Appendix A
Identifying Knots

Identifying unambiguously any knot type is, currently, impossible. The design of an algorithm that will make this possible is one of the biggest challenges for knot theorists in the years to come. Mathematically, knot types are usually identified by analysing the topological properties of the knot complement, i.e. the manifold formed by removing the knot from the space in which it is embedded ($S^3 - \mathcal{K}$), or the topological properties of the Seifert surface $\mathcal{S}$ constructed from a knot diagram representation of a knot (Adams 1994). In practice, these procedures are hardly translated into automated algorithms that can be repeated over thousands of knots. For this reason it is often preferred to use simpler methods, although less reliable. One of these methods is the computation of polynomials, such as the Alexander polynomial.

The construction of the Alexander polynomial starts from a knot diagram a knot (see Fig. A.1a). Being a knot diagram a 2D representation of a 3D object, it is not unique but depends on the perspective chosen for the projection. On the other hand, it can be shown that two knot diagrams of the same knot are equivalent, i.e. can be transformed into one another via a sequence of moves called “Reidemeister moves” (see Fig. A.1). In addition, knot diagrams that cannot be transformed into one another belong to different knot types. There are several quantities that can be calculated from knot diagrams, e.g. the minimal crossing number, the Dowker code, the bridge number or the Alexander and Jones polynomials. Here, I will describe the computation of the Alexander polynomial $\Delta(t)$.

The procedure is the following: (1) Assign a direction to the contour and mark the n crossings and n arcs between crossings (see Fig. A.1a); (2) Determine the sign of the crossings using the standard right-hand rule (see Fig. A.1b); (3) Construct an $n \times n$ matrix M, where the entries of the xth row are

$$M(x, i) = 1 - t;\ \ M(x, j) = -1;\ \ M(x, i) = t; \tag{A.1}$$

if the xth crossing is positive, or

$$M(x, i) = 1 - t;\ \ M(x, j) = t;\ \ M(x, i) = -1; \tag{A.2}$$

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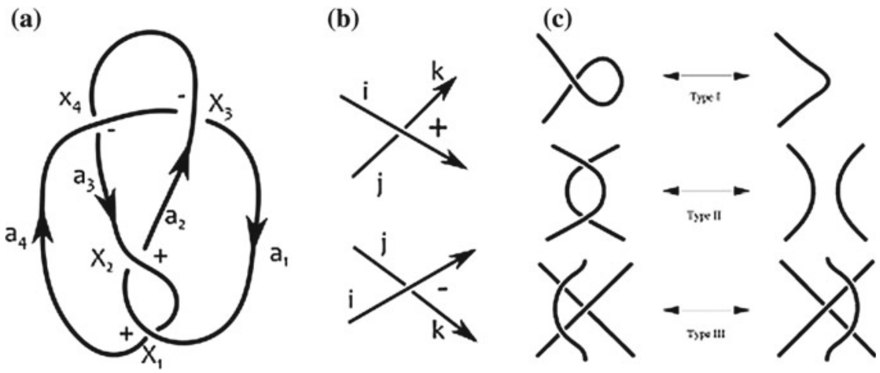

Fig. A.1 **a** Knot diagram of a 4_1 knot. **b** Positive and negative crossings. **c** The three types of Redemeister moves, from Orlandini and Whittington (2007)

if the xth crossing is negative, and where i, j, k are the strands forming the crossing such that i passes over j and k. (4) Delete one row and one column, i.e. take one minor of the matrix, and (5) compute its determinant $\Delta(t)$.

From the knot represented in Fig. A.1 one gets:

$$M(t) = \begin{pmatrix} 1-t & 0 & -1 & t \\ -1 & t & 1-t & 0 \\ -1 & 1-t & 0 & t \\ 0 & t & -1 & 1-t \end{pmatrix} \to M'(t) = \begin{pmatrix} t & 1-t & 0 \\ 1-t & 0 & t \\ t & -1 & 1-t \end{pmatrix} \quad \text{(A.3)}$$

whose determinant is $\Delta(t) = -t^2 + 3t - 1$: the Alexander polynomial of a figure of eight (4_1) knot.

Because Alexander polynomials of different knot diagrams of the same knot can differ up to $\pm t^m$ with $m \in \mathbb{Z}$, it is common practice to compute $\Delta(-1)$ (and identify $\Delta(-1)$ with $-\Delta(-1)$) so to avoid ambiguities. The Alexander polynomial is a good knot invariant for a number of practical applications, in particular when dealing with many randomly generated configurations. On the other hand, it cannot distinguish between, for instance, the 8_{20} knot and the Granny $3_1\#3_1$ knot or between a knot and its mirror image and it is, therefore, to be used with caution.

References

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Orlandini, E., Whittington, S.G.: Statistical topology of closed curves: some applications in polymer physics. Rev. Mod. Phys. **79**(2), 611 (2007)