APPENDIX A

The Topology of Subsets of $\mathbb{R}^n$

In this appendix, we briefly review some notions from topology that are used throughout the book. The exposition is intended as a quick review for readers with some previous exposure to these topics.

1. Open and Closed Sets and Limit Points

The natural **distance function** on $\mathbb{R}^n$ is defined such that for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$\text{dist}(\mathbf{a}, \mathbf{b}) = |\mathbf{a} - \mathbf{b}| = \sqrt{\langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle}.$$

Its most important property is the *triangle inequality*:

Proposition A.1 (The Triangle Inequality). *For all* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$,

$$\textit{dist}(\mathbf{a}, \mathbf{c}) \leq \textit{dist}(\mathbf{a}, \mathbf{b}) + \textit{dist}(\mathbf{b}, \mathbf{c}).$$

Proof. The Schwarz inequality (Lemma 1.12) says that $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq |\mathbf{v}||\mathbf{w}|$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Thus,

$$\begin{aligned} |\mathbf{v} + \mathbf{w}|^2 &= |\mathbf{v}|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + |\mathbf{w}|^2 \\ &\leq |\mathbf{v}|^2 + 2|\mathbf{v}| \cdot |\mathbf{w}| + |\mathbf{w}|^2 = (|\mathbf{v}| + |\mathbf{w}|)^2. \end{aligned}$$

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So $|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}|$. Applying this inequality to the vectors pictured in Fig. A.1 proves the triangle inequality. □

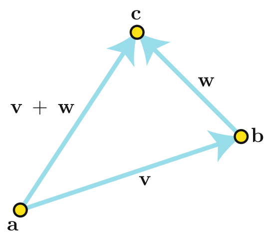

FIGURE A.1. Proof of the triangle inequality

Topology begins with precise language for discussing whether a subset of Euclidean space contains its boundary points. First, for $\mathbf{p} \in \mathbb{R}^n$ and $r > 0$, we denote the **ball about p of radius** r by

$$B(\mathbf{p}, r) = \{\mathbf{q} \in \mathbb{R}^n \mid \text{dist}(\mathbf{p}, \mathbf{q}) < r\}.$$

In other words, $B(\mathbf{p}, r)$ contains all points closer than a distance r from $\mathbf{p}$.

DEFINITION A.2.
*A point $\mathbf{p} \in \mathbb{R}^n$ is called a **boundary point** of a subset $S \subset \mathbb{R}^n$ if for all $r > 0$, the ball $B(\mathbf{p}, r)$ contains at least one point in S and at least one point not in S. The collection of all boundary points of S is called the **boundary** of S.*

Sometimes, but not always, boundary points of S are contained in S. For example, consider the "open upper half-plane"

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\},$$

and the "closed upper half-plane"

$$\overline{H} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}.$$

The x-axis, $\{(x, 0) \mid x \in \mathbb{R}^2\}$, is the boundary of H and also of $\overline{H}$. So H contains none of its boundary points, while $\overline{H}$ contains all of its boundary points. This distinction is so central that we introduce vocabulary for it:

DEFINITION A.3.
Let $S \subset \mathbb{R}^n$ be a subset.

(1) *S is called **open** if it contains none of its boundary points.*
(2) *S is called **closed** if it contains all of its boundary points.*

In the previous example, H is open, while $\overline{H}$ is closed. If part of the x-axis is adjoined to H (say the positive part), the result is neither closed nor open, since it contains some of its boundary points but not all of them.

A set $S \subset \mathbb{R}^n$ and its complement $S^c = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} \notin S\}$ clearly have the same boundary. If S contains none of these common boundary points, then S^c must contain all of them, and vice versa. So we learn a relationship between a subset and its complement:

LEMMA A.4.
A set $S \subset \mathbb{R}^n$ is closed if and only if its complement, S^c, is open.

The following provides a useful alternative definition of "open":

LEMMA A.5.
A set $S \subset \mathbb{R}^n$ is open if and only if for all $\mathbf{p} \in S$, there exists $r > 0$ such that $B(\mathbf{p}, r) \subset S$.

PROOF. If S is not open, then it contains at least one of its boundary points, and no ball about such a boundary point is contained in S. Conversely, suppose that there is a point $\mathbf{p} \in S$ such that no ball about $\mathbf{p}$ is contained in S. Then $\mathbf{p}$ is a boundary point of S, so S is not open. □

The proposition says that if you live in an open set, then so do all of your sufficiently close neighbors. How close is sufficient depends on how close you live from the boundary. For example, the set

$$S = (0, \infty) \subset \mathbb{R}$$

is open because for every $x \in S$, the ball $B(x, x/2) = (x/2, 3x/2)$ lies inside of S; see Fig. A.2. When x is close to 0, the radius of this ball is small.

Similarly, for every $\mathbf{p} \in \mathbb{R}^n$ and $r > 0$, the ball $B = B(\mathbf{p}, r)$ is itself open, because about every $\mathbf{q} \in B$, the ball of radius $\frac{r - \text{dist}(\mathbf{p},\mathbf{q})}{2}$ lies in B by the triangle inequality; see Fig. A.3.

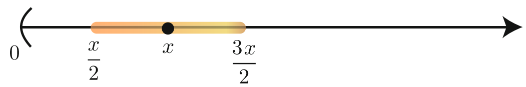

FIGURE A.2. The set $(0, \infty) \subset \mathbb{R}$ is open because it contains a ball about each of its points

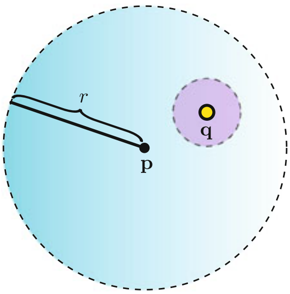

FIGURE A.3. The set $B(\mathbf{p}, r) \subset \mathbb{R}^n$ is open because it contains a ball about each of its points

Lemma A.6.
The union of a collection of open sets is open. The intersection of a finite collection of open sets is open. The intersection of a collection of closed sets is closed. The union of a finite collection of closed sets is closed.

The collection of all open subsets of $\mathbb{R}^n$ is called the **topology** of $\mathbb{R}^n$. It is surprising how many important concepts are topological, that is, definable purely in terms of the topology of $\mathbb{R}^n$. For example, the notion of whether a subset is closed is topological. The distance between points of $\mathbb{R}^n$ is not topological. The notion of **convergence** is topological by the second definition below, although it may not initially seem so at first:

Definition A.7.
An infinite sequence $\{\mathbf{p}_1, \mathbf{p}_2, \ldots\}$ of points of $\mathbb{R}^n$ is said to ***converge*** *to $\mathbf{p} \in \mathbb{R}^n$ if either of the following equivalent conditions holds:*

(1) $\lim_{n\to\infty} dist(\mathbf{p}, \mathbf{p}_n) = 0$.
(2) *For every open set, U, containing $\mathbf{p}$, there exists an integer N such that $\mathbf{p}_n \in U$ for all $n > N$ (Fig. A.4).*

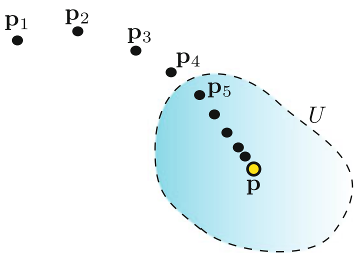

Figure A.4. A convergent sequence is eventually inside of every open set containing its limit

Definition A.8.
A point $\mathbf{p} \in \mathbb{R}^n$ is called a ***limit point*** *of a subset $S \subset \mathbb{R}^n$ if there exists an infinite sequence of points of S that converges to $\mathbf{p}$.*

Every point $\mathbf{p} \in S$ is a limit point of S, as evidenced by the redundant infinite sequence $\{\mathbf{p}, \mathbf{p}, \mathbf{p}, \ldots\}$. Every point of the boundary of S is a limit point of S as well. In fact, the collection of limit points of S equals the union of S and the boundary of S. Therefore, *a set $S \subset \mathbb{R}^n$ is closed if and only if it contains all of its limit points*, since this is the same as requiring it to contain all of its boundary points.

It is possible to show that a sequence converges without knowing its limit, just by showing that the terms get closer and closer to each other:

Definition A.9.
An infinite sequence of points $\{\mathbf{p}_1, \mathbf{p}_2, \ldots\}$ *in* $\mathbb{R}^n$ *is called a* ***Cauchy sequence*** *if for every* $\epsilon > 0$*, there exists an integer* N *such that* $dist(\mathbf{p}_i, \mathbf{p}_j) < \epsilon$ *for all* $i, j > N$*.*

It is straightforward to prove that every convergent sequence is Cauchy. A fundamental property of Euclidean space is the converse:

Proposition A.10.
Every Cauchy sequence in $\mathbb{R}^n$ *converges to some point of* $\mathbb{R}^n$*.*

We end this section with an important *relative* notion of open and closed:

Definition A.11.
Let $V \subset S \subset \mathbb{R}^n$ *be subsets.*

(1) V *is called* ***open in*** S *if there exists an open subset of* $\mathbb{R}^n$ *whose intersection with* S *equals* V*.*
(2) V *is called* ***closed in*** S *if there exists a closed subset of* $\mathbb{R}^n$ *whose intersection with* S *equals* V*.*

For example, setting $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, notice that the hemisphere $\{(x, y, z) \in S^2 \mid z > 0\}$ is open in S^2, because it is the intersection with S^2 of the following open set: $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$.

It is straightforward to show that V is open in S if and only if $\{\mathbf{p} \in S \mid \mathbf{p} \notin V\}$ is closed in S. The following is a useful equivalent characterization of "open in" and "closed in":

Proposition A.12.
Let $V \subset S \subset \mathbb{R}^n$ *be subsets.*

(1) V *is open in* S *if and only if for all* $\mathbf{p} \in V$*, there exists* $r > 0$ *such that* $\{\mathbf{q} \in S \mid dist(\mathbf{p}, \mathbf{q}) < r\} \subset V$*.*
(2) V *is closed in* S *if and only if every* $\mathbf{p} \in S$ *that is a limit point of* V *is contained in* V*.*

Part (1) says that if you live in a set that's open in S, then so do all of your sufficiently close neighbors in S. Part (2) says that if you live in a set that's closed in S, then you contain all of your limit points that belong to S. For example, the interval $[0, 1)$ is neither open nor closed in $\mathbb{R}$, but is open in $[0, 2]$ and is closed in $(-1, 1)$.

Let $\mathbf{p} \in S \subset \mathbb{R}^n$. A **neighborhood of p in** S means a subset of S that is open in S and contains $\mathbf{p}$. For example, $(1 - \epsilon, 1 + \epsilon)$ is a neighborhood of 1 in $(0, 2)$ for every $\epsilon \in (0, 1]$. Also, $[0, \epsilon)$ is a neighborhood of 0 in $[0, 1]$ for every $\epsilon \in (0, 1]$.

The collection of all subsets of S that are open in S is called the **topology** of S. It has many natural properties. For example, the relative version of Lemma A.6 is true: the union of a collection of subsets of S that are open in S is itself open in S, and similarly for the other statements.

In the remainder of this appendix, pay attention to which properties of a set S are topological, that is, definable in terms of only the topology of S. For example, the notion of a sequence of points of S converging to $\mathbf{p} \in S$ is topological. Why? Because convergence means that the sequence is eventually inside of any neighborhood of $\mathbf{p}$ in $\mathbb{R}^n$; this is the same as being eventually inside of any neighborhood of $\mathbf{p}$ in S, which has only to do with the topology of S. The idea is to forget about the ambient $\mathbb{R}^n$, and regard S as an independent object, with a topology and hence a notion of convergence.

2. Continuity

Let $S_1 \subset \mathbb{R}^{n_1}$ and $S_2 \subset \mathbb{R}^{n_2}$. A function $\mathbf{f} : S_1 \to S_2$ is called *continuous* if it maps nearby points to nearby points; more precisely:

Definition A.13.
*A function $\mathbf{f} : S_1 \to S_2$ is called **continuous** if for every infinite sequence $\{\mathbf{p}_1, \mathbf{p}_2, \ldots\}$ of points in S_1 that converges to a point $\mathbf{p} \in S_1$, the sequence $\{\mathbf{f}(\mathbf{p}_1), \mathbf{f}(\mathbf{p}_2), \ldots\}$ converges to $\mathbf{f}(\mathbf{p})$.*

For example, the "step function" $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

is not continuous. Why? Because the sequence

$$\{1/2, 1/3, 1/4, \ldots\}$$

in the domain of f converges to 0, but the sequence of images

$$\{f(1/2) = 1, f(1/3) = 1, f(1/4) = 1, \ldots\}$$

converges to 1 rather than to $f(0) = 0$.

Notice that $\mathbf{f}$ is continuous if and only if it is continuous when regarded as a function from S_1 to $\mathbb{R}^{n_2}$. It is nevertheless useful to forget about the ambient Euclidean spaces, and regard S_1 and S_2 as independent objects. This vantage point leads to the following beautiful, although less intuitive, way to define continuity:

Proposition A.14.
For a function $\mathbf{f} : S_1 \to S_2$, the following are equivalent:

(1) *f is continuous.*
(2) *For every set U that's open in S_2, $\mathbf{f}^{-1}(U)$ is open in S_1.*
(3) *For every set U that's closed in S_2, $\mathbf{f}^{-1}(U)$ is closed in S_1.*

Here, $\mathbf{f}^{-1}(U)$ denotes the set $\{\mathbf{p} \in S_1 \mid \mathbf{f}(\mathbf{p}) \in U\}$. The above step function fails this continuity test, because

$$f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) = (-\infty, 0],$$

which is not open in $\mathbb{R}$.

It is now clear that continuity is a topological concept, since this alternative definition involved only the topologies of S_1 and S_2.

Familiar functions from $\mathbb{R}$ to $\mathbb{R}$, such as polynomial, rational, trigonometric, exponential, and logarithmic functions, are all continuous on their domains. Furthermore, the composition of two continuous functions is continuous.

We next wish to describe what it means for S_1 and S_2 to be "topologically the same." There should be a bijection between them that pairs open sets with open sets. More precisely:

Definition A.15.
*A function $\mathbf{f} : S_1 \to S_2$ is called a **homeomorphism** if $\mathbf{f}$ is bijective and continuous and $\mathbf{f}^{-1}$ is continuous. If such a function exists, then S_1 and S_2 are said to be **homeomorphic**.*

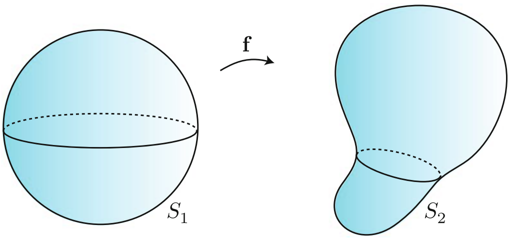

Figure A.5. Homeomorphic sets

Homeomorphic sets have the same "essential shape," such as the two subsets of $\mathbb{R}^3$ pictured in Fig. A.5. The hypothesis that $\mathbf{f}^{-1}$ is continuous is necessary. To see this, consider the function $\mathbf{f} : [0, 2\pi) \to S^1 \subset \mathbb{R}^2$ defined as $\mathbf{f}(t) = (\cos t, \sin t)$. It is straightforward to check that $\mathbf{f}$ is continuous and bijective, but $\mathbf{f}^{-1}$ is not continuous. (Why not?) We will see in Sect. 4 that $[0, 2\pi)$ is not homeomorphic to S^1, since only the latter is compact.

3. Connected and Path-Connected Sets

DEFINITION A.16.
A subset $S \subset \mathbb{R}^n$ is called ***path-connected*** *if for every pair $\mathbf{p}, \mathbf{q} \in S$, there exists a continuous function $\mathbf{f} : [0,1] \to S$ with $\mathbf{f}(0) = \mathbf{p}$ and $\mathbf{f}(1) = \mathbf{q}$.*

The terminology comes from visualizing the image of such a function $\mathbf{f}$ as a path in S beginning at $\mathbf{p}$ and ending at $\mathbf{q}$.

For example, the disk $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is path-connected, since every pair $\mathbf{p}, \mathbf{q} \in A$ can be connected by the straight line segment between them, explicitly parametrized as

$$\mathbf{f}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}).$$

But the disjoint union of two disks,

$$B = \{\mathbf{p} \in \mathbb{R}^2 \mid \operatorname{dist}(\mathbf{p}, (-2,0)) < 1 \text{ or } \operatorname{dist}(\mathbf{p}, (2,0)) < 1\},$$

is not path-connected, because no continuous path exists between points in different disks; see Fig. A.6.

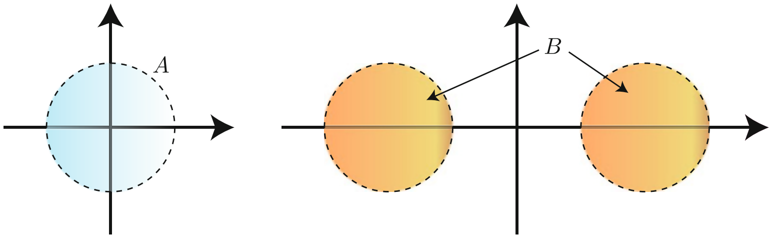

FIGURE A.6. A is path-connected, while B is not

In the non-path-connected example, the right disk is **clopen** (both open and closed) in B, and therefore so is the left disk. In other words, B decomposes into the disjoint union of two subsets that are both clopen in B.

DEFINITION A.17.
A set $S \subset \mathbb{R}^n$ is called ***connected*** *if there is no subset of S (other than all of S and the empty set) that is* ***clopen in*** $\mathbf{S}$ *(both open in S and closed in S).*

Such a separation of a path-connected set is impossible:

PROPOSITION A.18.
Every path-connected set $S \subset \mathbb{R}^n$ is connected.

PROOF. We first prove that the interval $[0,1]$ has no clopen subsets other than itself and the empty set. Suppose $A \subset [0,1]$ is another such set. Let t denote the infimum of A. Since A is closed, $t \in A$. Since A is open, there exists $r > 0$ such that all points of $[0,1]$ with distance $< r$ from t lie in A. This contradicts the fact that t is the infimum of A unless $t = 0$. Therefore, $0 \in A$. Since the complement, A^c, of A is also clopen, the same argument proves that $0 \in A^c$, which is impossible.

Next, let $S \subset \mathbb{R}^n$ be a path-connected set. Suppose that $A \subset S$ is a clopen subset. Suppose there exist points $\mathbf{p}, \mathbf{q} \in S$ such that $\mathbf{p} \in A$ and $\mathbf{q} \notin A$. Since S is path-connected, there exists a continuous function $\mathbf{f} : [0,1] \to S$ with $\mathbf{f}(0) = \mathbf{p}$ and $\mathbf{f}(1) = \mathbf{q}$. Then $\mathbf{f}^{-1}(A)$ is a clopen subset of $[0,1]$ that contains 0 but not 1, contradicting the previous paragraph. □

In practice, to prove that a property is true at all points of a connected set, it is often convenient to prove that the set of points where the property holds is nonempty, open, and closed. For example:

PROPOSITION A.19.
If $S \subset \mathbb{R}^n$ is connected, and $f : S \to \mathbb{R}$ is a continuous function that attains only integer values, then f is constant.

PROOF. Let $y_0 \in \mathbb{Z}$ denote an integer value attained by f. The nonempty set $f^{-1}(y_0) = \{x \in S \mid f(x) = y_0\}$ is closed in S because the singleton set $\{y_0\}$ is closed in $\mathbb{R}$. It is also open in S because it equals $f^{-1}(U)$, where $U = \left(y_0 - \frac{1}{2}, y_0 + \frac{1}{2}\right)$ is a neighborhood of y_0 in $\mathbb{R}$ that is small enough not to include any other integers. Since S is connected, $f^{-1}(y_0)$ must equal all of S, so f does not attain any other values. □

Connectedness and path-connectedness are topological notions. In particular, If $S_1 \subset \mathbb{R}^{n_1}$ and $S_2 \subset \mathbb{R}^{n_2}$ are homeomorphic, then either both are path-connected or neither is path-connected. Similarly, either both are connected or neither is connected.

4. Compact Sets

The notion of compactness is fundamental to topology. We begin with the most intuitive definition.

DEFINITION A.20.
*A subset $S \subset \mathbb{R}^n$ is called **bounded** if $S \subset B(\mathbf{p}, r)$ for some $\mathbf{p} \in \mathbb{R}^n$ and some $r > 0$. Further, S is called **compact** if it is closed and bounded.*

Compact sets are those that contain their limit points and lie in a finite chunk of Euclidean space. Unfortunately, this definition is not topological, since "bounded" cannot be defined without referring to the distance function on $\mathbb{R}^n$. In particular, boundedness is not preserved by homeomorphisms, since the bounded set $(0,1)$ is homeomorphic to the unbounded set $\mathbb{R}$. Nevertheless, compactness is a topological notion, as is shown by the following alternative definition:

Definition A.21.
Let $S \subset \mathbb{R}^n$.

(1) *An **open cover** of S is a collection, $\mathbb{O}$, of sets that are open in S and whose union equals S.*
(2) *S is called **compact** if every open cover, $\mathbb{O}$, of S has a finite subcover, meaning a finite subcollection $\{U_1, \ldots, U_n\} \subset \mathbb{O}$ whose union equals S.*

The equivalence of our two definitions of compactness is called the **Heine–Borel theorem**. The easy half of its proof goes like this: Suppose that S is not bounded. Then the collection

$$\{\mathbf{p} \in S \mid \text{dist}(\mathbf{0}, \mathbf{p}) < k\},$$

for $k = 1, 2, 3, \ldots$, is an open cover of S with no finite subcover. Next suppose that S is not closed, which means it is missing a limit point $\mathbf{q} \in \mathbb{R}^n$. Then the collection $\{\mathbf{p} \in S \mid \text{dist}(\mathbf{p}, \mathbf{q}) > 1/k\}$, for $k = 1, 2, 3, \ldots$, is an open cover of S with no finite subcover.

The other half of the proof is substantially more difficult. We content ourselves with a few examples.

The open interval $(0,1) \subset \mathbb{R}$ is not compact because it is not closed, or because

$$\mathbb{O} = \{(0, 1/2), (0, 2/3), (0, 3/4), (0, 4/5) \ldots\}$$

is an open cover of $(0,1)$ that has no finite subcover.

The closed interval $[0,1]$ is compact because it is closed and bounded. It is somewhat difficult to prove directly that every open cover of $[0,1]$ has a finite subcover; attempting to do so will increase your appreciation of the Heine–Borel theorem.

Our second definition of compactness is topological. Therefore, if $S_1 \subset \mathbb{R}^{n_1}$ and $S_2 \subset \mathbb{R}^{n_2}$ are homeomorphic, then either both are compact or neither is compact.

There is a third useful equivalent characterization of compactness, which depends on the notion of *subconvergence*.

Proposition and Definition A.22.

(1) *An infinite sequence of points* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots\}$ *in* $\mathbb{R}^n$ *is said to* ***subconverge*** *to* $\mathbf{p} \in \mathbb{R}^n$ *if there is an infinite subsequence* $\{\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \mathbf{p}_{i_3}, \ldots\}$ *(with* $i_1 < i_2 < i_3 < \cdots$*) that converges to* $\mathbf{p}$*.*

(2) *A subset* $S \subset \mathbb{R}^n$ *is compact if and only if every infinite sequence of points in* S *subconverges to some* $\mathbf{p} \in S$*.*

For example, the sequence $\{1/2, 2/3, 3/4, \ldots\}$ in $S = (0,1)$ subconverges only to $1 \notin S$, which gives another proof that $(0,1)$ is not compact. One consequence of this "subconvergence" characterizations of compactness is the following:

Proposition A.23.
If $S \subset \mathbb{R}^n$ *is compact and* $\mathbb{O}$ *is an open cover of* S*, then there exists a number* $\delta > 0$ *(called the* ***Lebesgue number*** *of* $\mathbb{O}$*) such that for any pair of points in* S *separated by a distance* $< \delta$*, there exists an open sets from the collection* $\mathbb{O}$ *that contains both points.*

Proof. Suppose to the contrary that there is no such number δ. Then for each positive integer n, there exist points $p_n, q_n \in S$ with $\text{dist}(p_n, q_n) < \frac{1}{n}$ that do not both belong to any member of $\mathbb{O}$. The sequences $\{p_n\}$ and $\{q_n\}$ must subconverge to points $p, q \in S$ respectively, and it is straightforward to show that $p = q$. Since $\mathbb{O}$ is an open cover, there exists $U \in \mathbb{O}$ such that $p \in U$. Since U is open, both subsequences are eventually inside U, contradicting the fact that for every n, p_n and q_n do not both belong to any single member of $\mathbb{O}$. □

The next proposition says that the continuous image of a compact set is compact.

Proposition A.24.
Let $S \subset \mathbb{R}^n$*. Let* $\mathbf{f} : S \to \mathbb{R}^m$ *be continuous. If* S *is compact, then the image,* $\mathbf{f}(S)$*, is compact.*

Proof. The function $\mathbf{f}$ is also continuous when regarded as a function from S to $\mathbf{f}(S)$. Let $\mathbb{O}$ be an open cover of $\mathbf{f}(S)$. Then $\mathbf{f}^{-1}(U)$ is open in S for every $U \in \mathbb{O}$, so $\mathbf{f}^{-1}(\mathbb{O}) := \{\mathbf{f}^{-1}(U) \mid U \in \mathbb{O}\}$ is an open cover of S. Since S is compact, there exists a finite subcover $\{\mathbf{f}^{-1}(U_1), \ldots, \mathbf{f}^{-1}(U_k)\}$ of $\mathbf{f}^{-1}(\mathbb{O})$. It is straightforward to check that $\{U_1, U_2, \ldots, U_k\}$ is a finite subcover of $\mathbb{O}$. □

Corollary A.25.
If $S \subset \mathbb{R}^n$ is compact and $f : S \to \mathbb{R}$ is continuous, then f attains is supremum and infimum.

The conclusion that f attains is supremum means two things. First, the supremum of $f(S)$ is finite (because $f(S)$ is bounded). Second, there is a point $\mathbf{p} \in S$ for which $f(\mathbf{p})$ equals this supremum (because $f(S)$ is closed).

Recommended Excursions

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Undergraduate Texts in Mathematics,
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