Appendix 1: Technical Notes on Set Theory

Cohen's Technique of Forcing

Cohen's technique of forcing involves constructing an *outer model* L[G] of L. The aim of this technique is to extend L by adding a new set G which is forced to have certain desired properties not possessed by the sets in L (e.g. the property of violating the continuum hypothesis). This is done in such a way that L[G] (i.e. the result of adjoining to L the set G and everything constructible from G together with the elements of L) is consistent with the axioms of ZFC.

Using the work of Gödel, Cohen (1963: 1143) assumed the existence of a model M which satisfies the axioms of ZF + V = L (i.e. ZF with the Axiom of Constructability). He took M to be a *countable standard transitive*[1] model. Countability is convenient and it follows from the Lowenheim–Skolem theorem that if a first-order set theory has a model at all, then it has a countable model. Cohen's focus on M being standard

[1] A set X is *countable* if its elements can be put in a one-to-one relation with the elements of ω.

A set X is *transitive* if every element of X is a subset of X.

A model M of a set theory is *standard* if the elements of M are well-founded sets ordered with the set membership relation.

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J. McDonnell, *The Pythagorean World*,
DOI 10.1007/978-3-319-40976-4

and transitive also made his proof easier, but these are not essential criteria for the technique of forcing in general.

For every object X which can be proved to exist in ZF + V = L, M contains an analogue object Y, which is often equal to X (Chow 2008: 6). For example, M contains all the ordinals and $\aleph_0$. However, the analogue Y is not *always* equal to X. For example, since M is countable, the set y which is the analogue of the power set of $\aleph_0$ in M will also be countable, so it cannot be the same as the power set of $\aleph_0$ in ZF + V = L (which Cantor showed to be uncountable). Many subsets are missing from y. We say that y is 'uncountable in M' even though it is countable in ZF + V = L. The notions of 'countability' and 'powerset' are relative to the model. Notions, such as 'the empty set', which do not depend on the model, are said to be *absolute*.

Cohen's aim (1963: 1144) was to extend M by adjoining a subset U of $\aleph_0$ that is missing from M and that violates the continuum hypothesis (e.g. contains $\aleph_2$ or more reals). The resulting model M[U] would be a model for ZF and the Axiom of Choice and the negation of the continuum hypothesis. He was concerned that, if U were chosen indiscriminately, then the set that plays the role of a particular cardinal number in M might not play the same role in M[U]. For example, the set $\aleph_2$ in M might become countable in M[U].[2] However he argued that, if U were chosen in a generic manner from M,[3] then no new information would be extracted from it in M[U] which was not already contained in M, so sets such as the cardinals would be preserved. Cohen wrote:

> we do not wish U to contain "special" information about M, which can only be seen from the outside… The U which we construct will be referred to as a "generic" set relative to M. The idea is that all the properties of U must be "forced" to hold merely on the basis that U behaves like a "generic" set in M. This concept of deciding when a statement about U is "forced" to hold is the key point of the construction. (Cohen 1966: 111)

Cohen formalised the notion of a set U being M-generic and of the forcing relation, through which satisfaction for the extension of M could be

[2] This behaviour is called *cardinal collapse*.

[3] For example, to be generic, U should contain infinitely many primes.

approached from inside M. He showed how to build the new set U one step at a time, tracking what new properties of M[U] would be "forced" to hold at each step (Chow 2008: 8). The result was a model which, together with Gödel's earlier work on L, proved the independence of the continuum hypothesis in ZFC.

Generating Large Cardinal Axioms Using Elementary Embeddings

The most general method for generating large cardinal axioms uses *elementary embeddings* (Woodin 2011b: 97–99). An elementary embedding j of V into a transitive class M is a mapping j: V → M which preserves all relations definable in the language of set theory (i.e. if arbitrary $a_0,\ldots,a_n$ in V satisfy an arbitrary first-order formula Φ in the language of set theory, then their mappings $j(a_0),\ldots,j(a_n)$ satisfy Φ in M, and conversely). A basic template for large cardinal axioms asserts that there exists such an embedding j which is not the identity. If j is not the identity then there will be a least ordinal which is not mapped to itself (i.e. such that $j(\alpha) \neq \alpha$) and this is called the *critical point* of j. Then the critical point of j is a measurable cardinal and the existence of the transitive class M and the elementary embedding j are witnesses for this.

By placing restrictions on M, by requiring it to be closer and closer to V (in a manner analogous to width reflection for inner models, but precisely defined in terms of the closure properties of M (Koellner 2011a: 10–12), we can generate stronger and stronger large cardinal axioms. By continuing this process to its natural limit; that is, by taking M = V, the axiom generated asserts the existence of a *Reinhardt* cardinal which is the critical point of a non-trivial elementary embedding j: V → V. However, this axiom has been shown to be inconsistent with ZFC by a theorem of Kunen (Kanamori 1994: 319). The strongest large cardinal axiom which evades Kunen's proof, and so is not known to be inconsistent with ZFC, is one asserting the existence of an *ω-huge* cardinal:

> A cardinal κ is an *ω-huge* cardinal if there exists λ > κ and an elementary embedding $j: V_{\lambda+1} \to V_{\lambda+1}$ such that κ is the critical point of j. (Woodin 2011c: 456)

The outlined procedure for generating large cardinal axioms does not show us how to build up to the large cardinal from below and so it is not intrinsic. Also, to be very clear, it does not guarantee the existence of the large cardinal or the relative consistency of the associated axiom with ZFC. It simply provides us with a means of pushing the level of cardinals as high as it can go, right up to the point where we bump into Kunen's theorem. A further sobering fact is that even the largest of large cardinal axioms is not enough to determine the continuum hypothesis. Levy and Solovay (1967) showed that all such large cardinal assumptions are relatively consistent with both the continuum hypothesis and its negation.

Large Cardinal Axioms and Definable Determinacy

In the early development of set theory, there was a focus on studying the properties of the hierarchy of definable sets of real numbers L(R)[4]—what is now called *descriptive* set theory (Koellner 2011b: 16–24). Many important results were proved. Bendixson (1883) and Cantor (1884) showed that all closed sets of reals have the *perfect set property*[5] and so satisfy the continuum hypothesis. This result was later extended to the first-order in a hierarchy of certain definable open sets. Another property of interest is Lesbesgue measurability. It was known that there are sets of reals which are not Lesbesgue measurable (giving rise to the Banach–Tarski paradox discussed in Sect. 4.2.3) but it was not known whether or not any definable sets of reals lacked this property. Again, proofs of Lesbesgue measurability were limited to the first-order in a hierarchy of certain definable open sets. The story was much the same

[4] See the discussion of Gödel's constructible universe L in Sect. 4.2.4 for a definition of definable sets. By starting with the real numbers R and iterating the definable power set operation along the ordinals, we get the hierarchy L(R) of definable sets of reals.

[5] See Koellner (2011b: 17–18) for a definition of the hierarchy of definable open and closed sets of reals. A set of reals is *perfect* if it is non-empty, closed and contains no isolated points. A set of reals has the *perfect set property* if it is either countable or contains a perfect subset. If a set of reals contains a perfect subset, it must have the same size as the continuum. Therefore, if a set of reals has the perfect set property, it must satisfy the continuum hypothesis (Koellner 2011b: 20).

for other regularity and structural properties of interest.[6] Beyond this, independence results developed in the 1960s (following the work of Gödel and Cohen) showed that many questions about the properties of definable sets of reals are not decidable in ZFC.

The situation was greatly clarified in the 1970s and 1980s with the development of new axioms for ZFC (Koellner 2011b: 24–30). One strand of research studied the implications of large cardinal axioms. The other strand, to do with axioms of *determinacy*, had its origins in a quite different area of mathematics—game theory. *The Axiom of Determinacy*, AD, is an assertion about a two-person infinite game. It says that for every set of reals A, there is a winning strategy for one of the players in the game associated with A.[7] AD is known to be inconsistent with the Axiom of Choice so, in the search for new axioms for ZFC, it is usual to consider restrictions of AD to certain definable sets of reals, giving rise to axioms of *definable determinacy* which are consistent with Choice. For example, by restricting AD to the hierarchy of definable sets of real numbers L(R), one gets the new axiom candidate $AD^{L(R)}$.

The underlying connection between determinacy and the regularity properties of sets of reals is not immediately obvious to outsiders, but to some set theorists "determinacy lies at the heart of the regularity properties and may be considered their true source" (Koellner 2011b: 26). What *is* obviously remarkable is the intricate web of results interweaving large cardinals, determinacy and regularity properties; as outlined by the examples below:

Theorem A.1.3.1 Assume ZFC + $AD^{L(R)}$. Then all the sets of reals in L(R) have the perfect set property, are Lesbesgue measurable and have the property of Baire.

Theorem A.1.3.2. (Woodin) Assume ZFC + "there exist ω-many Woodin cardinals with a measurable cardinal above them all". Then $AD^{L(R)}$.

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[6] Note that there are many such properties other than the ones mentioned in the text; for example, the property of Baire and the uniformisation property (Koellner 2011b).

[7] See Woodin (2010a: 4) for details.

Theorem A.1.3.3. (Woodin) Assume $AD^{L(R)}$. Then there is an inner model N of the theory ZFC + "there are ω-many Woodin cardinals".

Theorem A.1.3.4. (Woodin) Assume ZFC + "there is a proper class of Woodin cardinals". Then the independence of statements in L(R) cannot be established by forcing.[8]

Many more such linked theorems are discussed by Koellner (2011b: 26–35). These theorems bring the two strands of research on new axioms together in an unexpected way. They show that definable determinacy is necessary and sufficient to prove the regularity properties of certain definable sets of reals; and that large cardinals are necessary and sufficient to prove definable determinacy.

Koellner (2011b: 30–38) has used these results to support the case for definable determinacy and, hence, for large cardinal axioms. Some of his main points are:

1. The regularity properties of structure theory are desirable properties for definable sets of reals (e.g. they circumvent the Banach–Tarski paradox).
2. $AD^{L(R)}$ lifts the regularity properties of structure theory to the level of L(R). Conversely, all theories which have this desirable consequence imply $AD^{L(R)}$.
3. For sufficiently strong large cardinal axioms, which themselves imply $AD^{L(R)}$, the results regarding regularity properties cannot be undone by forcing from L(R).
4. $AD^{L(R)}$ is implied by many theories which have proved useful in strengthening ZFC and "there is reason to believe that $AD^{L(R)}$ lies in the *overlapping consensus* of all sufficiently strong, natural theories" (Koellner 2011b: 36).
5. There is promising research to indicate that the case can be strengthened beyond L(R).

[8] This is a very loose statement of an important theorem by Woodin. For a precise statement, see Theorem 12 of Koellner (2006).

6. Definable determinacy and large cardinal axioms are at root equivalent: "We have here a case where intrinsically plausible principles from completely different domains reinforce each other" (Koellner 2009b: 111).

Although the results are remarkable, not all set theorists find the case for definable determinacy compelling. They argue that descriptive set theory is only one area and focusing on another area might produce competing, incompatible axioms. Part of Koellner's argument is the claim that this is unlikely but, of course, it depends on what areas one finds interesting and what weight one gives to intrinsically plausible results from different areas. For example, Shelah (2003: 211–213) thinks that Gödel's Axiom of Constructibility V = L is an interesting axiom with some appealing consequences, and it is known to be incompatible with one of the axioms of definable determinacy.

Defining Effective Completeness

Theorem A.1.3.4 from the previous section offers the hope of overcoming forcing. It says that, assuming sufficiently strong large cardinal assumptions, the truth value of statements in L(R) cannot be changed by forcing. As stated in Koellner and Woodin (2009: 1158): "Woodin cardinals 'seal' or 'freeze' the theory of L(R)". Furthermore, the results of descriptive set theory show that the same large cardinal assumptions effectively resolve all of the open problems relating to the regularity and structural properties of definable sets of reals. So, in this sense, we can achieve effective completeness at the level of L(R). Recall that L(R) is the same as $L(V_{\omega+1})$. This means that we can achieve effective completeness at a level beyond, and inclusive of, the level of second-order arithmetic, $V_{\omega+1}$, but below the level of third-order arithmetic, $V_{\omega+2}$. It would be desirable to extend this result to $V_{\omega+2}$ and higher levels. However, the continuum hypothesis is a statement in $V_{\omega+2}$ and we know that it cannot be resolved by any standard large cardinal axioms, so something more will be needed. To define effective completeness at higher levels of the set hierarchy, it is necessary to introduce some technical notions.

Firstly, there is the notion of *the Levy hierarchy of formulas*. A formula in a language that contains the language of set theory is of type Σ_0 (or, equivalently, Π_0) if it is built from atomic formulas through the use of logical connectives and bounded quantifiers $\forall x \in y$ and $\exists x \in y$. A formula is of type Σ_1 if it is of the form $\exists \mathbf{x}\ \varphi$ where $\mathbf{x}$ is a list of variables and φ is of type Π_0. A formula is of type Π_1 if it is of the form $\forall \mathbf{x}\ \varphi$ where φ is of type Σ_0. In general, a formula is of type Σ_n, n>1, if it is of the form $\exists \mathbf{x}\ \varphi$ where φ is of type Π_{n-1}. And a formula is of type Π_n, n>1, if it is of the form $\forall \mathbf{x}\ \varphi$ where φ is of type Σ_{n-1}. In set theory, an assertion is called Π_2 if it is of the form: "For every infinite ordinal α, $V_\alpha \models \varphi$ ", for some Σ_1-sentence φ, where $\models$ is the symbol for consequence in the language. Similarly, an assertion is called Σ_2 if it is of the form: "There exists an infinite ordinal α such that $V_\alpha \models \varphi$ ", for some Π_1-sentence φ. A Σ_2-sentence is expressible as the negation of a Π_2-sentence.

Secondly, there is the notion of *a Σ^2_n sentence*. A Σ^2_n sentence is one of the form "$V_{\omega+2} \models \varphi$", where φ is a formula which is Σ_n in the Levy hierarchy. The continuum hypothesis is equivalent to a Σ^2_1 sentence. Hence, it is a sentence at the level of third-order arithmetic, $V_{\omega+2}$ and at the first level of the Levy hierarchy.

Thirdly, there is the notion of stratifying the universe of sets in terms of the cardinal partition, *H(κ)*, rather than the familiar ordinal partition, V_α. Here, κ is an infinite cardinal and $H(\kappa)$ is the set of all sets whose transitive closure has cardinality less than κ (i.e. the set of all sets which have cardinality less than κ, whose members have cardinality less than κ, whose members' members have cardinality less than κ, etc.) (Koellner 2011c: 13). For example, $H(\omega) = V_\omega$ and the theories of the structures $H(\omega_1)$ and $V_{\omega+1}$ are mutually interpretable. If the continuum hypothesis is true, then $H(\omega_2)$ and $V_{\omega+2}$ are mutually interpretable, otherwise $H(\omega_2)$ is less rich than $V_{\omega+2}$. So, in investigating the continuum hypothesis, it makes sense to consider $H(\omega_2)$ first.

Fourthly, and lastly, there is the notion of *Ω-completeness*. This will take a bit more explanation. We know that in order to resolve the continuum hypothesis, a new notion is needed. Part of the answer might be to introduce a logic which is stronger than first-order logic and is well behaved in the sense that its consequences cannot be altered by forcing in the presence of large cardinal axioms (Koellner 2011c: 15–17). Ω-logic is such a logic.[9] It has a notion of consequence, denoted $\models_\Omega$, which is robust to forcing under the assumption of a proper class of Woodin cardinals. It also has a quasi-syntactic proof relation, denoted $\vdash_\Omega$, defined in terms of universally Baire sets of reals which act as witnesses for the "proofs" in the logic (Woodin 2001b: 684). Ω-logic is known to be sound. Hence, if a sentence φ is Ω-provable in a set theory T, then it is necessarily an Ω-consequence of T. However, it is not known whether Ω-logic is complete; that is, whether every sentence φ which is an Ω-consequence of a set theory T is necessarily Ω-provable in T. The conjecture that Ω-logic is both sound and complete is called *the Ω conjecture*. This is so important that I will highlight it:

The Ω conjecture says that Ω-logic is both sound and complete.

A theory T is said to be *Ω-complete* for a collection of sentences Γ if for each $\varphi \in \Gamma$, either $T \models_\Omega \varphi$ or $T \models_\Omega \neg\varphi$ (Koellner 2011c: 17). Hence, the Ω-completeness of a theory for some collection of sentences means that those sentences are decidable in Ω-logic.

Exploring the Effective Completability of Set Theory

Using the notions introduced in Sect. A.1.4, Theorem A.1.3.4 asserting the invariance of the theory of L(R) under set forcing can be rephrased as:

Theorem A.1.5.1. (Woodin) Assume ZFC and that there is a proper class of Woodin cardinals. Then ZFC is Ω-complete for the collection of sentences of the form "L(R) $\models \varphi$". (Koellner 2011c: 17)

Because of the nature of the continuum hypothesis, we know that there is no standard large cardinal axiom LCA such that ZFC + LCA is Ω-complete for Σ^2_1 sentences. However, there is the possibility that a supplementary axiom could achieve this outcome and, surprisingly, it

[9] For more information on Ω-logic and its relation to full first- and second-order logic, see Koellner (2010).

turns out that simply assuming the continuum hypothesis as an additional axiom *does* achieve it. Thus:

Theorem A.1.5.2. (Woodin) Assume ZFC and that there is a proper class of measurable Woodin cardinals. Then ZFC + CH is Ω-complete for $\Sigma^2{}_1$. (Koellner 2011c: 29)

Moreover, up to Ω-equivalence, CH is the unique $\Sigma^2{}_1$-statement that is Ω-complete for $\Sigma^2{}_1$ (Koellner and Woodin 2009: 1161).

Of course, assuming the continuum hypothesis as an axiom doesn't count as resolving it. On the contrary, for many years, Woodin promoted a parallel case in which the continuum hypothesis is *negated* (Woodin 2001b). Thus, assuming that there is a proper class of Woodin cardinals and that the Strong Ω Conjecture[10] holds, Woodin (1999) proved that there is an axiom A such that ZFC + A is Ω-complete for the structure $H(\omega_2)$ and that any such axiom has the feature that the continuum hypothesis is *false* in the associated theory. The axiom A is not unique, there are many such Ω-complete theories (Koellner and Woodin 2009). However, Woodin found a particular axiom, called *Axiom (*)*, which gives a "maximal" theory in the sense that it satisfies all Π_2-sentences that can possibly hold (Koellner 2011c: 14). Assuming Axiom (*), the continuum hypothesis is false and $2\string^ \aleph_0 = \aleph_2$. Woodin (2001b: 687) saw Axiom (*) as being a candidate for the generalisation of $AD^{L(R)}$ to the structure $H(\omega_2)$; that is, settling in Ω-logic the full theory of $H(\omega_2)$ just as $AD^{L(R)}$ had settled the open problems of descriptive set theory in L(R).

So now, we have two cases for extending our Ω-complete theories: Case 1, in which the continuum hypothesis is assumed as an axiom and Case 2, which assumes Axiom (*) and has the consequence that the continuum hypothesis is false. Case 1 is only Ω-complete for $\Sigma^2{}_1$ sentences, whereas Case 2 is Ω-complete for the whole of the structure $H(\omega_2)$. Case 2 uses the additional assumption that the Strong Ω Conjecture holds.

Does this mean that there is a bifurcation of the universe of sets at the level of the continuum hypothesis? This question can't really be answered at the local level since there is always the possibility that

[10] The Strong Ω Conjecture is the Ω Conjecture plus a conjecture about an extended form of the AD. See Koellner (2011c: 16–17).

further extensions of the theory will favour one path over the other. It is known that if the Strong Ω Conjecture holds, then one cannot have an Ω-complete theory of the whole of the structure $V_{\omega+2}$:

Theorem A.1.5.3. (Woodin) Assume ZFC and that there is a proper class of Woodin cardinals. Assume the Strong Ω Conjecture. Then there is no recursively enumerable theory A such that ZFC + A is Ω-complete for Σ^2_3. (Koellner and Woodin 2009: 1162)

So this path is effectively closed off at the Σ^2_3 level. On the other hand, the following theorem shows that the continuum hypothesis is not sufficient to provide an Ω-complete theory at the Σ^2_2 level:

Theorem A.1.5.4. (Jensen, Shelah) ZFC + CH is not Ω-complete for Σ^2_2. (Koellner and Woodin 2009: 1162)

So this path is also closed.

One can consider further local scenarios at the Σ^2_2 level.[11] I take it that the upshot of these is that, if the Strong Ω Conjecture holds, then it is likely that there is a "best" theory in which the continuum hypothesis has a definite answer, even though there is not yet a compelling case to tell us which way that answer will fall (Koellner 2011c: 33). My impression is that the local approach is nearing the end of its usefulness. There were remarkable successes at the level of $V_{\omega+1}$ which were extended with great effort to fragments of $V_{\omega+2}$ but without any clear result for the continuum hypothesis. Cases multiply and it becomes hard to keep track of the arguments. The justification for the "best" cases becomes less compelling. Perhaps this situation will improve with further research but, regardless, the local approach has inherent limitations.

If the Strong Ω Conjecture does not hold, then there is the possibility that we could have an Ω-complete theory for $V_{\omega+2}$. In fact, it might be possible to achieve an Ω-complete theory for any specified level V_α and to piece these together into a coherent theory of the entire universe of sets. If this were achieved, then the theory would be a candidate for the effective

[11] See, for example, Koellner (2011c: 27–33).

completion of set theory along the lines envisaged by Gödel (1946: 151) (see Sect. 4.5.1). However, Koellner and Woodin (2009) have shown that such a theory could not be unique. If there is an Ω-complete theory of V_α for $\alpha \geq (\omega+2)$, then there is another, equally good but incompatible Ω-complete theory which differs on the truth value of the continuum hypothesis.[12] This seriously raises the spectre of pluralism. The existence of two, or more, coherent but incompatible Ω-complete theories of V would undermine the universalist's concept of truth. It would not be possible to combine such theories in the manner suggested by Martin (2001). Such a scenario is considered by Steel (2004) and discussed at length by Maddy (2005: 369–373). The theories would all be universal and equivalent in terms of the mathematics that would result (i.e. choosing one theory over another would not result in any behavioural or methodological differences). In choosing one theory, the others would be accessible as generic extensions. The continuum hypothesis (and other undecidable statements) would be meaningless in an absolute sense, but acquire a relative value as a matter of convention.

This scenario is highly speculative but does serve to crystallise a plausible pluralist case. Note that a multiverse of this kind does not conform to the radical pluralist view of what a multiverse might consist of (i.e. all consistent theories). It locks in a universalist view at least to the level of ZFC + "there is a proper class of measurable Woodin cardinals" which is essential for the application of the notion of Ω-completeness. In this way, it avoids the problems which plague the radical pluralist view.

Woodin (2011d) has used a global perspective to define and investigate a multiverse of a related kind, which he calls *the generic multiverse*. Ultimately, he sets up the generic multiverse position as a strawman, only to show that it is untenable under fundamental principles of set theory, *provided that the Ω Conjecture is true*. Here, once again, we see the importance of the Ω Conjecture.

Woodin's generic multiverse is generated from V by closing under generic extensions (forcing) and under generic refinements (inner models of a universe which the given universe is a generic extension of). The associated generic multiverse view of truth is that a sentence is true if

[12] Note that the continuum hypothesis is just a particular case; there are other suitable sentences which could be forced to differ between Ω-complete theories of V_α.

and only if it holds in each universe of the generic multiverse. Letting the background theory be ZFC + "there is a proper class of measurable Woodin cardinals", this view entails that the continuum hypothesis is indeterminate whilst $AD^{L(R)}$ is true (and, hence, all the results of descriptive set theory, described previously, are locked in).

Woodin (2011d: 19–21) proves that for each Π_2-sentence φ, the following are equivalent:

1. φ holds across the generic multiverse;
2. "φ is true in Ω-logic" holds across the generic multiverse;
3. "φ is true in Ω-logic" holds in at least one universe of the generic multiverse.

This means that if φ is Ω-valid in one universe, then it holds in all the universes of the generic multiverse. Notably, a sentence asserting the existence of a proper class of Woodin cardinals (and its consequences) and a sentence asserting the truth of the Ω conjecture would be invariant across the generic multiverse. Woodin says: "the notion of truth is the same as defined relative to each universe of the multiverse, so from the perspective of evaluating truth all the universes of the multiverse are equivalent" (Woodin 2011d: 15). He sees this as an important point in favour of the generic multiverse view. For one thing, it implies that a statement (such as a consistency statement) can't be indeterminate in one universe and determinate in another, as might happen under the radical pluralist view. Thus, truth is not so sensitive to the meta-universe in which the generic multiverse is being defined and the pluralist isn't forced to retreat to an infinite regression of meta-universes in order to define his position.

Next, Woodin (2011d: 17–18) formulates two related multiverse laws:

1. The set of Π_2-multiverse truths cannot be recursive in the set of multiverse truths of V_α for any specifiable α; and
2. The set of Π_2-multiverse truths is not definable in V_α across the multiverse for any specifiable α.

These laws capture the idea that any reasonable concept of multiverse truth cannot be reduced to, or defined in, an initial fragment V_α of the universe V. That would lead to "a brand of formalism that denies the transfinite by reducing truth about the universe of sets to truth about a simple fragment" (Woodin 2011d: 17). Woodin's multiverse laws are in the spirit of the reflection principles of set theory.

Woodin (2011d: 23–25) goes on to show that, assuming both the Ω Conjecture and the existence of a proper class of Woodin cardinals, both multiverse laws are violated by the generic multiverse position. Specifically, he proves that both laws are violated at the level of $V_{\delta 0+1}$ where δ_0 is the least Woodin cardinal. This result can be sharpened to $V_{\omega+2}$ for the first multiverse law, but the import is the same in any case: the generic multiverse position is untenable because the set of all its truths is reducible to the set of truths of some initial fragment of V.

The pluralists' best response, short of placing restrictions on the generic multiverse, is to deny the truth of the Ω Conjecture. Woodin ([2011d: 28] and [2011b: 111–112]) argues that there is evidence for the Ω Conjecture, albeit nothing conclusive. Firstly, the Ω Conjecture can be shown to be consistent with the theory ZFC + "There is a proper class of Woodin cardinals". Secondly, the assertion of its Ω-satisfiability is a Σ_2-assertion which is invariant across the generic multiverse and there are no known examples of Σ_2-statements that are provably absolute and not settled by large cardinal axioms. Thirdly, "recent results indicate that if [the Inner Model] program can succeed at the level of supercompact cardinals then no large cardinal hypothesis whatsoever can refute the Ω Conjecture" [Woodin 2011d: 28]. These are the results from the theory of Ultimate L (as discussed in Sect. 4.5.3 in the main text).

Fending Off a Potential Threat to the Universalist View

In "The Realm of the Infinite" (2011b), Woodin begins by arguing that we implicitly accept the meaningfulness of the large finite and that there is nothing to prevent us from accepting the meaningfulness of the universe of sets (as discussed in Sect. 4.6 in the main text). In the next part of his paper, he gives further arguments for accepting the truth of set theory, this time in opposition to a hypothetical Sceptic who denies any genuine meaning to a conception of uncountable sets (i.e. he accepts the meaningfulness of the large finite and the completed infinity of the natural numbers, but not of uncountable sets, and certainly not of large cardinals).

Woodin lays out the evidence for large cardinals and makes his prediction that the theory ZFC + "there exist infinitely many Woodin cardinals" will never be shown to be inconsistent (even though, analogously to the large finite case, an inconsistency *could* conceivably be found in the physical realm). The Sceptic accepts the evidence for the consistency of the theory, and agrees with the prediction, but doesn't accept that it is the result of the truth of the axioms (or, equivalently, the existence of the large cardinals). He suggests that the infinite realm is so inclusive that any reasonably defined large cardinal axiom would turn out to be consistent, provided that there isn't a simple proof of its inconsistency. In particular, he points to the Reinhardt cardinal axiom, which is the strongest possible large cardinal axiom generated by the elementary embedding template (as discussed in Sect. A.1.2). Kunen produced a simple proof of its inconsistency with ZFC, but that proof depended on using the Axiom of Choice, and it is still possible that it is consistent with ZF. Furthermore, if the theory ZF + "there is a weak Reinhardt cardinal" *is* consistent, then it implies the consistency of ZFC + "there is a proper class of ω-huge cardinals" and thereby accounts for all the consistency predictions of Woodin's accepted large cardinal hierarchy. So this one claim—the formal consistency of the Reinhardt axiom with ZF—would account for all the claims of set theory.

The Skeptic's claim creates a serious problem for Woodin. The consistency of the Reinhardt cardinal axiom could not be grounded in the *existence* of a Reinhardt cardinal because then we would have an immediate bifurcation of set theory into a universe in which the Axiom of Choice holds and one in which it does not hold. Anyway, there is no fine structure theory for a Reinhardt cardinal, so no witnesses to its existence. Rather, the set theorist might try to account for its formal consistency by showing that the theory ZF + "there is a weak Reinhardt cardinal" is equiconsistent with a theory of the form ZFC + LCA, for some large cardinal axiom LCA; in the same way that ZF + AD can be shown to be equiconsistent with ZFC + "there exist infinitely many Woodin cardinals", even though AD itself is incompatible with the Axiom of Choice (Woodin 2011b: 100–101). However, this approach will not be possible because the Reinhardt cardinal axiom is stronger than any large cardinal axiom not known to refute the Axiom of Choice and, so, will not be equiconsistent with any such theory. Next, Woodin explores the idea that the generic multiverse view of truth might be able to account for the prediction that weak Reinhardt cardinals are consistent with ZF (Woodin 2011b: 107–108). Again, this approach will not be possible because, as we saw previously, the generic multiverse view of truth is untenable under the assumption of the Ω Conjecture because it violates the multiverse laws.

Next, Woodin plays his trump card—Ultimate L. If a solution to Ultimate L could be found, then one corollary would be that the Reinhardt cardinal axiom is inconsistent with ZF (Woodin 2011b: 115–116). So, rather than contorting ZFC to account for the consistency of weak Reinhardt cardinals with ZF, it could be proved that they are, in fact, inconsistent. This would comprehensively refute the Skeptic's claim. More than that, it would provide "for the *first time* an example of a natural large cardinal axiom proved to be inconsistent as a result of a deep structural analysis" (Woodin 2011b: 116). Such an inconsistency result could never be achieved by Inner Model Theory because of its incremental nature. But Ultimate L would provide us with the means to explore the boundary between possible and impossible large cardinal axioms, strengthening its claim to being the ultimate arbiter of mathematical truth and existence. Woodin's hunch is that it would enable us to eliminate essentially all the large cardinal axioms known to contradict the Axiom of Choice, tantamount to a proof of the Axiom of Choice (Woodin 2011c: 470). Then, according to Woodin's view, any suggested set theory would need to be justified by first establishing its equiconsistency with a specific level of Ultimate L. Ulimate L, in its turn, would be justified by our understanding of the hierarchy of large cardinal axioms as "true axioms about the universe of sets" (Woodin 2011b: 96).

Woodin thinks that one of the main pieces of evidence supporting the truth of large cardinal axioms is our ability to construct their fine structure theory. Using Inner Model Theory, evidence for each large cardinal axiom is built up from below. Starting from Gödel's L, minimal extenders are added at each stage to witness the existence of the next large cardinal in the hierarchy. However, we know that this process cannot be extended beyond the level of an inner model with one supercompact cardinal. The reason is that

Woodin's investigation of this case has shown that such a model would automatically inherit all stronger large cardinals. Then these stronger large cardinal axioms would require some further extrinsic justification. Woodin suggests that the answer to this dilemma is to use the structural analogy of the theory of L(R) under $AD^{L(R)}$ to the theory of $L(V_{\lambda+1})$ under the assumption of an ω-huge cardinal to reveal the structure theory of ω-huge cardinals without constructing a full inner model theory (Woodin 2011c: 469–470).

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Appendix 2: Problems with Plenitude

Overview

This appendix considers the problems that arise if too much plenitude is postulated. It refers to David Lewis' *On the Plurality of Worlds* (1986) for a fair statement of the problems and a defence of plenitude. Lewis' main reason for embracing plenitude is that it allows a great benefit in terms of unity and economy of theory. He thinks that logical space for philosophers can be developed in an analogous way to set theory for mathematicians, thus enabling philosophers to use a vast realm of possibilia to reduce the required number of primitives and premises (1986: 4). To pave the way, he has shown in *Parts of Classes* (1991) how the constructions of set theory can be reproduced in mereology (i.e. the study of parts and wholes).

How Much Plenitude?

Lewis defends the thesis of modal realism which he states as:

> the thesis that the world we are part of is but one of a plurality of worlds, and that we who inhabit this world are only a few out of all the inhabitants of all the worlds. (1986: vii)

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J. McDonnell, *The Pythagorean World*,
DOI 10.1007/978-3-319-40976-4

According to this thesis, there are many possible worlds, some just like ours and some completely alien to our understanding. A possible world has parts which are possible individuals. A world is the mereological sum of all the possible individuals that are its parts.[13] Spatiotemporal relations make individuals cohere in a world.

Lewis advocates a principle of recombination. Parts of any world can be duplicated, multiplied, and combined into a new world. There is just one proviso:

> The only limit on the extent to which a world can be filled with duplicates of possible individuals is that the parts of a world must be able to fit together with some possible size and shape of spacetime. Apart from that, anything can coexist with anything, and anything can fail to coexist with anything. (Lewis 1986: 89–90)

The proviso limits the ways in which space-time may be occupied (cf. set theory, in which there is no limit to the ways in which sets can be construed and the result is Cantor's hierarchy of infinities). If space-time is a continuum, then it cannot accommodate more individuals than the cardinality of the real numbers. Lewis is prepared to contemplate different possible space-times but he is not prepared to let the cardinality of possible individuals yield consequences about the possible size of space-time. Therefore, the cardinality of possible individuals is limited.

Forrest and Armstrong (1984) show that, without the proviso, the principle of recombination leads to paradoxes for Lewis' possible worlds analogous to the paradoxes of set theory. For example, consider the world which contains all the possible worlds as parts. This "biggest world" exists according to Lewis' mereology of possible worlds. Suppose that it has cardinality K. Forrest and Armstrong show how to construct a world with cardinality greater than K by extracting parts of the biggest world, duplicating and recombining them. However, by hypothesis, no world can have cardinality greater than K. Fortunately, the proviso prevents this argument from going through: the "bigger than biggest" world is too big to fit into the biggest world and, so, too big to fit into any possible space-time.

[13] The mereological sum of several things is the least inclusive thing that includes all of them as parts.

Lewis relies on the proviso to prevent the construction of "bigger than biggest" worlds. This forces him to contemplate a cleavage between the structure of mathematics and the structure of possible worlds. There will be some set-theoretical objects that are bigger than any possible world. Nevertheless (at some cost in terms of unity and economy of theory), Lewis sticks to the principle of plenitude in listing his conception of what there is altogether: "the possible worlds, the possible individuals that are their parts, and the mathematical objects, even if those should turn out to be pure sets not made out of the parts of the worlds" (1986: 111). Lewis thinks that we should accept mathematics and he rejects the suggestion that any mathematical objects could be fictional. He is less concerned by Benacerraf's dilemma than some:

> If we are prepared to expand our existential beliefs for the sake of theoretical unity, and if thereby we come to believe the truth, then we attain knowledge. We can know that there exist countless objects causally isolated from us and unavailable to our inspection. (1986: 109)

According to Lewis, mathematics and possible worlds give us knowledge beyond the reach of our causal acquaintance. We arrive at this knowledge by reasoning from general principles (such as recombination). Indeed, the same principles are involved in constructing sets and possible worlds; it's just that possible worlds have an additional constraint.

The constraint on possible worlds derives from the intuition that they are just like our world in many relevant ways. They can be described as "concrete", just like our world and so can be distinguished from the "abstract" world of mathematics, even though Lewis cannot find a principled distinction between concrete and abstract (1986: 81–86). The question arises as to where to draw the line. Do concrete individuals have to fit into finite-dimensional space-times? Lewis thinks that that is too restrictive and he hopes that there is a natural break higher up: "high enough to make room for all the possibilities we really need to believe in" (1986: 103). The break must be *natural* because, if it were arbitrary, then we could always imagine a possible world in which the boundary is moved. It is hard to see how this natural break would be defined and Lewis does not offer an answer. The constraint on possible worlds leads to an inevitable clash of intuitions.

The Convergence of Physical and Mathematical Objects

I think that Lewis' goals of unity and simplicity would be much better served if he abandoned the concrete/abstract distinction altogether or, at least, removed the structural distinction between concrete and abstract possibilia (in other words, if he allowed the structure of all possible worlds to be isomorphic to some mathematical structure and vice versa). Of course, this is a difficult option for those who see possible worlds as a combination of concrete parts because it means either abandoning the higher reaches of set theory (which Lewis will not do) or finding some physical interpretation for mathematical objects like large cardinals (impossible for Lewis).

A similar dilemma is faced by philosophers of mathematics who favour Aristotelian realism (see e.g. [Franklin 2014]). Aristotelian realists take the view that the world consists of particulars and universals and that mathematical objects are universals which are *in re* (i.e. in the things themselves). In their view, mathematics studies structures which are instantiated in real objects such as bathroom tiles (groups) and coastlines (fractals). This leaves them with the problem of what to say about uninstantiated mathematical objects. In the modal version of Aristotelian realism, the response is that some mathematical objects can be constructed combinatorially from concepts sourced in the real world—they are possibilia similar to Lewis' possible worlds. Still, the assumption is that they *could* be instantiated in concrete things and this puts a constraint on what mathematical objects are supported by this view.

Some mathematical objects don't seem to have relevance to the physical world at all. For example, it is hard to envisage any role for the inaccessible numbers of set theory in any physical process. We would not reach that sort of cardinality even in a multiverse even if an infinite number of universes had been continually generating baby universes since past infinity. An inaccessible number is just that: it cannot be reached by the iteration of constructive processes such as taking unions of collections of smaller sets, or taking power sets, or taking any combination of operations available in set theory. If one imagines translating all mathematical objects into the background language of set theory, then the subset which is relevant to known physical processes is tiny. Feferman (2006b: 446) has suggested that it can be described by the predicative part.[14]

One option is to restrict mathematics to the point where it can be physicalised in some sense, by claiming that some part of it is inconsistent or fictional. For example, one might hold (with Nelson [2011]) that only the finite part of mathematics is consistent. This could be combined with a view of the universe as a digital computer operating on a discrete system of binary information (as held by Fredkin [1992] and Wolfram [2002]). In this view, there would be a core part of mathematics—the finite part—which is consistent, firmly grounded in physical intuition and real, and physics would only be concerned with that part. Physics and mathematics would both be translatable into a language of finite information processing. Presumably, Aristotelian realists could adopt this ultrafinitist position and, indeed, Franklin (2014: 133) questions whether infinities are really needed in applied mathematics. However, he seems to prefer to adopt a version of modal realism in which some of the truths of mathematics refer to actual physical instantiations whilst others refer to possible instantiations, and the possible instantiations can be infinite. This reinstates the problem of where to draw the line. Beyond the finite, there is no principled place to draw the line between physics and mathematics. Aristotelian realism supports an epistemology in which our knowledge of mathematical objects is explained in terms of our knowledge of everyday objects but it does not account for our knowledge of infinity.

Hartry Field is a philosopher who abolishes the distinction between concrete and abstract objects by denying the existence of abstract entities. He champions the view that mathematics is fictional and of only instrumental use in physical theories. As a test case, Field (1980) tries to identify the 'physical' part of Newtonian gravitational theory (i.e. the part which has empirical support and/or corresponds to physical structure) and separate it from the 'surplus' part (i.e. the part which provides logical and instrumental support). He develops a nominalised version of the theory which refers only to space-time points and regions, not to

[14] According to predicativists, the natural numbers form a definite totality but the collection of all sets of natural numbers (i.e. the power set) does not.

abstract mathematical objects. He defines bridge laws which map statements from the nominalised theory to its mathematical counterpart (and vice versa). According to Field, the advantage of using the mathematical theory is that it allows us to use all the deductive power of mathematics to demonstrate consequences which might have been obscure or difficult to prove in the nominalised theory. The mathematics doesn't lead to any new assertions about the physical world—all the physical consequences of the theory can be derived from the underlying nominalised theory. However, in order for the nominalised theory to retain the full power of the mathematical theory, Field is forced to build almost all the structure and complexity of the real number system into space-time itself. Effectively, he adds pseudo-mathematical structure to the nominalised theory until it converges towards the structure of the mathematical theory. Thus, his attempt to nominalise Newtonian gravitational theory leads to a counter-productive blurring of the concrete/abstract boundary.

I argue that the concrete/abstract boundary should be abolished by embracing all objects as abstract objects. What exists are minds (abstract individuals) and thoughts (abstract classes[15]). In this view, physical objects are a subset of mathematical objects but are not essentially different. Both physical and mathematical objects are the objects of intention of some mind (i.e. they are thoughts). The structure of the world instantiates all of mathematics, all possible forms, and physical reality condenses out by a process of self-actualisation in thought. I assume that the world is fully rational and that there are minds which are able to understand, test and validate their theories about its structure at all levels. Thus, any set is oversee-able by some mind. Gödel's incompleteness theorems show that every level of the set-theoretical hierarchy contains true statements which can only be proved using the resources of higher levels. The hierarchy of effective field theories in physics is similarly incomplete. A coherent and fully rational theory of physics and mathematics together requires a mind which embraces all of mathematics but transcends it. That Mind is the encompassing mind which is the mereological sum of all individual minds and their thoughts. It is a proper class.

[15] Not all classes are objects, some are proper classes.

Comments on Lewis' *Parts of Classes*

In Lewis' metaphysical framework, reality divides exhaustively into individuals and classes (1991: 7). An individual is anything that has no members but is a member of some class. A class is either a set or a proper class. A singleton is a class that has no parts except itself—it is a mereological atom. Every individual has a singleton and so does every set. The only things that lack singletons are the proper classes. A class is any mereological sum of singletons. A proper class does not have a singleton but it is the mereological sum of singletons.

The notion of a singleton is a primitive. Lewis finds it mysterious (1991: 29–35). Singletons have only a single member so they do not fall under the usual description of a set as being a collection of many into one. He says: "Our utter ignorance about the nature of the singletons amounts to utter ignorance about the nature of classes generally" (1991: 31). I say that a singleton is a thought: it is the result of a mind taking some individual as an object of intention. Classes are not in spacetime and they are not located where their members are, but they are not nowhere, rather, they are associated with minds.

According to Lewis, any class includes the null set, but the null set is not a subclass or a part (1991: 10–11). Classes do not have the null set as part because the null set is not a class, it is an individual. Any individual will do for the null set. Lewis redefines the null set as the mereological sum of all individuals (1991: 14). It is everywhere rather than nowhere. This fits well with my framework. The null set is the mereological sum of all monads and so is the encompassing mind bare of its thoughts. It is the One which is transcendentally present in all beings and is their source (Sect. 5.4: 159).

Paradoxes of Plenitude

What about Forrest and Armstrong's paradox which arises when we allow unrestricted plenitude for possible worlds? Their argument does not get off the ground in my framework because it assumes that there is a big world which contains all the possible worlds as parts. In my framework, there is no world of all worlds, just as there is no set of all sets, because each individual monad is a possible world and the class of monads is a proper class.

Lewis (1986: 104–108) considers a second paradox of plenitude concerning the set of worlds that characterises the content of somebody's thought.[16] There is no natural bound on the cardinality of the set of somebody's thoughts and the argument uses this fact to construct a possible world whose cardinality exceeds the cardinality of the set of all possible worlds. As before, this argument fails in my framework because there is no set of all possible worlds. However, some interesting points arise in Lewis' discussion. For example, Lewis claims that:

> Not just any set of worlds is a set that might possibly give the content of someone's thought. Most sets of world, in fact all but an infinitesimal minority of them, are not eligible contents of thought. (1986: 105)

I do not agree with this. Any set defines a possible thought although not all possible thoughts define a set. Lewis is committed to the view that there are unthinkable contents. I hold that all content is thinkable by some mind. Lewis' discussion of unthinkable content seems only to show that there is content which it is perverse for a mind to think about its world, given its beliefs and desires. He does not show that such content is unthinkable. I can think "I want a saucer of mud" even if I cannot put myself into a state in which I would want to turn that thought into action. Another mind in another possible world might actually want a saucer of mud!

Actuality

Lewis' indexical analysis of actuality enables him to dodge some potential problems of plenitude, including ethical problems. According to Lewis:

[16] Here, "somebody" is not considered to have human limitations (e.g. it might refer to a god) and so it should be read as "some mind", maybe an infinite mind.

> every world is *actual* at itself, and thereby all worlds are on a par. This is *not* to say that all worlds are actual—there's no world at which that is true. (1986: 93)

For us, our world is actual and other possible worlds are not actual. Therefore, we should concern ourselves with evils in this world, even if there would be the same sum total of good and evil throughout the worlds no matter what we do in this one (1986: 123–128). Reasonably, it is the good and evil in our actual world which we care about. My framework also denies that other worlds are actual, so I can refer to Lewis' arguments in defence of these problems.

However, my analysis of actuality differs significantly from Lewis'. In quantum monadology, the actual world is the one and only, necessary, best possible world. In Lewis' modal realism, actuality is a contingent matter—it is not the case that one world alone is absolutely actual. First, Lewis asks how we would know that we are absolutely actual. Then, he writes:

> And yet we *do* know for certain that the world we are part of is the actual world—just as certainly as we know that the world we are part of is the very world we are part of. (1986: 93)

I take this knowledge that each of us has—of our own existence and actuality—as the most certain knowledge which we can have. It poses a problem for monadology in that it is not clear how an actual monad can know that it is actual rather than merely possible. One response is to say that it is a special property which actual monads have—to be immediately acquainted with their own actuality—but this is not satisfactory as an explanation. Another response is to invoke the concrete/abstract divide—the actual world is concrete and possible worlds are abstract—but this distinction is not available in the idealistic interpretation of monadology which I have adopted.

I propose to make a distinction between actual and possible monads in terms of consciousness—actual monads are conscious whereas potential monads are like philosophical zombies. Consciousness is a property of actual monads because they follow the critical path from beginning to end and complete the neural circuit in which meaning is communicated to each individual mind by the encompassing mind.

The Construction of Possible Worlds

Lewis' view of the construction of possible worlds is a static, mereological view involving the combination of parts into wholes. I don't find this way of constructing possible worlds satisfactory. It misses the dynamic, process component of the construction of possible worlds which imposes a hierarchy of priority on thoughts; the ordinal hierarchy as opposed to the cardinal hierarchy which Lewis draws from in his imaginative experiments. Spatiotemporal relations are not what makes a possible world cohere. It is the laws which enable monads to have harmonious relations. Possible worlds have to be built up coherently according to local laws.

Lewis seems to think that possible worlds help us to say what it means for a false theory of nature to be close to the truth: "A theory is close to the truth to the extent that our world resembles some world where that theory is exactly true" (1986: 24). This doesn't seem right. Our world obeys a hierarchy of laws at different scales which result in harmony. A world in which, for example, Newton's laws were exactly true would not be like our world at all. For one thing, quantum mechanics is responsible for the stability of fundamental matter in our world. Lewis' worlds seem to be more like the dreams (imaginings) of monads at a particular scale, more wish than possible fact, not taking total harmony into account. As he says:

> imaginability is a poor criteria of possibility. We can imagine the impossible, provided we do not imagine it in perfect detail and all at once. We cannot imagine the possible in perfect detail and all at once, not if it is at all complicated. (1986: 90)

I can imagine a hundred-foot elephant walking around the earth but I know that it is not possible according to our physical laws and I doubt whether it is possible at all. It is not enough for possible worlds to be imaginable, they have to be actualisable.

I think that to properly define the construction of possible worlds, it will be necessary to go beyond the mereological approach of *Parts of Classes* to embed mathematics as a bare structure in a bigger theory of concepts which has static (cardinal, to do with size) and dynamic (ordinal, to do with priority) components.

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J. McDonnell, *The Pythagorean World*,
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J. McDonnell, *The Pythagorean World*,
DOI 10.1007/978-3-319-40976-4