \alpha^{-1}(\alpha(r)) &= r, \text{ for all } r \in [0, a) \\ \alpha(\alpha^{-1}(r)) &= r, \text{ for all } r \in [0, b). \end{aligned}$$

[14] For further reading, see [2, 4, 5, 6].

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Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-3-319-42031-8

Moreover, $\alpha^{-1}(\cdot)$ is a class $\mathscr{K}$ function. If $\alpha(\cdot)$ is a class $\mathscr{K}_\infty$ function, so is also $\alpha^{-1}(\cdot)$. If $\beta(\cdot,\cdot)$ is a class $\mathscr{KL}$ function and $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions, the function thus defined

$$\begin{aligned} \gamma : [0, a) \times [0, \infty) &\to [0, \infty) \\ (r, t) &\mapsto \alpha_1(\beta(\alpha_2(r), t)) \end{aligned}$$

is a class $\mathscr{KL}$ function.

The Theorems of Lyapunov. Consider an autonomous nonlinear system

$$\dot{x} = f(x) \tag{B.1}$$

in which $x \in \mathbb{R}^n$, $f(0) = 0$ and $f(x)$ is locally Lipschitz. The stability, or asymptotic stability, properties of the equilibrium $x = 0$ of this system can be tested via the well-known criterion of Lyapunov, which, using comparison functions, can be expressed as follows. Let B_d denote the open ball of radius d in $\mathbb{R}^n$, i.e.,

$$B_d = \{x \in \mathbb{R}^n : \|x\| < d\}.$$

Theorem B.1 (Direct Theorem) *Let $V : B_d \to \mathbb{R}$ be a C^1 function such that, for some class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, defined on $[0, d)$,*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \qquad \textit{for all } x \in B_d. \tag{B.2}$$

If

$$\frac{\partial V}{\partial x} f(x) \leq 0 \qquad \textit{for all } x \in B_d, \tag{B.3}$$

the equilibrium $x = 0$ of (B.1) is stable.

If, for some class $\mathscr{K}$ function $\alpha(\cdot)$, defined on $[0, d)$,

$$\frac{\partial V}{\partial x} f(x) \leq -\alpha(\|x\|) \qquad \textit{for all } x \in B_d, \tag{B.4}$$

the equilibrium $x = 0$ of (B.1) is locally asymptotically stable.

If $d = \infty$ and, in the above inequalities, $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are class $\mathscr{K}_\infty$ functions, the equilibrium $x = 0$ of (B.1) is globally asymptotically stable.

Remark B.1 The usefulness of the comparison functions, in the statement of the theorem, is motivated by the following simple arguments. Suppose (B.3) holds. Then, so long as $x(t) \in B_d$, $V(x(t))$ is non-increasing, i.e., $V(x(t)) \leq V(x(0))$. Pick $\varepsilon < d$ and define $\delta = \overline{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)$. Then, using (B.2), it is seen that, if $\|x(0)\| < \delta$,

$$\underline{\alpha}(\|x(t)\|) \leq V(x(t)) \leq V(x(0)) \leq \overline{\alpha}(\|x(0)\|) \leq \overline{\alpha}(\delta) = \underline{\alpha}(\varepsilon)$$

which implies $\|x(t)\| \leq \varepsilon$. This shows that $x(t)$ exists for all t and the equilibrium $x = 0$ is stable.

Suppose now that (B.4) holds. Define $\gamma(r) = \alpha(\overline{\alpha}^{-1}(r))$, which is a class $\mathscr{K}$ function. Using the estimate on the right of (B.2), it is seen that $\alpha(\|x\|) \geq \gamma(V(x))$ and hence

$$\frac{\partial V}{\partial x} f(x) \leq -\gamma(V(x)).$$

Since $V(x(t))$ is a continuous function of t, non-increasing and nonnegative for each t, there exists a number $V^* \geq 0$ such that $\lim_{t\to\infty} V(x(t)) = V^*$. Suppose V^* is strictly positive. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\gamma(V(x(t)) \leq -\gamma(V^*) < 0.$$

Integration with respect to time yields

$$V(x(t)) \leq V(x(0)) - \gamma(V^*)t$$

for all t. This cannot be the case, because for large t the right-hand side is negative, while the left-hand side is nonnegative. From this it follows that $V^* = 0$ and therefore, using the fact that $V(x)$ vanishes only at $x = 0$, it is concluded that $\lim_{t\to\infty} x(t) = 0$. Note also that identical arguments hold for the analysis of the asymptotic properties of a time-dependent system

$$\dot{x} = f(x, t)$$

so long $f(0, t) = 0$ for all $t \geq 0$ and $V(x)$ is independent of t. ◁

Sometimes, in the design of feedback laws, while it is difficult to obtain a system whose equilibrium $x = 0$ is globally asymptotically stable, it is relatively easier to obtain a system in which trajectories are bounded (maybe for a specific set of initial conditions) and have suitable decay properties. Instrumental, in such context, is the notion of *sublevel set* of a Lyapunov function $V(x)$ which, for a fixed nonnegative real number c, is defined as

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}.$$

The function $V(x)$, which is *positive definite* (i.e., is positive for all nonzero x and zero at $x = 0$) is said to be *proper* if, for each $c \in \mathbb{R}$, the sublevel set Ω_c is a *compact* set. Now, it is easy to check that the function $V(x)$ is proper if and only the inequality on the left-hand side of (B.2) holds for all $x \in \mathbb{R}^n$, with a function $\underline{\alpha}(\cdot)$ which is of class $\mathscr{K}_\infty$. Note also that, if $V(x)$ is proper, for any $c > 0$ it is possible to find a numbers $c_1 > 0$ and $c_2 > 0$ such that

$$B_{c_1} \subset \Omega_c \subset B_{c_2}.$$

A typical example of how sublevel sets can be used to analyze boundedness and decay of trajectories is the following one. Let r_1 and r_2 be two positive numbers, with $r_2 > r_1$. Suppose $V(x)$ is a function satisfying (B.2), with $\underline{\alpha}(\cdot)$ a class $\mathscr{K}_\infty$ function. Pick any pair of positive numbers c_1, c_2, such that

$$\Omega_{c_1} \subset B_{r_1} \subset B_{r_2} \subset \Omega_{c_2}.$$

and let $S_{c_1}^{c_2}$ denote the "annular" compact set

$$S_{c_1}^{c_2} = \{x \in \mathbb{R}^n : c_1 \leq V(x) \leq c_2\}.$$

Suppose that, for some $a > 0$,

$$\frac{\partial V}{\partial x} f(x) \leq -a \qquad \text{for all}\, x \in S_{c_1}^{c_2}.$$

Then, for each initial condition $x(0) \in B_{r_2}$, the trajectory $x(t)$ of (B.1) is defined for all t and there exists a finite time T such that $x(t) \in B_{r_1}$ for all $t \geq T$. In fact, take any $x(0) \in B_{r_2} \setminus \Omega_{c_1}$. Such $x(0)$ is in $S_{c_1}^{c_2}$. So long as $x(t) \in S_{c_1}^{c_2}$, the function $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t) \leq -a$$

and hence

$$V(x(t)) \leq V(x(0)) - at \leq c_2 - at.$$

Thus, at a time $T \leq (c_2 - c_1)/a$, $x(T)$ is on the boundary of the set Ω_{c_1}. On the boundary of Ω_{c_1} the derivative of $V(x(t))$ with respect to time is negative and hence the trajectory enters the set Ω_{c_1} and remains there for all $t \geq T$.

It is well known that the criterion for asymptotic stability provided by the previous theorem has a *converse*, namely, the existence of a function $V(x)$ having the properties indicated in Theorem B.1 is *implied* by the property of asymptotic stability of the equilibrium $x = 0$ of (B.1). In particular, the following result holds.

Theorem B.2 (Converse Theorem) *Suppose the equilibrium $x = 0$ of (B.1) is locally asymptotically stable. Then, there exist $d > 0$, a C^1 function $V : B_d \to \mathbb{R}$, and class $\mathscr{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold. If the equilibrium $x = 0$ of (B.1) is globally asymptotically stable, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (B.2) and (B.4) hold with $d = \infty$.*

It is well known that, for a nonlinear system, the property of asymptotic stability of the equilibrium $x = 0$ does not necessarily imply *exponential* decay to zero of $\|x(t)\|$. If the equilibrium $x = 0$ of system (B.1) is globally asymptotically stable and, moreover, there exist numbers $d > 0$, $M > 0$ and $\lambda > 0$ such that

$$x(0) \in B_d \quad \Rightarrow \quad \|x(t)\| \leq M e^{-\lambda t} \|x(0)\| \quad \text{for all}\, t \geq 0$$

it is said that this equilibrium is *globally asymptotically and locally exponentially* stable. In this context, the following criterion is useful.

Lemma B.1 *The equilibrium $x = 0$ of nonlinear system (B.1) is globally asymptotically and locally exponentially stable if and only if there exists a smooth function $V(x) : \mathbb{R}^n \to \mathbb{R}$, three class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and three real numbers $\delta > 0$, $\underline{a} > 0$, $a > 0$, such that*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -\alpha(\|x\|)$$

for all $x \in \mathbb{R}^n$ and

$$\underline{\alpha}(s) = \underline{a}s^2, \qquad \alpha(s) = as^2$$

for all $s \in B_\delta$.

B.2 Input-to-State Stability and the Theorems of Sontag

In the analysis of *forced* nonlinear systems, the property of *input-to-state stability*, introduced and thoroughly studied by E.D. Sontag, plays a role of paramount importance.[15] Consider a forced nonlinear system

$$\dot{x} = f(x, u) \tag{B.5}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0, 0) = 0$ and $f(x, u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0, \infty) \to \mathbb{R}^m$ of (B.5) can be any piecewise continuous bounded function. The space of all such functions is endowed with the so-called supremum norm $\|u(\cdot)\|_\infty$, which is defined as

$$\|u(\cdot)\|_\infty = \sup_{t \geq 0} \|u(t)\|.$$

Definition B.1 System (B.5) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, called a *gain function*, such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_\infty) \tag{B.6}$$

for all $t \geq 0$.

---

[15]The concept of input-to-state stability, its properties and applications have been introduced in the sequence of papers [10, 11, 14]. A summary of the most relevant aspect of the theory can also be found in [5, p. 17–31].

Since, for any pair $\beta > 0, \gamma > 0$, $\max\{\beta, \gamma\} \le \beta + \gamma \le \max\{2\beta, 2\gamma\}$, an alternative way to say that a system is input-to-state stable is to say that there exists a class $\mathscr{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (B.5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \max\{\beta(\|x(0)\|, t), \gamma(\|u(\cdot)\|_\infty)\} \tag{B.7}$$

for all $t \ge 0$. Note also that, letting $\|u(\cdot)\|_{[0,t]}$ denote the supremum norm of the restriction of $u(\cdot)$ to the interval $[0, t]$, namely

$$\|u(\cdot)\|_{[0,t]} = \sup_{s\in[0,t]} \|u(s)\| ,$$

the bound (B.6) can be also expressed in the alternative form[16]

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_{[0,t]}) , \tag{B.8}$$

and the bound (B.7) in the alternative form

$$\|x(t)\| \le \max\{\beta(\|x(0)\|, t), \gamma(\|u(\cdot)\|_{[0,t]})\} ,$$

both holding for all $t \ge 0$.

The property of input-to-state stability can be given a characterization which extends the well-known criterion of Lyapunov for asymptotic stability. The key tool for such characterization is the notion of *ISS-Lyapunov function*.

Definition B.2 A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (B.5) if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\chi(\cdot)$ such that

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n \tag{B.9}$$

and

$$\frac{\partial V}{\partial x} f(x, u) \le -\alpha(\|x\|) \quad \text{for all } (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \text{ satisfying } \|x\| \ge \chi(\|u\|). \tag{B.10}$$

An equivalent form in which the notion of an ISS-Lyapunov function can be described is the following one.

---

[16] In fact, since $\|u(\cdot)\|_{[0,t]} \le \|u(\cdot)\|_\infty$ and $\gamma(\cdot)$ is increasing, (B.8) implies (B.6). On the other hand, since $x(t)$ depends only on the restriction of $u(\cdot)$ to the interval $[0, t]$, one could in (B.6) replace $u(\cdot)$ with an input $\bar{u}(\cdot)$ defined as $\bar{u}(s) = u(s)$ for $0 \le s \le t$ and $\bar{u}(s) = 0$ for $s > t$, in which case $\|\bar{u}(\cdot)\|_\infty = \|u(\cdot)\|_{[0,t]}$, and observe that (B.6) implies (B.8).

Lemma B.2 *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (B.5) if and only if there exist class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathscr{K}$ function $\sigma(\cdot)$ such that (B.9) holds and*

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \quad \textit{for all } (x, u) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{B.11}$$

The existence of an ISS-Lyapunov function turns out to be a necessary and sufficient condition for input-to-state stability.

Theorem B.3 *System (B.5) is input-to-state stable if and only if there exists an ISS-Lyapunov function. In particular, if such function exists, then an estimate of the form (B.6) holds with $\gamma(r) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(r)))$.*

The following elementary examples describe how the property of input-to-state stability can be checked and an estimate of the gain function can be evaluated.

Example B.1 A stable linear system

$$\dot{x} = Ax + Bu$$

is input-to-state stable, with a linear gain function. In fact, let P denote the unique positive-definite solution of the Lyapunov equation $PA + A^{\mathrm{T}}P = -I$ and observe that $V(x) = x^{\mathrm{T}}Px$ satisfies

$$\frac{\partial V}{\partial x}(Ax + Bu) \leq -\|x\|^2 + c\|x\|\,\|u\|$$

for some $c > 0$. Pick $0 < \varepsilon < 1$ and set $\ell = c/(1-\varepsilon)$. Then, it is easy to see that

$$\|x\| \geq \ell\|u\| \quad \Rightarrow \quad \frac{\partial V}{\partial x}(Ax + Bu) \leq -\varepsilon\|x\|^2.$$

The system is input-to-state, with $\chi(r) = \ell r$. Since $\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2$, we obtain following the estimate for the (linear) gain function

$$\gamma(r) = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,\ell r.$$

◁

Example B.2 Let $n = 1$, $m = 1$ and consider the system

$$\dot{x} = -ax^k + bx^p u,$$

in which $k \in \mathbb{N}$ is *odd*, $a > 0$ and $p \in \mathbb{N}$ is such that $p < k$. Pick $V(x) = \frac{1}{2}x^2$ and note that, since $k + 1$ is even,

$$\frac{\partial V}{\partial x}(-ax^k + bx^p u) \le -a|x|^{k+1} + |b||x|^{p+1}|u|$$

Pick $0 < \varepsilon < a$ and define (recall that $k > p$)

$$\chi(r) = \left(\frac{|b|r}{a - \varepsilon}\right)^{\frac{1}{k-p}}.$$

Then, it is easy to see that

$$\|x\| \ge \chi(\|u\|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}(-ax^k + bx^p u) \le -\varepsilon|x|^{k+1}.$$

The system is input-to-state stable, with $\gamma(r) = \chi(r)$.

Note that the condition $k > p$ is essential. In fact, the following system, in which $k = p = 1$,

$$\dot{x} = -x + xu$$

is not input-to-state stable. Under the bounded (constant) input $u(t) = 2$ the state $x(t)$ evolves as a solution of $\dot{x} = x$ and hence diverges to infinity. ◁

The notion of input-to-state stability lends itself to a number of alternative (equivalent) characterizations, among which one of the most useful can be expressed as follows.

Theorem B.4 *System (B.5) is input-to-state stable if and only if there exist class $\mathcal{K}$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any bounded input and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ satisfies*

$$\begin{aligned} \|x(\cdot)\|_\infty &\le \max\{\gamma_0(\|x(0)\|), \gamma(\|u(\cdot)\|_\infty)\} \\ \limsup_{t\to\infty} \|x(t)\| &\le \gamma(\limsup_{t\to\infty} \|u(t)\|). \end{aligned}$$

B.3 Cascade-Connected Systems

In this section we investigate the *asymptotic* stability of the equilibrium $(z, \xi) = (0, 0)$ of a pair of cascade-connected subsystems of the form

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi), \end{aligned} \tag{B.12}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $f(0, 0) = 0$, $g(0) = 0$, and $f(z, \xi)$, $g(\xi)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$ (see Fig. B.1).

Since similar cascade connections occur quite often in the analysis (and feedback design) of nonlinear systems, it is important to understand under what conditions

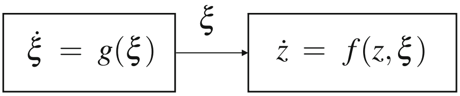

Fig. B.1 A cascade connection of systems

the stability properties of the two components subsystems determine the stability of the cascade. If both systems were linear systems, the cascade would be a system modeled as

$$\begin{aligned}\dot{z} &= Fz + G\xi \\ \dot{\xi} &= A\xi,\end{aligned}$$

and it is trivially seen that if both F and G have all eigenvalues in $\mathbb{C}^-$, the cascade is an asymptotically stable system. The nonlinear counterpart of such property, though, requires some extra care.

The simplest scenario, in this respect, is one in which one is interested in seeking only local stability. In this case, the following result holds.[17]

Lemma B.3 *Suppose the equilibrium $z = 0$ of*

$$\dot{z} = f(z, 0) \tag{B.13}$$

is locally asymptotically stable and the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is stable. Then the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is stable. If the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is locally asymptotically stable, then the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is locally asymptotically stable.

It must be stressed, though, that in this lemma only the property of *local* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ is considered. In fact, by means of a simple counterexample, it can be shown that the *global* asymptotic stability of $z = 0$ as an equilibrium of (B.13) and the *global* asymptotic stability of $\xi = 0$ as an equilibrium of $\dot{\xi} = g(\xi)$ *do not* imply, in general, *global* asymptotic stability of the equilibrium $(z, \xi) = (0, 0)$ of the cascade. As a matter of fact, the cascade connection of two such systems may even have finite escape times. To infer global asymptotic stability of the cascade, a (strong) extra condition is needed, as shown below.

Example B.3 Consider the case in which

$$\begin{aligned}f(z, \xi) &= -z + z^2\xi \\ g(\xi) &= -\xi.\end{aligned}$$

Clearly $z = 0$ is a globally asymptotically equilibrium of $\dot{z} = f(z, 0)$ and $\xi = 0$ is a globally asymptotically equilibrium of $\dot{\xi} = g(\xi)$. However, this system has finite escape times. To show that this is the case, consider the differential equation

[17] More details and proofs of the results stated in this section can be found in [5, p. 11–17 and 31–36].

$$\dot{\tilde{z}} = -\tilde{z} + \tilde{z}^2 \tag{B.14}$$

with initial condition $\tilde{z}(0) = z_0$. Its solution is

$$\tilde{z}(t) = \frac{-z_0}{z_0 - 1 - z_0 \exp(-t)} \exp(-t).$$

Suppose $z_0 > 1$. Then, the $\tilde{z}(t)$ escapes to infinity in finite time. In particular, the maximal (positive) time interval on which $\tilde{z}(t)$ is defined is the interval $[0, t_{\max}(z_0))$ with

$$t_{\max}(z_0) = \ln\left(\frac{z_0}{z_0 - 1}\right).$$

Now, return to system (B.12), with initial condition (z_0, ξ_0) and let ξ_0 be such that

$$\xi(t) = \exp(-t)\xi_0 \geq 1 \qquad \text{for all } t \in [0, t_{\max}(z_0)).$$

Clearly, on the time interval $[0, t_{\max}(z_0))$, we have

$$\dot{z} = -z + z^2\xi \geq -z + z^2.$$

By comparison with (B.14), it follows that

$$z(t) \geq \tilde{z}(t).$$

Hence $z(t)$ escapes to infinity, at a time $t^* \leq t_{\max}(z_0)$. The lesson learned from this example is that, even if $\xi(t)$ exponentially decreases to 0, this may not suffice to prevent finite escape time in the upper system. The state $z(t)$ escapes to infinity at a time in which the effect of $\xi(t)$ on the upper equation is still not negligible. ◁

The following results provide the extra condition needed to ensure global asymptotic stability in the cascade.

Lemma B.4 *Suppose the equilibrium $z = 0$ of (B.13) is asymptotically stable, and let S be a subset of the domain of attraction of such equilibrium. Consider the system*

$$\dot{z} = f(z, \xi(t)). \tag{B.15}$$

in which $\xi(t)$ is a continuous function, defined for all $t \geq 0$ and suppose that $\lim_{t\to\infty} \xi(t) = 0$. *Pick $z_0 \in S$, and suppose that the integral curve $z(t)$ of (B.15) satisfying $z(0) = z_0$ is defined for all $t \geq 0$, bounded, and such that $z(t) \in S$ for all $t \geq 0$. Then* $\lim_{t\to\infty} z(t) = 0$.

This last result implies, in conjunction with Lemma B.3, that if the equilibrium $z = 0$ of (B.13) is globally asymptotically stable, if the equilibrium $\xi = 0$ of the lower subsystem of (B.12) is globally asymptotically stable, and all trajectories of the composite system (B.12) *are bounded*, the equilibrium $(z, \xi) = (0, 0)$ of (B.12) is globally asymptotically stable.

To be in a position to use this result in practice, one needs to determine conditions under which the boundedness property holds. This is indeed the case if the upper subsystem of the cascade, viewed as a system with state z and input ξ, is input-to-state stable. In view of this, it can be claimed that if the upper subsystem of the cascade is input-to-state stable and the lower subsystem is globally asymptotically stable (at the equilibrium $\xi = 0$), the cascade is globally asymptotically stable (at the equilibrium $(z, \xi) = (0, 0)$).

As a matter fact, a more general result holds, which is stated as follows.

Theorem B.5 *Suppose that system*

$$\dot{z} = f(z, \xi), \tag{B.16}$$

viewed as a system with input ξ and state z is input-to-state stable and that system

$$\dot{\xi} = g(\xi, u), \tag{B.17}$$

viewed as a system with input u and state ξ is input-to-state stable as well. Then, system

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi, u) \end{aligned}$$

is input-to-state stable.

Example B.4 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2\xi_1\xi_2 \\ \dot{x}_2 &= -x_2 + \xi_1^2 - x_1\xi_1\xi_2 \\ \dot{\xi}_1 &= -\xi_1^3 + \xi_1 u_1 \\ \dot{\xi}_2 &= -\xi_2 + u_2. \end{aligned}$$

The subsystem consisting of the two top equations, seen as a system with state $x = (x_1, x_2)$ and input $\xi = (\xi_1, \xi_2)$ is input-to-state stable. In fact, let this system be written as

$$\dot{x} = f(x, \xi)$$

and consider the candidate ISS-Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

for which we have

$$\frac{\partial V}{\partial x} f(x, \xi) = -(x_1^2 + x_2^2) + x_2\xi_1^2 \leq -x_1^2 - \frac{1}{2}x_2^2 + \frac{1}{2}\xi_1^4 \leq -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|\xi\|^4.$$

Thus, the function $V(x)$ satisfies the condition indicated in Lemma B.2, with

$$\alpha(r) = \frac{1}{2}r^2\,, \qquad \sigma(r) = \frac{1}{2}r^4.$$

The subsystem consisting of the two bottom equations is composed of two separate subsystems, both of which are input-to-state stable, as seen in Examples B.1 and B.2. Thus, the overall system is input-to-state stable. ◁

B.4 Limit Sets

Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{B.18}$$

with $x \in \mathbb{R}^n$. It is well known that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, for all $x_0 \in \mathbb{R}^n$ the solution of (B.18) with initial condition $x(0) = x_0$, denoted by $x(t, x_0)$, exists on some open interval of the point $t = 0$ and is unique.

Definition B.3 Let $x_0 \in \mathbb{R}^n$ be fixed. Suppose that $x(t, x_0)$ is defined for all $t \geq 0$. A point x is said to be an ω-limit *point* of the motion $x(t, x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k, x_0) = x.$$

The ω-limit *set* of a point x_0, denoted $\omega(x_0)$, is the union of all ω-limit points of the motion $x(t, x_0)$.

It is obvious from this definition that an ω-limit point *is not* necessarily a limit of $x(t, x_0)$ as $t \to \infty$, because the solution in question may not admit any limit as $t \to \infty$ (see for instance Fig. B.2).[18]

However, it is known that, if the motion $x(t, x_0)$ is *bounded*, then $x(t, x_0)$ asymptotically approaches the set $\omega(x_0)$, as specified in the lemma that follows.[19] In this respect, recall that a set $S \subset \mathbb{R}^n$ is said to be *invariant* under (B.18) if for all initial conditions $x_0 \in S$ the solution $x(t, x_0)$ of (B.18) exists for all $t \in (-\infty, +\infty)$ and

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[18]Figures B.2, B.3, B.4 are reprinted from *Annual Reviews in Control*, Vol. 32, A. Isidori and C.I. Byrnes, Steady-state behaviors in nonlinear systems with an application to robust disturbance rejection, pp. 1–16, Copyright (2008), with permission from Elsevier.

[19]See [1, p. 198].

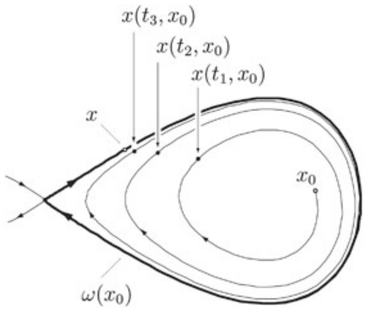

Fig. B.2 The ω-limit set of the point x_0

$x(t, x_0) \in S$ for all such t.[20] Moreover, the *distance* of a point $x \in \mathbb{R}$ from a set $S \subset \mathbb{R}^n$, denoted $\mathrm{dist}(x, S)$, is the nonnegative real number defined as

$$\mathrm{dist}(x, S) = \inf_{z \in S} \|x - z\|.$$

Lemma B.5 *Suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$. Then, $\omega(x_0)$ is a nonempty connected compact set, invariant under (B.18). Moreover,*

$$\lim_{t \to \infty} \mathrm{dist}(x(t, x_0), \omega(x_0)) = 0.$$

Example B.5 Consider the classical (stable) Van der Pol oscillator, written in state-space form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu(1 - x_1^2)x_2 \end{aligned} \tag{B.19}$$

in which, as it is well known, the damping term $\mu(1 - x_1^2)y$ can be seen as a model of a nonlinear resistor, negative for small x_1 and positive for large x_1 (see [6]). From the phase portrait of this system (depicted in Fig. B.3 for $\mu = 1$) it is seen that all motions except the trivial motion occurring for $x_0 = 0$ are bounded in positive time and approach, as $t \to \infty$, the limit cycle $\mathscr{L}$. As consequence, $\omega(x_0) = \mathscr{L}$ for any $x_0 \neq 0$, while $\omega(0) = \{0\}$. ◁

An important useful application of the notion of ω-limit set of a point is found in the proof of the following result, commonly known as LaSalle's invariance principle.[21]

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[20] We recall, for the sake of completeness, that a set S is said to be *positively invariant*, or *invariant in positive time* (respectively, *negatively invariant* or *invariant in negative time*) if for all initial conditions $x_0 \in X$, the solution $x(t, x_0)$ exists for all $t \geq 0$ and $x(t, x_0) \in X$ for all $t \geq 0$ (respectively exists for all $t \leq 0$ and $x(t, x_0) \in X$ for all $t \leq 0$). Thus, a set is invariant if it is both positively invariant and negatively invariant.

[21] See e.g., [6].

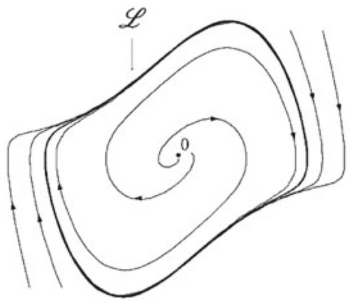


Fig. B.3 The phase portrait of the Van der Pol oscillator

Theorem B.6 *Consider system (B.18). Suppose there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n,$$

for some pair of class $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ and such that

$$\frac{\partial V}{\partial x} f(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{B.20}$$

Let $\mathscr{E}$ denote the set

$$\mathscr{E} = \{x \in \mathbb{R}^n : \frac{\partial V}{\partial x} f(x) = 0\}. \tag{B.21}$$

Then, for each x_0, the integral curve $x(t, x_0)$ of (B.18) passing through x_0 at time $t = 0$ is bounded, and

$$\omega(x_0) \subset \mathscr{E}.$$

Proof A direct consequence of (B.20) is that, for any x_0, the motion $x(t, x_0)$ is bounded in positive time. In fact, this property yields $V(x(t, x_0)) \leq V(x_0)$ for all $t \geq 0$ and this in turn implies (see Remark B.1)

$$\|x(t, x_0)\| \leq \underline{\alpha}^{-1}(\overline{\alpha}(\|x_0\|)).$$

Thus, the limit set $\omega(x_0)$ is nonempty, compact and invariant. The nonnegative-valued function $V(x(t, x_0))$ is non-increasing for $t \geq 0$. Thus, there is a number $V_0 \geq 0$, possibly dependent on x_0, such that

$$\lim_{t \to \infty} V(x(t, x_0)) = V_0.$$

By definition of limit set, for each point $x \in \omega(x_0)$, there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that $\lim_{k\to\infty} x(t_k, x_0) = x$. Thus, since $V(x)$ is continuous,

$$V(x) = \lim_{k\to\infty} V(x(t_k, x_0)) = V_0.$$

In other words, the function $V(x)$ takes the same value V_0 at any point $x \in \omega(x_0)$. Now, pick any initial condition $\bar{x}_0 \in \omega(x_0)$. Since the latter is invariant, we have $x(t, \bar{x}_0) \in \omega(x_0)$ for all $t \in \mathbb{R}$. Thus, along this particular motion, $V(x(t, \bar{x}_0)) = V_0$ and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} V(x(t, \bar{x}_0)) = \frac{\partial V}{\partial x} f(x)\Big|_{x=x(t,\bar{x}_0)}.$$

This, implies

$$x(t, \bar{x}_0) \in \mathscr{E}, \qquad \text{for all } t \in \mathbb{R}$$

and, since $\bar{x}_0$ is any point in $\omega(x_0)$, proves the theorem. ◃

This theorem is often used to determine the asymptotic properties of the integral curves of (B.18). In fact, in view of Lemma B.5, it is seen that if a function $V(x)$ can be found such that (B.20) holds, any trajectory of (B.18) is bounded and converges, asymptotically, to an invariant set that is entirely contained in the set $\mathscr{E}$ defined by (B.21). In particular, if system (B.18) has an equilibrium at $x = 0$ and it can be determined that, in the set $\mathscr{E}$, the only possible invariant set is the point $x = 0$, then the equilibrium in question is globally asymptotically stable.

Example B.6 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1(1 + x_1 x_2)\,, \end{aligned}$$

pick $V(x) = x_1^2 + x_2^2$, and observe that

$$\frac{\partial V}{\partial x} f(x) = -2(x_1 x_2)^2.$$

The function on the right-hand side is not negative definite, but it is negative semi-definite, i.e., satisfies (B.20). Thus, trajectories converge to bounded sets that are invariant and contained in the set

$$\mathscr{E} = \{x \in \mathbb{R}^2 : x_1 x_2 = 0\}.$$

Now, it is easy to see that no invariant set may exist, other than the equilibrium, entirely contained in the set $\mathscr{E}$. In fact, if a trajectory of the system is contained in $\mathscr{E}$ for all $t \in \mathbb{R}$, this trajectory must be a solution of

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1\,. \end{aligned}$$

This system, a harmonic oscillator, has only one trajectory entirely contained in $\mathscr{E}$, the trivial trajectory $x(t) = 0$. Thus, the equilibrium point $x = 0$ is the only possible invariant set contained in $\mathscr{E}$ and therefore this equilibrium is globally asymptotically stable. ◃

Returning to the analysis of the properties of limit sets, let $B \subset \mathbb{R}^n$ be a fixed bounded set and suppose that *all* motions with initial condition $x_0 \in B$ are bounded in positive time. Since any motion $x(t, x_0)$ asymptotically approaches the limit set $\omega(x_0)$ as $t \to \infty$, it seems reasonable to look at the set

$$\Omega = \bigcup_{x_0 \in B} \omega(x_0)\,, \tag{B.22}$$

as to a "target" set that is asymptotically approached by motions of (B.18) with initial conditions in B. However, while it is true that the distance of $x(t, x_0)$ from the set (B.22) tends to 0 as $t \to \infty$ for any x_0, the convergence to such set may fail be *uniform* in x_0, even if the set B is compact. In this respect, recall that—by definition—the distance of $x(t, x_0)$ from a set S tends to 0 as $t \to \infty$ if for every ε there exists T such that

$$\text{dist}(x(t, x_0), S) \leq \varepsilon\,, \qquad \text{for all}\, t \geq T. \tag{B.23}$$

The number T in this expression depends on ε but also on x_0.[22] The distance of $x(t, x_0)$ from S is said to tend to 0, as $t \to \infty$, *uniformly* in x_0 on B, if for every ε there exists T, which depends on ε *but not* on x_0, such that (B.23) holds for all $x_0 \in B$.

Example B.7 Consider again the Example B.5, in which the set Ω defined by (B.22) consists of the union of the equilibrium point $\{0\}$ and of the limit cycle $\mathscr{L}$ and let B be a compact set satisfying $B \supset \mathscr{L}$. All $x_0 \in B$ are such that $\text{dist}(x(t, x_0), \Omega) \to 0$ as $t \to \infty$. However, the convergence is not uniform in x_0. In fact, observe that, if $x_0 \neq 0$ is inside $\mathscr{L}$, the motion $x(t, x_0)$ is bounded in negative time and remains inside $\mathscr{L}$ for all $t \leq 0$ (as a matter of fact, it converges to 0 as $t \to -\infty$). Pick any $x_1 \neq 0$ inside $\mathscr{L}$ such that $\text{dist}(x_1, \mathscr{L}) > \varepsilon$ and let T_1 be the minimal time needed to have $\text{dist}(x(t, x_1), \mathscr{L}) \leq \varepsilon$ for all $t \geq T_1$. Let $T_0 > 0$ be fixed and define $x_0 = x(-T_0, x_1)$. If T_0 is large, x_0 is close to 0, and the minimal time T needed to have $\text{dist}(x(t, x_0), \Omega) \leq \varepsilon$ for all $t \geq T$ is $T = T_0 + T_1$. Since the time T_0 can be taken arbitrarily large, it follows that the time $T > 0$ needed to have $\text{dist}(x(t, x_0), \Omega) \leq \varepsilon$ for all $t \geq T$ can be made arbitrarily large, even if x_0 is taken within a compact set. ◃

Uniform convergence to the target set is important for various reasons. On the one hand, for practical purposes it is important to have a fixed bound on the time needed to get within an ε-distance of that set. On the other hand, uniform convergence plays a relevant role in the existence Lyapunov functions, an indispensable tool in analysis and design of feedback systems. While convergence to the set (B.22) is not guaranteed to be uniform, there is a larger set—though—for which such property holds.

[22] In fact it is likely that, the more is x_0 distant from S, the longer one has to wait until $x(t, x_0)$ becomes ε-distant from S.

Definition B.4 Let B be a bounded subset of $\mathbb{R}^n$ and suppose $x(t, x_0)$ is defined for all $t \geq 0$ and all $x_0 \in B$. The ω-limit set of B, denoted $\omega(B)$, is the set of all points x for which there exists a sequence of pairs $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k, x_k) = x.$$

It is clear from the definition that, if B consists of only one single point x_0, all x_k's in the definition above are necessarily equal to x_0 and the definition in question returns the definition of ω-limit set of a point. It is also clear that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. In fact, all such points have the property indicated in the definition, with all the x_k's being taken equal to x_0. Thus, in particular, if all motions with $x_0 \in B$ are bounded in positive time,

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B).$$

However, the converse inclusion is not true in general.

Example B.8 Consider again the system in Example B.5, and let B be a compact set satisfying $B \supset \mathscr{L}$. We know that $\{0\}$ and $\mathscr{L}$, being ω-limit sets of points of B, are in $\omega(B)$. But it is also easy to see that any other point inside $\mathscr{L}$ is a point of $\omega(B)$. In fact, let $\bar{x}$ be any of such points and pick any sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$. Since $x(t, \bar{x})$ remains inside $\mathscr{L}$ (and hence in B) for all negative values of t, it is seen that $x_k := x(-t_k, \bar{x})$ is a point in B for all k. The sequence $\{x_k, t_k\}$ is such that $x(t_k, x_k) = \bar{x}$ and therefore the property required for $\bar{x}$ to be in $\omega(B)$ is trivially satisfied. This shows that $\omega(B)$ includes not just $\{0\}$ and $\mathscr{L}$, but also all points of the open region surrounded by $\mathscr{L}$. $\triangleleft$

The relevant properties of the ω-limit set of a set, which extend those presented earlier in Lemma B.5, can be summarized as follows.[23]

Lemma B.6 *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Moreover, the distance of $x(t, x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.*

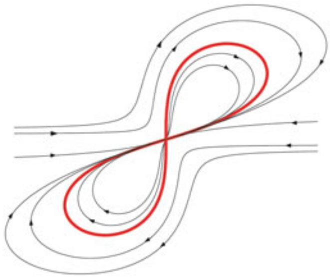
Thus, as it is the case for the ω-limit set of a point, the ω-limit set *of a bounded set B* is compact and invariant. Being invariant, the set $\omega(B)$ is filled with motions which exist for all $t \in (-\infty, +\infty)$ and all such motions, since this set is compact, are bounded in positive and in negative time. Moreover, this set is uniformly approached by motions with initial conditions $x_0 \in B$. We conclude the section with another property, that will be used later to define the concept of *steady-state behavior* of a system.[24]

Lemma B.7 *If B is a compact set invariant for (B.18), then $\omega(B) = B$.*

[23] For a proof see, e.g., [3, 7, 8].

[24] For a proof, see [17].

Fig. B.4 The phase portrait of system (B.24)



B.5 Limit Sets and Stability

It is well known that, in a nonlinear system, an equilibrium point which attracts all motions with initial conditions in some open neighborhood of this point is not necessarily stable in the sense of Lyapunov. A classical example showing that convergence to an equilibrium does not imply stability is provided by the following 2-dimensional system.[25]

Example B.9 Consider the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{pmatrix} \tag{B.24}$$

in which $f(0, 0) = g(0, 0) = 0$ and

$$\begin{pmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{pmatrix} = \frac{1}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)} \begin{pmatrix} x_1^2(x_2 - x_1) + x_2^5 \\ x_2^2(x_2 - 2x_1) \end{pmatrix}$$

for $(x_1, x_2) \neq (0, 0)$. The phase portrait of this system is the one depicted in Fig. B.4. This system has only one equilibrium at $(x, y) = (0, 0)$ and any initial condition $(x_1(0), x_2(0))$ in the plane produces a motion that asymptotically tends to this point. However, it is not possible to find, for every $\varepsilon > 0$, a number $\delta > 0$ such that every initial condition in a disc of radius δ produces a motion which remains in a disc of radius ε for all $t \geq 0$. ◁

It is also known—though—that if the convergence to the equilibrium is *uniform*, then the equilibrium in question *is stable*, in the sense of Lyapunov. This property is a consequence of the fact that $x(t, x_0)$ depends continuously on x_0 (see for example [2, p. 181]).

We have seen before that bounded motions of (B.18) with initial conditions in a bounded set B asymptotically approach the compact invariant set $\omega(B)$. Thus, the question naturally arises to determine whether or not this set is also stable in the sense of Lyapunov. In this respect, we recall that the notion of *asymptotic stability*

[25] See [2, pp. 191–194] and [9].

of a closed invariant set $\mathscr{A}$ is defined as follows. The set $\mathscr{A}$ is asymptotically stable if:

(i) for every $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{dist}(x_0, \mathscr{A}) \leq \delta \quad \Rightarrow \quad \text{dist}(x(t, x_0), \mathscr{A}) \leq \varepsilon \quad \text{for all } t \geq 0.$$

(ii) there exists a number $d > 0$ such that

$$\text{dist}(x_0, \mathscr{A}) \leq d \quad \Rightarrow \quad \lim_{t\to\infty} \text{dist}(x(t, x_0), \mathscr{A}) = 0.$$

It is not difficult to show (see [12] or [15]) that if the set $\mathscr{A}$ is also bounded and hence compact, and the convergence in (ii) is *uniform* in x_0, then property (ii) implies property (i). This yields the following important property of the set $\omega(B)$.

Lemma B.8 *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (B.18). Suppose also that $\omega(B)$ is contained in the interior of B. Then, $\omega(B)$ is asymptotically stable, with a domain of attraction that contains B.*

B.6 The Steady-State Behavior of a Nonlinear System

We use the concepts introduced in the previous section to define a notion of *steady state* for a nonlinear system.

Definition B.5 Consider system (B.18) with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose that X is positively invariant. The motions of this system are said to be *ultimately bounded* if there is a bounded subset $B \subset X$ with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B$ for all $t \geq T$ and all $x_0 \in X_0$.

Motions with initial conditions in a set B having the property indicated in the previous definition are indeed bounded and hence it makes sense to consider the limit set $\omega(B)$, which—according to Lemma B.6—is nonempty and has all the properties indicated in that lemma. What it is more interesting, though, is that—while a set B having the property indicated in the previous definition is clearly not unique—the set $\omega(B)$ is a unique well-defined set.

Lemma B.9 [26]*Let the motions of (B.18) be ultimately bounded and let B' be any other bounded subset of X with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B'$ for all $t \geq T$ and all $x_0 \in X_0$. Then, $\omega(B) = \omega(B')$.*

[26]See [17] for a proof.

It is seen from this that, in any system whose motions are ultimately bounded, all motions asymptotically converge to a well-defined compact invariant set, which is filled with trajectories that are bounded in positive and negative time. This motivates the following definition.

Definition B.6 Suppose the motions of system (B.18), with initial conditions in a closed and positively invariant set X, are ultimately bounded. A *steady-state motion* is any motion with initial condition in $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady-state locus* of (B.18) and the restriction of (B.18) to $\omega(B)$ is the *steady-state behavior* of (B.18). $\triangleleft$

This definition characterizes the steady-state *behavior* of a nonlinear *autonomous* system, such as system (B.18). It can be used to characterized the steady-state *response* of a *forced* nonlinear system

$$\dot{z} = f(z, u) \tag{B.25}$$

so long as the input u can be seen as the output of an *autonomous* "input generator"

$$\begin{aligned} \dot{w} &= s(w) \\ u &= q(w). \end{aligned} \tag{B.26}$$

In this way, the concept of steady-state response (to specific classes of inputs) can be extended to nonlinear systems.

The idea of seeing the steady-state response of a forced system as a particular response of an augmented autonomous system has been already exploited in Sect. A.5, in the analysis of the steady-state response of a stable linear system to harmonic inputs. In the present setting, the results of such analysis can be recast as follows. Let (B.25) be a stable linear system, written as

$$\dot{z} = Az + Bu\,, \tag{B.27}$$

in which $z \in \mathbb{R}^n$, and let (B.26) be the "input generator" defined in (A.26). The composition of (B.25) and (B.26) is the autonomous linear system (compare with (A.28))

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{B.28}$$

Pick a set W_c defined as

$$W_c = \{w \in \mathbb{R}^2 : \|w\| \leq c\}$$

in which c is a fixed number, and consider the set $X = W_c \times \mathbb{R}^n$. The set X is a closed set, positively invariant for the motion of (B.28). Moreover, since the lower subsystem of (B.28) is a linear asymptotically stable system driven by a bounded input, the motions of system (B.28), with initial conditions taken in X, are ultimately bounded. In fact, let Π be the solution of the Sylvester equation (A.29) and recall that the difference $z(t) - \Pi w(t)$ tends to zero as $t \to \infty$. Then, any bounded set B of the form

$$B = \{(w, z) \in W_c \times \mathbb{R}^n : \|z - \Pi w\| \le d\}$$

in which d is any positive number, has the property requested in the definition of ultimate boundedness. It is easy to check that

$$\omega(B) = \{(w, z) \in W_c \times \mathbb{R}^n : z = \Pi w\},$$

that is, $\omega(B)$ is the graph of the restriction of the linear map $x = \Pi w$ to the set W_c. The set $\omega(B)$ is invariant for (B.28), and the restriction of (B.28) to the set $\omega(B)$ characterizes the steady-state response of (B.27) to harmonic inputs of fixed angular frequency ω, and amplitude not exceeding c.

A totally similar result holds if the input generator is a nonlinear system of the form (B.26), whose initial conditions are chosen in a *compact invariant* set W. The fact that W is invariant for the dynamics of (B.26) implies, as a consequence of Lemma B.8, that the steady-state locus of (B.26) is the set W itself, i.e., that the input generator is "in steady state".[27] The composition of (B.26) and (B.27) yields an augmented system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= Az + Bq(w), \end{aligned} \tag{B.29}$$

in which $(w, z) \in X := W \times \mathbb{R}^n$. Note that, since W is invariant for (B.26), the set X is invariant for (B.29).

Since the inputs generated by (B.26) are bounded and the lower subsystem of (B.29) is input-to-state stable, the motions of system (B.29), with initial conditions taken in X, are ultimately bounded. In fact, since W is compact and invariant, there exists a number U such that $\|q(w(t))\| \le U$ for all $t \in \mathbb{R}$ and all $w(0) \in W$. Therefore, standard arguments can be invoked to deduce the existence of positive numbers K, λ and M such that

$$\|z(t)\| \le Ke^{-\lambda t}\|z(0)\| + MU$$

for all $t \ge 0$. From this, it is immediate to check that any bounded set B of the form

$$B = \{(w, z) \in W \times \mathbb{R}^n : \|z\| \le (1 + d)MU\}$$

[27] Note that the set W_c considered in the previous example had exactly this property.

in which d is any positive number, has the property requested in the definition of ultimate boundedness. This being the case, it can be shown that the steady-state locus of (B.29) is the graph of the (nonlinear) map[28]

$$\begin{aligned} \pi : W &\to \mathbb{R}^n \\ w &\mapsto \pi(w) = \int_{\infty}^{0} e^{-A\tau} Bq(\bar{w}(\tau, w)) d\tau \,, \end{aligned} \tag{B.30}$$

i.e.,

$$\omega(B) = \{(w, z) \in W \times \mathbb{R}^n : z = \pi(w)\} \,,$$

To check that this is the case, observe first of all that—since $q(\bar{w}(t, w))$ is by hypothesis a bounded function of t and all eigenvalues of A have negative real part—the integral on the right-hand side of (B.30) is finite for every $w \in W$. Then, observe that the graph of the map $z = \pi(w)$ is invariant for (B.29). In fact, pick an initial state for (B.29) on the graph of this map, i.e., a pair (w_0, z_0) satisfying $z_0 = \pi(w_0)$ and compute the solution $z(t)$ of the lower equation of (B.29), via the classical variation of constants formula, to obtain

$$\begin{aligned} z(t) &= e^{At} \int_{-\infty}^{0} e^{-A\tau} Bq(\bar{w}(\tau, w_0)) d\tau + \int_{0}^{t} e^{A(t-\tau)} Bq(\bar{w}(\tau, w_0)) d\tau \\ &= \int_{\infty}^{0} e^{-A\theta} Bq(\bar{w}(\theta + t, w_0)) d\theta = \int_{\infty}^{0} e^{-A\theta} Bq(\bar{w}(\theta, w(t, w_0))) d\theta . \end{aligned}$$

This shows that $z(t) = \pi(w(t))$ and proves the invariance of the graph of $\pi(\cdot)$ for (B.29). Since the graph of $\pi(\cdot)$ is a compact set invariant for (B.29), this set is necessarily a subset of the steady-state locus of (B.29). Finally, observe that, since the eigenvalues of A have negative real part, all motions of (B.29) whose initial conditions are not on the graph of $\pi(\cdot)$ are unbounded in negative time and therefore cannot be contained in the steady-state locus, which by definition is a bounded invariant set. Thus, the only points in the steady-state locus are precisely the points of the graph of $\pi(\cdot)$.

This result shows that the steady-state response of a stable linear system to an input generated by a nonlinear system of the form (B.26), with initial conditions $w(0)$ taken in a compact invariant set W, can be expressed in the form

$$z_{ss}(t) = \pi(w(t))$$

in which $\pi(\cdot)$ is the map defined in (B.30).

Remark B.2 Note that the motions of the autonomous input generator (B.26) are not necessarily periodic motions, as it was the case for the input generator (A.26). For

[28]In the following formula, $\bar{w}(t, w)$ denotes the integral curve of $\dot{w} = s(w)$ passing through w at time $t = 0$. Note that, as a consequence of the fact that W is closed and invariant, $\bar{w}(t, w)$ is defined for all $(t, w) \in \mathbb{R} \times W$.

instance, the system in question could be a stable Van der Pol oscillator, with W defined as the set of all points inside and on the boundary of the limit cycle. In this case, it is possible to think of the steady-state response of (B.27) not just as of the (single) periodic input obtained when the initial condition of (B.26) is taken on the limit cycle, but also as of all (non periodic) inputs obtained when the initial condition is taken in the interior of W. ◁

Consider now the case of a general nonlinear system of the form (B.25), in which $z \in \mathbb{R}^n$, with input u supplied by a nonlinear input generator of the form (B.26). Suppose that system (B.25) is input-to-state stable and that the initial conditions of the input generator are taken in compact invariant set W. It is easy to see that the motions of the augmented system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, q(w)), \end{aligned} \tag{B.31}$$

with initial conditions in the set $X = W \times \mathbb{R}^n$, are ultimately bounded. In fact, since W is a compact set, there exists a number $U > 0$ such that

$$\|u(\cdot)\|_\infty = \|q(w(\cdot))\|_\infty \leq U$$

for all $w(0) \in W$. Since (B.25) is input-to-state stable, there exist a class $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathscr{K}$ function $\gamma(\cdot)$ such that

$$\|z(t)\| \leq \beta(\|z(0)\|, t) + \gamma(\|u(\cdot)\|_\infty) \leq \beta(\|z(0)\|, t) + \gamma(U)$$

for all $t \geq 0$. Since $\beta(\cdot,\cdot)$ is a class $\mathscr{KL}$ function, for any compact set Z and any number $d > 0$, there exists a time T such that $\beta(\|z(0)\|, t) \leq d\gamma(U)$ for all $z(0) \in Z$ and all $t \geq T$. Thus, it follows that the set

$$B = \{(w, z) \in W \times \mathbb{R}^n : \|z\| \leq (1 + d)\gamma(U)\}$$

has the property requested in the definition of ultimate boundedness.

Since the motions of the augmented system (B.31) are ultimately bounded, its steady-state locus $\omega(B)$ is well defined. As a matter of fact, it is possible to prove that also in this case the set in question is the *graph of a map* defined on W.

Lemma B.10 *Consider a system of the form (B.31) with $(w, z) \in W \times \mathbb{R}^n$. Suppose its motions are ultimately bounded. If W is a compact set invariant for $\dot{w} = s(w)$, the steady-state locus of (B.31) is the graph of a (possibly set-valued) map defined on W.*

Proof Since W is compact and invariant for (B.26), $\omega(W) = W$. As a consequence, for all $\bar{w} \in W$ there is a sequence $\{w_k, t_k\}$ with w_k in W for all k such that $\bar{w} = \lim_{k\to\infty} w(t_k, w_k)$. Set $x = \mathrm{col}(w, z)$ and let $x(t, x_0)$ denote the integral curve of (B.31) passing through x_0 at time $t = 0$. Pick any point $z_0 \in \mathbb{R}^n$ and let $x_k =$

$\text{col}(w_k, z_0)$. All such x_k's are in a compact set. Hence, by definition of ultimate boundedness, there is a bounded set B and a integer $k^* > 0$ such that $x(t_{k^*+\ell}, x_k) \in B$ for all $\ell \geq 0$ and all k. Set $\bar{x}_\ell = x(t_{k^*}, x_\ell)$ and $\tau_\ell = t_{k^*+\ell} - t_{k^*}$, for $\ell \geq 0$, and observe that, by construction, $x(\tau_\ell, \bar{x}_\ell) = x(t_{k^*+\ell}, x_\ell)$, which shows that all $x(\tau_\ell, \bar{x}_\ell)$'s are in B, a bounded set. Hence, there exists a subsequence $\{x(\tau_h, \bar{x}_h)\}$ converging to a point $\hat{x} = \text{col}(\hat{w}, \hat{z})$, which is a point of $\omega(B)$ because all $\bar{x}_h$'s are in B. Since system (B.31) is upper triangular, necessarily $\hat{w} = \bar{w}$. This shows that, for any point $\bar{w} \in W$, there is at least one point $\hat{z} \in \mathbb{R}^n$ such that $(\bar{w}, \hat{z}) \in \omega(B)$. ◁

It should be stressed that the map whose graph characterizes the steady-state locus of (B.31) may fail to be single-valued and, also, may fail to be continuously differentiable, as shown in the examples below.

Example B.10 Consider the system

$$\dot{z} = -z^3 + zu\,, \tag{B.32}$$

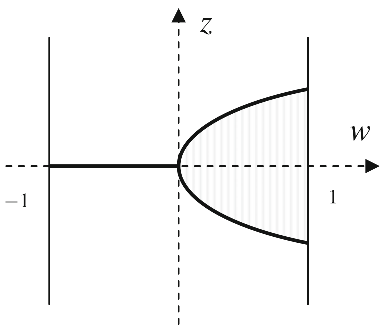
which is input-to-state stable, with input u provided by the input generator

$$\begin{aligned} \dot{w} &= 0 \\ u &= w \end{aligned}$$

for which we take $W = \{w \in \mathbb{R} : |w| \leq 1\}$. Thus, $u(t) = w(t) = w(0) := w_0$. If $w_0 \leq 0$, system (B.32) has a globally asymptotically stable equilibrium at $z = 0$. If $w_0 > 0$, system (B.32) has one unstable equilibrium at $z = 0$ and two locally asymptotically stable equilibria at $z = \pm\sqrt{w_0}$. For every fixed $w_0 > 0$, trajectories of (B.32) with initial conditions satisfying $|z_0| > \sqrt{w_0}$ asymptotically converge to either one of the two asymptotically stable equilibria, while the compact set

$$\{(w, z);\, w = w_0,\, |z| \leq \sqrt{w_0}\}$$

Fig. B.5 The steady-state locus of system (B.32)



is invariant. As a consequence, the steady-state locus of the augmented system

$$\begin{aligned}\dot{z} &= -z^3 + zw \\ \dot{w} &= 0\end{aligned}$$

is the graph of the set-valued map

$$\pi \ : \ w \in W \quad \mapsto \quad \pi(w) \subset \mathbb{R}$$

defined as (Fig. B.5)

$$\begin{aligned}-1 \le w \le 0 \quad &\Rightarrow \quad \pi(w) = \{0\} \\ 0 < w \le 1 \quad &\Rightarrow \quad \pi(w) = \{z \in \mathbb{R} : |z| \le \sqrt{w}\}.\end{aligned}$$ ◁

Example B.11 Consider the system

$$\dot{z} = -z^3 + u$$

which is input to state stable, with input u provided by the harmonic oscillator

$$\begin{aligned}\dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 \\ u &= w_1\end{aligned}$$

for which we take $W = \{w \in \mathbb{R}^2 : \|w\| \le 1\}$. It can be shown[29] that, for each $w(0) \in W$, there is one and only one value $z(0) \in \mathbb{R}$ from which the motion of the resulting augmented system (B.31) is bounded both in positive and negative time. The set of all such pairs identifies a single-valued map $\pi : W \to \mathbb{R}$, whose graph characterizes the steady-state locus of the system. The map in question is continuously differentiable at any nonzero w, but it is only continuous at $w = 0$. ◁

If the map whose graph characterizes the steady-state locus of (B.31) *is single-valued*, the steady-state response of an input-to-state stable system of the form (B.25) to an input generated by a system of the form (B.26) can be expressed as

$$z_{ss}(t) = \pi(w(t)) \,,$$

in which $\pi(\cdot)$ is a map defined on W. In general, it is not easy to give explicit expressions of this map (such as the one considered earlier in (B.30)). However, if $\pi(\cdot)$ is continuously differentiable, a very expressive implicit characterization is possible. In fact, recall that the steady-state locus of (B.31) is by definition an invariant set, i.e., $z(t) = \pi(w(t))$ for all $t \in \mathbb{R}$ along any trajectory of (B.31) with initial condition satisfying $z(0) = \pi(w(0))$. Along all such trajectories,

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = f(z(t), q(w(t))) = f(\pi(w(t)), q(w(t))).$$

[29] See [16].

If $\pi(w)$ is continuously differentiable, then

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{\partial \pi}{\partial w}\Big|_{w=w(t)} \frac{\mathrm{d}w(t)}{\mathrm{d}t} = \frac{\partial \pi}{\partial w}\Big|_{w=w(t)} s(w(t))$$

and hence it is seen that $\pi(w)$ satisfies the partial differential equation

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), q(w)) \qquad \text{for all } w \in W. \tag{B.33}$$

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