General Appendices

Appendix I
The Gamma Matrices in Various Dimensions

In this appendix, we summarize some of the representations of the gamma matrices γ^μ in $D = 2, 4, 10$ dimensional spacetimes, satisfying the anti-commutation relations:[1]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \mathrm{diag}[-1, 1, \ldots, 1], \quad \mu = 0, 1, \ldots, (D-1). \tag{I.1}$$

Four Dimensional Spacetime

Some of the properties of the gamma matrices, based on their anti-commutations relations in (I.1) in 4D are given here in Box I.1.

[1] It should be noted that no gamma matrix may be chosen to be the identity matrix I without making the other gamma matrices non-zero. The reason for this is very simple. If, for example γ^μ, for a fixed μ, is chosen to be I, then for any other gamma matrix γ^ν, for $\nu \neq \mu$, the anti-commutation relation (I.1) implies that $2I\gamma^\nu = 0$, i.e., *all* the other gamma matrices γ^ν, for $\nu \neq \mu$, are *zero*. The latter, in turn, is inconsistent with (I.1) as the anti-commutator of the zero matrix with itself is zero.

E.B. Manoukian, *Quantum Field Theory II*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-33852-1

Box I.1: Some properties of the gamma matrices

$$\gamma^{\mu}\gamma^{\nu} = -\eta^{\mu\nu}I + (1/2)\,[\gamma^{\mu}, \gamma^{\nu}], \quad \mathrm{Tr}\,[\gamma^{\mu}] = 0,$$

$$\left(\gamma^{0}\right)^2 = I, \quad \left(\gamma^{i}\right)^2 = -I, \quad i = 1, 2, 3.$$

$$\eta_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = -4I,$$

$$\eta_{\mu\nu}\gamma^{\mu}\big(\gamma^{\sigma}\big)\gamma^{\nu} = 2\gamma^{\sigma},$$

$$\eta_{\mu\nu}\gamma^{\mu}\big(\gamma^{\sigma}\gamma^{\lambda}\big)\gamma^{\nu} = 4\,\eta^{\sigma\lambda},$$

$$\eta_{\mu\nu}\gamma^{\mu}\big(\gamma^{\sigma}\gamma^{\lambda}\gamma^{\rho}\big)\gamma^{\nu} = 2\,\gamma^{\rho}\gamma^{\lambda}\gamma^{\sigma},$$

$$\Big[\gamma^{\mu}, \Big[\gamma^{\sigma}, \gamma^{\rho}\Big]\Big] = 4\left(\gamma^{\sigma}\eta^{\mu\rho} - \gamma^{\rho}\eta^{\mu\sigma}\right),$$

$$\mathrm{Tr}\,[\gamma^{\mu}\gamma^{\nu}] = -4\,\eta^{\mu\nu},$$

$$\mathrm{Tr}\left[\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right] = 4\big(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}\big),$$

$$\mathrm{Tr}\left[\gamma^{5}\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right] = -4\,\mathrm{i}\,\varepsilon^{\alpha\beta\mu\nu},$$

$\varepsilon^{\alpha\beta\mu\nu}$ totally anti-symmetric with $\varepsilon^{0123} = +1$.

Tr [odd number of γ's] = 0.

$$\gamma^{5} = \mathrm{i}\,\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}, \quad \mathrm{Tr}\,[\gamma^{5}] = 0, \quad \mathrm{Tr}\,[\gamma^{5}\gamma^{\mu}] = 0,$$

$$\left(\gamma^{5}\right)^2 = I, \quad \{\gamma^{5}, \gamma^{\mu}\} = 0,$$

$$\left(\gamma^{\mu}a_{\mu}\right)^2 = -I\left[\mathbf{a}^2 - (a^0)^2\right], \quad \left(\boldsymbol{\gamma}\cdot\mathbf{a}\right)^2 = -I\,\mathbf{a}^2,$$

$$\mathbf{a} = \left(a_1, a_2, a_3\right), \quad a_0 = -a^0, \quad a_i = a^i, \quad i = 1, 2, 3.$$

Now we introduce various representations of the gamma matrices in 4D.

◇ Dirac representation:

$$\gamma^{0} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} \mathbf{0} & \sigma^{i} \\ -\sigma^{i} & \mathbf{0} \end{pmatrix}, \; i = 1, 2, 3, \quad \gamma^{5} \equiv \mathrm{i}\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

◇ Chiral Representation:

$$\gamma^{0} = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} \mathbf{0} & \sigma^{i} \\ -\sigma^{i} & \mathbf{0} \end{pmatrix}, \; i = 1, 2, 3, \quad \gamma^{5} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

◊ Majorana representation:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} \mathrm{i}\sigma^3 & 0 \\ 0 & \mathrm{i}\sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} -\mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}.$$

Two Dimensional Spacetime

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Ten Dimensional Spacetime

$$\gamma^0 = \begin{pmatrix} 0 & -\mathrm{i}I_{16} \\ \mathrm{i}I_{16} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\Gamma^i \\ -\Gamma^i & 0 \end{pmatrix}, \; i = 1, \ldots, 9,$$

$$\gamma_{\mathrm{C}} = (-\mathrm{i})\gamma^0\gamma^1 \ldots \gamma^9 = \begin{pmatrix} I_{16} & 0 \\ 0 & -I_{16} \end{pmatrix},$$

where

$$\Gamma^1 = \mathrm{i}\,\sigma^2 \times \sigma^2 \times \sigma^2 \times \sigma^2, \qquad \Gamma^2 = \mathrm{i}\,I \times \sigma^1 \times \sigma^2 \times \sigma^2,$$
$$\Gamma^3 = \mathrm{i}\,I \times \sigma^3 \times \sigma^2 \times \sigma^2, \qquad \Gamma^4 = \mathrm{i}\,\sigma^1 \times \sigma^2 \times I \times \sigma^2,$$
$$\Gamma^5 = \mathrm{i}\,\sigma^3 \times \sigma^2 \times I \times \sigma^2, \qquad \Gamma^6 = \mathrm{i}\,\sigma^2 \times I \times \sigma^1 \times \sigma^2,$$
$$\Gamma^7 = \mathrm{i}\,\sigma^2 \times I \times \sigma^3 \times \sigma^2, \qquad \Gamma^8 = \mathrm{i}\,I \times I \times I \times \sigma^1, \qquad \Gamma^9 = \mathrm{i}\,I \times I \times I \times \sigma^3.$$

Appendix II
Some Basic Fields in 4D

Under a homogeneous Lorentz transformation,

$$x'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}, \quad \partial'_{\mu} = \Lambda_{\mu}{}^{\nu}\, \partial_{\nu}, \tag{II.1}$$

$$\Lambda^{\mu}{}_{\rho}\, \eta_{\mu\nu}\, \Lambda^{\nu}{}_{\lambda} = \eta_{\rho\lambda}, \quad (\Lambda^{-1})^{\nu}{}_{\mu} = \Lambda_{\mu}{}^{\nu}, \quad [\eta_{\rho\lambda}] = \mathrm{diag}\,[-1, 1, 1, 1\,]. \tag{II.2}$$

We note that the first equality in (II.2) rewritten in matrix form reads

$$\Lambda^{\top}\, \eta\, \Lambda = \eta, \tag{II.3}$$

and since $\det \eta = -1$, we may infer that for the transformations in (II.1) that

$$\det\, \Lambda = +1. \tag{II.4}$$

It is not -1 as this transformation includes neither time nor space reflections. From this and the fact that $\partial x'^{\mu}/\partial x^{\nu} = \Lambda^{\mu}{}_{\nu}$, we learn that the Jacobian of the transformation $x' \to x$ is given by $\det \Lambda = 1$. This establishes the Lorentz invariance of the volume element in Minkowski spacetime:

$$(\mathrm{d}x) = (\mathrm{d}x'), \qquad \text{where } (\mathrm{d}x) \equiv \mathrm{d}x^{0}\, \mathrm{d}x^{1}\, \mathrm{d}x^{2}\, \mathrm{d}x^{3}. \tag{II.5}$$

This together with the definition of a Lorentz scalar $\Phi(x)$ by the condition

$$\Phi'(x') = \Phi(x), \tag{II.6}$$

under the above Lorentz transformations, guarantees the invariance of integrals of the form

$$\mathscr{A} = \int (\mathrm{d}x)\, \Phi(x), \tag{II.7}$$

E.B. Manoukian, *Quantum Field Theory II*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-33852-1

such as the action integral, and lead to the development of Lorentz invariant theories. In Chap. 2, we have the more ambitious programme of developing supersymmetric invariant actions involving superfields.

Under an infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}, \quad \delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}. \tag{II.8}$$

For explicit expressions of $\Lambda^{\mu}{}_{\nu}$ and $\delta\omega^{\nu\mu}$, see, e.g., [1], Chap. 16.

The Langrangian density of a Hermitian scalar field interacting with an external source K is given by

$$\mathscr{L}(x) = -\frac{1}{2}\,\partial_{\mu}\varphi(x)\partial^{\mu}\varphi(x) - \frac{m^2}{2}\,\varphi^2(x) + K(x)\varphi(x), \tag{II.9}$$

and the vacuum-to-vacuum transition amplitude is given by

$$\langle\, 0_+ \,|\, 0_- \rangle = \exp\Big[\,\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x\,')K(x)\Delta_+(x-x\,')K(x\,')\Big], \tag{II.10}$$

from which the propagator is obtained by functional differentiations as follows:

$$\frac{\delta}{\delta K(x\,')}(-\mathrm{i})\frac{\delta}{\delta K(x)}\langle\, 0_+ \,|\, 0_- \rangle\Big|_{K=0} = \Delta_+(x-x\,') = \mathrm{i}\,\langle\, 0 \,|\, \big(\varphi(x)\varphi^{\dagger}(x\,')\big)_+ \,|\, 0\rangle, \tag{II.11}$$

where

$$\Delta_+(x-x\,') = \int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p\,(x-x')}}{p^2+m^2-\mathrm{i}\,\epsilon}, \tag{II.12}$$

and we have used the notation $\langle\, 0\,|\,.\,|\,0\,\rangle$ for $\langle\, 0_+|\,.\,|0_-\rangle$ for $K=0$.

Of particular interest is the functional (path) integral expression in terms of classical fields given by

$$\int(\mathscr{D}\phi)\exp\mathrm{i}\Big[\int(\mathrm{d}x)(\mathrm{d}x\,')\big(-\frac{1}{2}\,\phi(x)M(x,x\,')\phi(x\,')\big) + \int(\mathrm{d}x)\,\phi(x)K(x)\Big] \tag{II.13}$$

$$= \langle\, 0_+ \,|\, 0_- \rangle \int(\mathscr{D}\phi)\exp\mathrm{i}\Big[\int(\mathrm{d}x)(\mathrm{d}x\,')\big(-\frac{1}{2}\,\phi(x)M(x,x\,')\phi(x\,')\big)\Big], \tag{II.14}$$

$$M(x,x\,') = [-\Box + m^2\,]\,\delta^{(4)}(x-x\,'), \tag{II.15}$$

$$M^{-1}(x,x\,') = \frac{1}{[-\Box + m^2 - \mathrm{i}\,\epsilon]}\,\delta^{(4)}(x-x\,') = \Delta_+(x-x\,'), \tag{II.16}$$

$$\int M(x,x')\,(\mathrm{d}x')\,M^{-1}(x',y) = \delta^{(4)}(x-y). \tag{II.17}$$

From (II.13)–(II.17), we may, in a compact matrix notation in spacetime variables, write the useful expression

$$\frac{1}{\sqrt{\det M}}\exp\Big[\frac{\mathrm{i}}{2}\,K\,M^{-1}K\Big] = \int(\mathscr{D}\phi)\exp\Big[\,\mathrm{i}\Big(-\frac{1}{2}\,\phi M\phi + \phi K\Big)\Big]. \tag{II.18}$$

For spin 1/2, the Dirac equation is given by

$$\Big[\frac{\gamma^{\mu}\partial_{\mu}}{\mathrm{i}} + m\Big]\psi(x) = 0, \tag{II.19}$$

which may be obtained from the Lagrangian density

$$\mathscr{L}(x) = -\frac{1}{2\,\mathrm{i}}\Big(\overline{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x) - (\partial^{\mu}\overline{\psi})(x)\psi(x)\Big) - m\overline{\psi}(x)\psi(x). \tag{II.20}$$

Under a homogeneous Lorentz transformation, which may include a 3D rotation, the Dirac equation reads

$$\Big[\frac{\gamma^{\nu}\partial'_{\nu}}{\mathrm{i}} + m\Big]K\psi(x) = 0, \qquad K\,\psi(x) = \psi\,'(x'), \quad x' = \Lambda x, \tag{II.21}$$

where the matrix K satisfies the relations

$$K^{\dagger}\,\gamma^{0}\,K = \gamma^{0}, \qquad \Lambda^{\mu}{}_{\nu}\gamma^{\nu} = K^{-1}\,\gamma^{\mu}\,K. \tag{II.22}$$

For infinitesimal transformations, as given in (II.8), we may set

$$K \;\simeq\; I + \frac{\mathrm{i}}{2}\,\delta\omega^{\mu\nu}\,S_{\mu\nu}, \tag{II.23}$$

where $S_{\mu\nu}$ is to be determined. By substituting this expression in (II.22), we obtain

$$[\,S^{\lambda\mu},\gamma^{\nu}\,] \;=\mathrm{i}\,\left(\eta^{\lambda\nu}\gamma^{\mu} - \eta^{\mu\nu}\gamma^{\lambda}\right), \tag{II.24}$$

whose solution is $S^{\lambda\mu} = \mathrm{i}\,[\,\gamma^{\lambda},\gamma^{\mu}\,]/4$, and we may write

$$K \;\simeq\; I + \frac{\mathrm{i}}{2}\,\delta\omega^{\mu\nu}\,S_{\mu\nu}, \quad S^{\lambda\mu} = \frac{\mathrm{i}}{4}\,[\,\gamma^{\lambda},\gamma^{\mu}\,]. \tag{II.25}$$

In the presence of an external electromagnetic field, the Dirac equation reads

$$\Big[\gamma^\mu\Big(\frac{\partial_\mu}{\mathrm{i}} - \mathrm{e}A_\mu(x)\Big) + m\Big]\psi(x) = 0, \tag{II.26}$$

from which one obtains the equation ($\overline{\psi} = \psi^\dagger\gamma^0$) for the charge conjugate field $\psi^{\mathscr{C}}$:

$$\Big[\gamma^\mu\Big(\frac{\partial_\mu}{\mathrm{i}} + \mathrm{e}A_\mu(x)\Big) + m\Big]\psi^{\mathscr{C}}(x) = 0, \quad \psi^{\mathscr{C}}(x) = \mathscr{C}\,\overline{\psi}^\top(x), \tag{II.27}$$

with opposite sign of the charge e, where

$$\mathscr{C} = \mathrm{i}\gamma^2\gamma^0. \tag{II.28}$$

The Lagrangian density of a (Hermitian) massless vector (the photon), may be taken as

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu. \tag{II.29}$$

For a photon with momentum k^μ, $k^2 = 0$, we may introduce polarization vectors $\mathrm{e}^\mu_\lambda = (0, \mathbf{e}_\lambda)$, with $\lambda = 1, 2$, such that $k_\mu \mathrm{e}^\mu_\lambda = 0$, $\mathrm{e}^{\mu *}_\lambda \mathrm{e}_{\lambda'\mu} = \delta_{\lambda\lambda'}$, satisfying the completeness relation

$$\eta^{\mu\nu} = \frac{k^\mu \underline{k}^\nu + \underline{k}^\mu k^\nu}{k\,\underline{k}} + \sum_{\lambda=\pm 1} \mathrm{e}^\mu_\lambda \mathrm{e}^{\nu *}_\lambda, \qquad k = (k^0, \mathbf{k}),\ \underline{k} = (k^0, -\mathbf{k}) \tag{II.30}$$

For completeness we also give the Lagrangian densities of massless spin 3/2 field (the gravitino), as well as of a (Hermitian) spin 2 field (the graviton).

For the 3/2 field, we may take ($\gamma\partial \equiv \gamma^\mu\partial_\mu$, $\overleftrightarrow{\partial} \equiv \overrightarrow{\partial} - \overleftarrow{\partial}$)

$$\mathscr{L} = -\frac{1}{2\mathrm{i}}\overline{\psi}_\mu\Big(\eta^{\mu\nu}\gamma\overleftrightarrow{\partial} - \big(\gamma^\mu\overleftrightarrow{\partial}^\nu + \gamma^\nu\overleftrightarrow{\partial}^\mu\big) + \gamma^\mu(-\gamma\overleftrightarrow{\partial})\gamma^\nu\Big)\psi_\nu, \tag{II.31}$$

while for the spin 2, we may take

$$\mathscr{L} = -\frac{1}{2}\partial^\sigma U_{\mu\nu}\,\partial_\sigma U^{\mu\nu} + \partial_\mu U^{\mu\nu}\,\partial_\sigma U^\sigma{}_\nu - \partial_\sigma U^{\sigma\mu}\,\partial_\mu U + \frac{1}{2}\partial^\mu U\,\partial_\mu U, \tag{II.32}$$

where $U = U^\mu{}_\mu$.[1]

[1] For details on all of the above see Sect. 4.7 of Vol. I [2].

References

1. Manoukian, E. B. (2006). *Quantum theory: A wide spectrum*. AA Dordrecht: Springer.
2. Manoukian, E. B. (2016). *Quantum field theory I: Foundations and abelian and non-abelian gauge theories*. Dordrecht: Springer.

Solutions to the Problems

Chapter 1

1.1. For the Minkowski metric $\eta_{\alpha\beta}$, $\partial_\nu \eta_{\alpha\beta} = 0$, $\eta_{\alpha\beta} = e_\alpha{}^\mu \, e_{\beta\mu}$. Also, by mere relabeling of dummy indices, we may write

$$\Gamma_{\nu\sigma}{}^\mu \, e_\alpha{}^\sigma \, e_{\beta\mu} = \Gamma_{\nu\mu}{}^\sigma \, e_\alpha{}^\mu \, e_{\beta\sigma}.$$

Accordingly, we obviously have

$$\partial_\nu (e_\alpha{}^\mu e_{\beta\mu}) + \Gamma_{\nu\sigma}{}^\mu \, e_\alpha{}^\sigma e_{\beta\mu} - \Gamma_{\nu\mu}{}^\sigma \, e_\alpha{}^\mu \, e_{\beta\sigma} = 0.$$

Upon expanding the latter equation we obtain

$$e_{\beta\mu}(\partial_\nu e_\alpha{}^\mu + \Gamma_{\nu\sigma}{}^\mu e_\alpha{}^\sigma) = -\, e_\alpha{}^\mu (\partial_\nu e_{\beta\mu} - \Gamma_{\nu\mu}{}^\sigma e_{\beta\sigma}),$$

or

$$e_{\beta\mu} \nabla_\nu \, e_\alpha{}^\mu = -\, e_\alpha{}^\mu \, \nabla_\nu \, e_{\beta\mu}.$$

The equality in question then follows upon rewriting the right-hand side of the above equation as

$$-\, e_\alpha{}^\mu \, \nabla_\nu \, e_{\beta\mu} = -e_\alpha{}^\mu \, \nabla_\nu \, e_\beta{}^\gamma \, g_{\gamma\,\mu} = -e_{\alpha\gamma} \, \nabla_\nu \, e_\beta{}^\gamma \equiv -e_{\alpha\mu} \, \nabla_\nu \, e_\beta{}^\mu,$$

using the fact that $\nabla_\nu g_{\gamma\mu} = 0$. Equivalently, we may write $\nabla_\nu \, e_\alpha{}^\mu \, e_{\beta\mu} = 0$.

1.2. (i) This is a special case of (1.1.40) obtained by contracting μ_1 and μ in the latter. We provide, however, a direct demonstration of this. Using the facts that $\nabla_\beta \, \xi^\alpha$ is a mixed tensor, $\nabla_\alpha \, \xi^\alpha$ is a scalar, and that $\Gamma_{\sigma\lambda}{}^\kappa$ is

E.B. Manoukian, *Quantum Field Theory II*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-33852-1

symmetric in (σ, λ), give

$$\begin{aligned}\nabla_\alpha (\nabla_\beta \xi^\alpha) &= \partial_\alpha \partial_\beta \xi^\alpha + (\partial_\alpha \Gamma_{\sigma\beta}{}^\alpha) \xi^\sigma + \Gamma_{\beta\sigma}{}^\lambda \partial_\lambda \xi^\sigma - \Gamma_{\beta\sigma}{}^\lambda \partial_\lambda \xi^\sigma \\ &\quad - \Gamma_{\beta\alpha}{}^\lambda \Gamma_{\sigma\lambda}{}^\alpha \xi^\sigma + \Gamma_{\alpha\sigma}{}^\alpha \partial_\beta \xi^\sigma + \Gamma_{\alpha\lambda}{}^\alpha \Gamma_{\sigma\beta}{}^\lambda \xi^\sigma, \\ \nabla_\beta (\nabla_\alpha \xi^\alpha) &= \partial_\beta \partial_\alpha \xi^\alpha + (\partial_\beta \Gamma_{\sigma\alpha}{}^\alpha) \xi^\sigma + \Gamma_{\alpha\sigma}{}^\alpha \partial_\beta \xi^\sigma,\end{aligned}$$

which upon subtraction gives the identity in (i).

(ii) $\nabla_\nu h^{\alpha\beta}$ is a mixed tensor field, $\nabla_\alpha h^{\alpha\beta}$ is a vector field. In a local Lorentz coordinate system at the point x in question, the connection vanishes but not its derivatives. Accordingly, we may write in the latter coordinate system:

$$\begin{aligned}\nabla_\alpha \nabla_\nu h^{\alpha\beta} &\to \partial_\alpha \partial_\nu h^{\alpha\beta} + \partial_\alpha \Big(\Gamma_{\sigma\nu}{}^\alpha h^{\sigma\beta} + \Gamma_{\nu\sigma}{}^\beta h^{\alpha\sigma} \Big) \\ &= \partial_\alpha \partial_\nu h^{\alpha\beta} + \partial_\alpha \Gamma_{\sigma\nu}{}^\alpha \, h^{\sigma\beta} + \partial_\alpha \Gamma_{\nu\sigma}{}^\beta \, h^{\alpha\sigma}, \\ \nabla_\nu \nabla_\alpha h^{\alpha\beta} &\to \partial_\nu \partial_\alpha h^{\alpha\beta} + \partial_\nu \Big(\Gamma_{\sigma\alpha}{}^\alpha h^{\sigma\beta} + \Gamma_{\alpha\sigma}{}^\beta h^{\alpha\sigma} \Big), \\ &= \partial_\nu \partial_\alpha h^{\alpha\beta} + + \partial_\nu \Gamma_{\sigma\alpha}{}^\alpha \, h^{\sigma\beta} + \partial_\nu \Gamma_{\alpha\sigma}{}^\beta \, h^{\alpha\sigma},\end{aligned}$$

from which

$$[\nabla_\alpha, \nabla_\nu] \, h^{\alpha\beta} \to \Big(\partial_\alpha \Gamma_{\sigma\nu}{}^\alpha - \partial_\nu \Gamma_{\sigma\alpha}{}^\alpha \Big) h^{\sigma\beta} + \Big(\partial_\alpha \Gamma_{\sigma\nu}{}^\beta - \partial_\nu \Gamma_{\alpha\sigma}{}^\beta \Big) h^{\alpha\sigma}.$$

From (1.1.50), (1.1.39), we recognize the coefficient of $h^{\sigma\beta}$ in the first term on the right-hand side, as $R_{\nu\sigma}$, and the coefficient of $h^{\alpha\sigma}$ as $R^\beta{}_{\sigma\alpha\nu}$ at the point x in such a coordinate system. As a tensor equation the statement of the problem holds true in a general coordinate system as well.

1.3. (i) According (A-1.9), for a matrix A: $\delta \det A = \det A \, \mathrm{Tr}[(A^{-1})\delta A]$. Hence for $A = [g_{\mu\nu}]$, $\delta g = g g^{\alpha\beta} \delta g_{\alpha\beta}$, since $[g^{\alpha\beta}]$ is the inverse of $[g_{\mu\nu}]$. The first equality then follows by multiplying the latter by minus, and noting that $\delta\sqrt{-g} = (1/2)\delta(-g)/\sqrt{-g}$. On the other hand $g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha{}_\beta$ which upon taking the variation of the latter gives, in particular, $g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta}$, implying the second equality.

(ii) The first equality in (i) above implies the first equality in (ii). On the other hand, we explicitly have $\Gamma_{\mu\sigma}{}^\sigma = (1/2) g^{\alpha\beta} \partial_\mu g_{\alpha\beta}$ which gives the second equality.

1.4. (i) The vanishing of the covariant derivative of the (inverse of the) metric $\nabla_\mu g^{\alpha\beta} = 0$, gives $\Gamma_{\mu\sigma}{}^\alpha g^{\sigma\beta} = -(\partial_\mu g^{\alpha\beta} + \Gamma_{\mu\sigma}{}^\beta g^{\sigma\alpha})$. Upon setting $\beta = \mu$, summing over it, and using part (ii) in Problem 1.3 establish the equality. Part (ii) is the content of part (ii) in Problem 1.3.

1.5. (i) $\sqrt{-g}\,\nabla_\mu \xi^\mu = \sqrt{-g}\,(\partial_\mu \xi^\mu + \Gamma_{\mu\sigma}{}^{\mu}\xi^\sigma)$ and the result follows upon using part (ii) of Problem 1.4.

(ii) Recall that $\partial_\nu \phi$ is a vector field, we may write $\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \sqrt{-g}\,\partial_\mu\phi\partial^\mu\phi$. Partial integration implies that the latter is equivalent to $-(\partial_\mu\sqrt{-g})\phi\,\partial^\mu\phi - \sqrt{-g}\,\phi\partial_\mu\partial^\mu\phi$. The result again follows from part (ii) of Problem 1.3, and the expression of the covariant derivative of a vector field.

(iii) The results follows from the application of part (i) of Problem 1.3 for $\sqrt{-g}$ followed by the variation of $g^{\mu\nu}$ (the coefficient of $\partial_\mu\phi\partial_\nu\phi$), keeping ϕ fixed as required.

1.6. From the just mentioned equation, the following transformation law of $\delta\Gamma_{\varrho\,\nu}{}^{\gamma}$ follows to be

$$\delta\,\Gamma'_{\varrho\,\nu}{}^{\gamma} = \frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\,\frac{\partial x^{\mu}}{\partial x'^{\varrho}}\,\frac{\partial x^{\kappa}}{\partial x'^{\nu}}\,\delta\,\Gamma_{\mu\,\kappa}{}^{\sigma},$$

which is the transformation rule of a tensor.

1.7. (i) The variation of the Riemann tensor is explicitly given by

$$\delta R^{\mu}{}_{\nu\,\rho\sigma} = \partial_\rho \delta\Gamma_{\nu\sigma}{}^{\mu} - \partial_\sigma\,\delta\Gamma_{\nu\rho}{}^{\mu} + \delta\Gamma_{\rho\kappa}{}^{\mu}\Gamma_{\nu\sigma}{}^{\kappa} + \Gamma_{\rho\kappa}{}^{\mu}\delta\Gamma_{\nu\sigma}{}^{\kappa}$$
$$-\delta\Gamma_{\sigma\kappa}{}^{\mu}\Gamma_{\nu\rho}{}^{\kappa} - \Gamma_{\sigma\kappa}{}^{\mu}\delta\Gamma_{\nu\rho}{}^{\kappa}.$$

In a local lorentz coordinate system, the connection vanishes *locally*, but *not* its variation since the latter is a tensor. Accordingly, in a local Lorentz coordinate system, at the point x in question, we may write $\delta R^{\mu}{}_{\nu\,\rho\sigma} = \partial_\rho\,\delta\Gamma_{\nu\sigma}{}^{\mu} - \partial_\sigma\,\delta\Gamma_{\nu\,\rho}{}^{\mu}$. Since $\delta\Gamma_{\nu\sigma}{}^{\mu}$, $\delta\Gamma_{\nu\,\rho}{}^{\mu}$ are tensors, a tensorial equation holding in every coordinate is obtained by replacing the partial derivatives by their covariant counterparts. This gives (i).

Part (ii) follows by writing $R_{\mu\nu\rho\sigma} = g_{\mu\lambda}R^{\lambda}{}_{\nu\rho\sigma}$ and using part (i).

Part (iii) is obtained by making a contraction between ρ and μ in (i). In a local Lorentz coordinate system, the first derivative of the metric is, locally, zero at the point in question, and we may write

$$\delta\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu\delta g_{\nu\sigma} + \partial_\nu\delta g_{\mu\sigma} - \partial_\sigma\delta g_{\mu\nu}).$$

Accordingly, in every coordinate system, $\delta\Gamma_{\mu\nu}{}^{\rho}$ has the structure as stated in the problem with covariant derivatives replacing partial derivatives, thus establishing part (iv).

1.8. Upon using the expression for $\delta R_{\mu\nu}$ in Problem 1.7, the expression for $g^{\mu\nu}\Gamma_{\mu\nu}{}^{\sigma}$ in part (i) of Problem 1.4, and the definition of a covariant derivative, we get

$$\sqrt{-g}\; g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{-g}\; g^{\mu\nu}\big(\partial_\rho \delta\Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\rho\mu}{}^{\sigma}\delta\Gamma_{\sigma\nu}{}^{\rho} - \Gamma_{\rho\nu}{}^{\sigma}\delta\Gamma_{\sigma\mu}{}^{\rho} \\ + \Gamma_{\rho\sigma}{}^{\rho}\delta\Gamma_{\mu\nu}{}^{\sigma}\big) - \sqrt{-g}\; g^{\mu\nu}\,\partial_\nu\delta\Gamma_{\mu\rho}{}^{\rho} - \partial_\nu(\sqrt{-g}\; g^{\mu\nu})\delta\Gamma_{\mu\rho}{}^{\rho}.$$

The last two terms constitute the total derivative $-\partial_\nu(\sqrt{-g}\; g^{\mu\nu}\delta\Gamma_{\mu\rho}{}^{\rho})$. On the other hand, by partial integration the sum of the remaining terms is equivalent to

$$-\sqrt{-g}\;(\partial_\rho g^{\nu\sigma} + \Gamma_{\rho\mu}{}^{\sigma}g^{\mu\nu} + \Gamma_{\rho\mu}{}^{\sigma}g^{\mu\nu})\,\delta\Gamma_{\nu\sigma}{}^{\rho} \equiv -\sqrt{-g}\;(\nabla_\rho g^{\nu\sigma})\,\delta\Gamma_{\nu\sigma}{}^{\rho},$$

which vanishes, since $\nabla_\rho g^{\mu\sigma} = 0$ [see (1.1.46)].

1.9. Using part (i) of Problems 1.3, and 1.8, we have

$$\delta(\sqrt{-g}\; g^{\mu\nu}R_{\mu\nu}) = \sqrt{-g}\Big(-\frac{1}{2}R\, g_{\mu\nu}\,\delta g^{\mu\nu} + R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\Big) \\ = \sqrt{-g}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R\Big)\delta g^{\mu\nu}.$$

1.10. (i) Note that: $\nabla_\mu T^{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n}$
$= \partial_\mu T^{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n} + \Gamma_{\mu\sigma}{}^{\mu_1}T^{\sigma\mu_2\ldots\mu_n} + \cdots + \Gamma_{\mu\sigma}{}^{\mu_n}T^{\mu_1\mu_2\ldots\sigma}$.
On the other hand,

$$\sqrt{-g}\,\partial_\mu T^{\mu_1\ldots\mu_n}S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n} = -\,(\sqrt{-g}\,\Gamma_{\mu\kappa}{}^{\kappa})T^{\mu_1\ldots\mu_n}S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n} \\ -\sqrt{-g}\,T^{\mu_1\ldots\mu_n}\partial_\mu S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n},\ (*)$$

up to a total derivative, where we have used part (ii) of Problem 1.3. By mere relabeling of dummy indices, one may also write

$$\Gamma_{\mu\kappa}{}^{\kappa}\, T^{\mu_1\ldots\mu_n}S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n} = \Gamma_{\mu\sigma}{}^{\mu}\, T^{\mu_1\ldots\mu_n}S_{\mu_1\ldots\mu_k}{}^{\sigma}{}_{\mu_{k+1}\ldots\mu_n},$$

in reference to the first term on the right-hand side of (*) above, also

$$\Gamma_{\mu\sigma}{}^{\mu_1}T^{\sigma\mu_2\ldots\mu_n}S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n} = T^{\mu_1\mu_2\ldots\mu_n}\Gamma_{\mu\mu_1}{}^{\sigma}S_{\sigma\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n},$$

in reference to the second term on the right-hand side of the expression for $\nabla_\mu T^{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n}$ above, and similarly for the other indices

$\mu_2, \ldots, \mu_n$. Accordingly, the left-hand side of (i) in the statement of the problem, may be rewritten, up to a total derivative, as

$$- \sqrt{-g}\, T^{\mu_1\ldots\mu_n}(\partial_\mu S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n} - \cdots + \Gamma_{\mu\sigma}{}^{\mu} S_{\mu_1\ldots\mu_k}{}^{\sigma}{}_{\mu_{k+1}\ldots\mu_n}$$
$$- \Gamma_{\mu\mu_n}{}^{\sigma} S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_{n-1}\sigma}) = -\sqrt{-g}\, T^{\mu_1\ldots\mu_n} \nabla_\mu S_{\mu_1\ldots\mu_k}{}^{\mu}{}_{\mu_{k+1}\ldots\mu_n},$$

establishing the first equality.

(ii) The derivation of the second one is essentially the same except the following term $\sqrt{-g}\, \Gamma_{\mu\sigma}{}^{\sigma} T^{\mu_1\ldots\mu_k\mu\mu_{k+1}\ldots\mu_n} S_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n}$ cancels out in the process, as is easily verified upon partial integration. Part (iii) is a special case of (i).

1.11. From part (iii) of Problem 1.10, we may rewrite the integrand of the action W_{matter} as: $-(1/2)\, \sqrt{-g}\, (-\phi\, g^{\mu\nu} \nabla_\mu(\partial_\nu \phi) + m^2\phi^2)$, leading to the field equation

$$(\Box - m^2)\, \phi = 0, \qquad \Box\, \phi = g^{\mu\nu} \nabla_\mu (\partial_\nu \phi) \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu\, \phi.$$

(ii) From part (iii) of Problem 1.5, we obtain

$$T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi - \frac{1}{2}\, g_{\mu\nu} (\partial^\rho \phi\, \partial_\rho \phi + m^2 \phi^2).$$

(iii) With suitable relabeling of dummy indices, we explicitly have

$$\nabla_\mu T^{\mu\nu} = (\Box\, \phi - m^2 \phi)\, \partial^\nu \phi + g^{\sigma\nu} (\nabla_\mu \partial_\sigma \phi - \nabla_\sigma \partial_\mu \phi)\, \partial^\mu \phi.$$

But $\nabla_\mu \partial_\sigma \phi = \partial_\mu \partial_\sigma \phi - \Gamma_{\mu\sigma}{}^{\lambda} \partial_\lambda \phi = \nabla_\sigma \partial_\mu \phi$, leading to

$$\nabla_\mu T^{\mu\nu} = (\Box\, \phi - m^2 \phi)\, \partial^\nu \phi,$$

which vanishes as a consequence of the field equation. Due to symmetry in (μ, ν), this also holds for $\nabla_\nu T^{\mu\nu} = 0$.

1.12. $\delta \mathscr{L}$ is worked out to be

$$\big(\partial^\mu \Lambda^\nu + \partial^\nu \Lambda^\mu - \eta^{\mu\nu} \partial.\Lambda\big)\big(\partial_\mu \Gamma_{\rho\nu}{}^{\rho} - \partial_\rho \Gamma_{\mu\nu}{}^{\rho}\big) + \xi^{\mu\nu}\big(\partial_\mu \partial_\rho \partial_\nu \Lambda^\rho - \partial_\rho \partial_\mu \partial_\nu \Lambda^\rho\big)$$
$$+ \eta^{\mu\nu}\big(\Gamma_{\mu\nu}{}^{\lambda} \partial_\lambda \partial_\rho \Lambda^\rho + \Gamma_{\lambda\rho}{}^{\rho} \partial_\mu \partial_\nu \Lambda^\lambda - \Gamma_{\rho\nu}{}^{\lambda} \partial_\lambda \partial_\mu \Lambda^\rho - \Gamma_{\lambda\mu}{}^{\rho} \partial_\rho \partial_\nu \Lambda^\lambda\big)$$
$$= -\big(\partial^\mu \Gamma_{\mu\lambda}{}^{\lambda} - \partial^\lambda \Gamma_{\mu}{}^{\mu}{}_{\lambda}\big) \partial_\rho \Lambda^\rho + \big(\partial^\mu \Gamma_{\mu\lambda}{}^{\lambda} - \partial^\lambda \Gamma_{\mu}{}^{\mu}{}_{\lambda}\big) \partial_\rho \Lambda^\rho = 0,$$

up to total derivatives, after cancelation of various terms.

1.13. Let us start from the right-hand side of (1.4.4) which in detail reads

$$\big(g_{\mu\lambda} + h_{\mu\lambda}\big) \partial_\nu \Lambda^\lambda + \big(g_{\nu\lambda} + h_{\nu\lambda}\big) \partial_\mu \Lambda^\lambda$$
$$+ \Lambda^\lambda \Big(\partial_\lambda h_{\mu\nu} + (g_{\mu\sigma} + h_{\mu\sigma}) \Gamma_{\nu\lambda}{}^{\sigma} + (g_{\nu\sigma} + h_{\nu\sigma}) \Gamma_{\mu\lambda}{}^{\sigma} - h_{\sigma\nu} \Gamma_{\mu\lambda}{}^{\sigma} - h_{\sigma\mu} \Gamma_{\nu\lambda}{}^{\sigma}\Big).$$

Note that the terms involving $h_{..}\Gamma_{..}{}^{\cdot}$ in the second line cancel out, while the corresponding ones with $g_{..}$ give $\partial_\lambda g_{\mu\nu}$ on account of the fact that $\nabla_\lambda g_{\mu\nu} = 0$. This establishes the statement of the problem. A more direct way of seeing this is that (1.4.3) as a tensor equation and must hold with the partial derivatives in it replaced by covariant ones. After having done this, make use of the fact that $\nabla_\lambda g_{\mu\nu} = 0$ to obtain (1.4.4).

1.14. $\Gamma_{\mu\nu}{}^{\sigma} = (1/2)g^{\sigma\lambda}\big(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\big)$. Now for $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, $\delta g^{\mu\nu} = -h^{\mu\nu} + h^{\mu\lambda}\, h_\lambda{}^{\nu}$, with the latter given in (1.4.6),

$$\delta\Gamma_{\mu\nu}{}^{\sigma} = -\frac{1}{2}\big(h^{\sigma\lambda} - h^{\sigma\rho}h_\rho{}^{\lambda}\big)\big(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\big)$$
$$+\frac{1}{2}g^{\sigma\lambda}\big(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\big) - \frac{1}{2}\,h^{\sigma\lambda}\big(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\big).$$

Although $\Gamma_{\mu\nu}{}^{\sigma}$ is not a tensor, $\delta\Gamma_{\mu\nu}{}^{\sigma}$ is a tensor (see Problem 1.6). Accordingly, in a local Lorentz coordinate system, at the point in question, $\partial_\lambda g_{\mu\nu} = 0$. Hence, in a general coordinate system we may simply replace partial derivatives by covariant ones to obtain

$$\delta\Gamma_{\mu\nu}{}^{\sigma} = \frac{1}{2}\,\big(\nabla_\mu h_\nu{}^{\sigma} + \nabla_\nu h_\mu{}^{\sigma} - \nabla^\sigma h_{\mu\nu}\big) - \frac{1}{2}\,h^{\sigma\lambda}\big(\nabla_\mu h_{\nu\lambda} + \nabla_\nu h_{\mu\lambda} - \nabla_\lambda h_{\mu\nu}\big).$$

1.15. For a given function f: $\delta(1/f) = -(1/f^2)\,\delta f$. Hence from Problem 1.3,(i), $\delta(1/\sqrt{-g}) = (1/2\sqrt{-g})g_{\rho\kappa}\delta g^{\rho\kappa}$, from which the statement of the problem follows.

1.16. With $\eta^{\mu\nu\lambda\sigma} = \varepsilon^{\mu\nu\lambda\sigma}/\sqrt{-g}$, $\eta_{\mu\nu\lambda\sigma} = \sqrt{-g}\,\varepsilon_{\mu\nu\lambda\sigma}$, we have the identity

$$\eta^{\mu\nu\rho\sigma}\eta_{\lambda\kappa\gamma\,\epsilon} = -\sum_{\mathrm{P}[\mu\nu\rho\sigma]} \mathrm{sgn}_{\mathrm{P}}\,\delta^{\mu}{}_{\lambda}\,\delta^{\nu}{}_{\kappa}\,\delta^{\rho}{}_{\gamma}\,\delta^{\sigma}{}_{\epsilon},$$

where $\sum_{\mathrm{P}[\mu\nu\rho\sigma]}\mathrm{sgn}_{\mathrm{P}}$ stands for a summation over all permutations of $\{\mu\nu\rho\sigma\}$ with corresponding signs attached. Upon writing

$$R_{\mu\nu\rho\sigma}R_{\lambda\kappa\gamma\,\epsilon}\,\eta^{\mu\nu\lambda\kappa}\,\eta^{\rho\sigma\gamma\,\epsilon} = R_{\mu\nu}{}^{\rho\sigma}R_{\lambda\kappa}{}^{\gamma\,\epsilon}\eta^{\mu\nu\lambda\kappa}\,\eta_{\rho\sigma\gamma\,\epsilon},$$

the previous identity leads to the expression given in the problem.

1.17. The Bianchi identity (1.1.44) reads

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\rho R_{\mu\nu\sigma\lambda} + \nabla_\sigma R_{\mu\nu\lambda\rho} = 0.$$

Upon multiplying the latter by $\eta^{\rho\sigma\alpha\lambda}$ and conveniently relabeling the indices gives:

$$0 = [\eta^{\rho\sigma\alpha\lambda} + \eta^{\lambda\rho\alpha\sigma} + \eta^{\sigma\lambda\alpha\rho}]\nabla_\lambda R_{\mu\nu\rho\sigma} = 3\,\eta^{\rho\sigma\alpha\lambda}\nabla_\lambda R_{\mu\nu\rho\sigma},$$

by finally using the totally anti-symmetric nature of $\eta^{\rho\sigma\alpha\lambda}$.

1.18. Since no indices in $\eta^{\mu\nu\alpha\beta}$, due to its complete antisymmetry, may be equal, it is sufficient to establish the above result for η^{0123}. In detail

$$\nabla_\lambda \eta^{0123} = \varepsilon^{0123}\partial_\lambda \frac{1}{\sqrt{-g}} + \Gamma_{\lambda 0}{}^{0}\eta^{0123} + \Gamma_{\lambda 1}{}^{1}\eta^{0123} + \Gamma_{\lambda 2}{}^{2}\eta^{0123} + \Gamma_{\lambda 3}{}^{3}\eta^{0123}$$
$$= \varepsilon^{0123}\Big(\partial_\lambda \frac{1}{\sqrt{-g}} + \frac{1}{\sqrt{-g}}\Gamma_{\lambda\sigma}{}^{\sigma}\Big).$$

From Problem 1.15 and Problem 1.3 (ii): $\partial_\lambda(1/\sqrt{-g}) = -(1/\sqrt{-g})\Gamma_{\lambda\sigma}{}^{\sigma}$, and the statement in the problem follows.

1.19. Starting from (1.9.10), we have by an elementary iteration procedure

$$\chi(\gamma(\overline{s})) \simeq \Big[1 + (\mathrm{i})\int_{\underline{s}}^{\overline{s}} \mathrm{d}s\, \dot\gamma^{\,a}(s)\, A_a\big(\gamma(s)\big)\Big]\, \chi(\gamma(\underline{s})),$$
$$\vdots$$
$$\chi(\gamma(\overline{s})) = \Big[1 + \sum_{n\geq 1}(\mathrm{i})^n \int_{\underline{s}}^{\overline{s}} \mathrm{d}s_n \int_{\underline{s}}^{s_n} \mathrm{d}s_{n-1}\cdots\int_{\underline{s}}^{s_2} \mathrm{d}s_1$$
$$\times\, \dot\gamma^{\,a_n}(s_n)\, A_{a_n}\big(\gamma(s_n)\big)\ldots\dot\gamma^{\,a_1}(s_1)\, A_{a_1}\big(\gamma(s_1)\big)\Big]\, \chi\big(\gamma(\underline{s})\big).$$

Upon using the path ordering notation

$$\mathscr{P}\Big(A_{a_{i_1}}\big(\gamma(s_{i_1})\big)\ldots A_{a_{i_k}}\big(\gamma(s_{i_k})\big)\Big) = A_{a_1}\big(\gamma(s_1)\big)\ldots A_{a_k}\big(\gamma(s_k)\big),$$

for $s_1 \geq \cdots \geq s_k$, where $\{i_1, \ldots, i_k\}$ is any permutation of $1, \ldots, k$, we obtain

$$\chi\big(\gamma(\overline{s})\big) = \mathscr{P}\Big(\sum_{n\geq 0}\frac{(\mathrm{i})^n}{n!}\Big[\int_{\underline{s}}^{\overline{s}} \mathrm{d}s\, \dot\gamma^{\,a}(s)\, A_a\big(\gamma(s)\big)\Big]^n\Big)\, \chi\big(\gamma(\underline{s})\big),$$

with $n!$ corresponding to the $n!$ possible permutations of the indices $\{1, 2, \ldots, n\}$, for a given n. The above leads to the expression of the holonomy in (1.9.12).

Chapter 2

2.1. From the second identity in (2.1.9), we have $\gamma^{\,0}K = (\gamma^{\,0}K^{-1})^\dagger$. Therefore

$$\overline{\epsilon}K\gamma^\mu K^{-1}\epsilon = \epsilon^\dagger(\gamma^{\,0}K^{-1})^\dagger\gamma^\mu K^{-1}\epsilon = (K^{-1}\epsilon)^\dagger\gamma^{\,0}\gamma^\mu K^{-1}\epsilon = (\overline{K^{-1}\epsilon})\gamma^\mu(K^{-1}\epsilon).$$

2.2. From (2.1.10), (2.1.11),

$$x'' = \Lambda'\big(\Lambda x + \frac{\mathrm{i}}{2}\,\overline{\epsilon}\gamma K\,\epsilon - b\big) + \frac{\mathrm{i}}{2}\overline{\epsilon}'\gamma\,K'\big(K\theta + \epsilon\big) - b',$$

which upon using the identity $\Lambda'\gamma = K'^{-1}\gamma\;K'$, given in (2.1.9), and by collecting terms, the transformation rule in (2.1.14) emerges. The transformation rule in (2.1.15) simply follows by replacing θ' by $K\theta + \epsilon$ in $\theta'' = K'\theta' + \epsilon'$.

2.3. (i) The first identity was derived in (2.2.18). The second follows upon taking the adjoint of the first, multiplying by γ^0 from the right, and using the properties $\{\gamma^5, \gamma^\sigma\} = 0,\ (\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$. The fourth follows upon taking the adjoint of the third and multiplying from the right by γ^0. The third follows by choosing, in turn,

$$(A_1 = \gamma^5\gamma^\mu,\ A_2 = I),\ (A_1 = \gamma^5,\ A_2 = \gamma^\mu),$$

in (2.2.16), and adding the results.

(ii) The first follows by multiplying the first identity in (2.2.22) by $(\overline{\theta}\gamma^5)_a$. The second follows by multiplying the first in (2.2.22) by $(\overline{\theta}\gamma^5\gamma^\mu)_a$ and using the identity $\overline{\theta}\gamma^\mu\theta = 0$ in (2.2.11). The third follows upon multiplying the third identity in (2.2.22) by $(\overline{\theta}\gamma^5)_a$. The third one also follows by multiplying the fourth identity in (2.2.22) by $(\gamma^5\theta)_a$. The fourth one follows by multiplying the third identity in (2.2.22) by $(\overline{\theta}\gamma^5\gamma^\sigma)_a$, using the identity $\gamma^\sigma\gamma^\mu = [\gamma^\sigma, \gamma^\mu]/2 - \eta^{\mu\nu}$, and, from the last equality in (2.2.11), that $\overline{\theta}\gamma^5[\gamma^\sigma, \gamma^\mu]\theta = 0$. Finally the fifth one follows upon multiplying the first one in (2.2.22) by $\overline{\theta}_a$.

(iii) These easily follow upon multiplying (2.2.21), in turn, by $\overline{\theta}\theta$, $\overline{\theta}\gamma^5\theta$, $\overline{\theta}\gamma^5\gamma^\sigma\theta$ and making use of the equalities in (2.2.23), (2.2.24).

2.4. (i) This identity follows simply upon multiplying (2.2.21) by θ_c, using the identities in (2.2.22), anti-symmetrizing with respect to the indices a, b, c, and finally using the Fierz identity (A-2.1) in Chap. 2. (ii) This identity follows by multiplying (2.2.21) by $\theta_c\,\theta_d$, using the identities in (2.2.25), (2.2.26), anti-symmetrizing with respect to the indices a, b, c, d, and finally using the Fierz identity (A-2.2).

2.5. Multiply the classic Fierz identity (A-2.3) by $(\gamma^5)_{a'a}\mathscr{C}_{dd'}$ to obtain

$$(\gamma^5\gamma^\mu)_{a'b}(\gamma_\mu\mathscr{C})_{cd'} = -(\gamma^5\mathscr{C})_{a'd'}\delta_{cb} - \frac{1}{2}(\gamma^5\gamma^\mu\mathscr{C})_{a'd'}(\gamma_\mu)_{cb}$$
$$-\frac{1}{2}(\gamma^\mu\mathscr{C})_{a'd'}(\gamma^5\gamma_\mu)_{cb} + \mathscr{C}_{a'd'}(\gamma^5)_{cb}.$$

Make a copy of this equation by making the replacements: $a' \to a$, $b \to d$, $d' \to k$, and another copy, this time making replacements:

$$a' \to k,\ b \to d,\ d' \to a,$$

and add the two resulting equations, using, in the process:

$(\gamma^5\gamma^\mu\mathscr{C})_{ka} = -(\gamma^5\gamma^\mu\mathscr{C})_{ak}$, $(\gamma^5\mathscr{C})_{ka} = -(\gamma^5\mathscr{C})_{ak}$, $(\gamma^\mu\mathscr{C})_{ka} = (\gamma^\mu\mathscr{C})_{ak}$, and the identity in question follows.

2.6. Using the reality of the matrices, γ^0, $\mathscr{C}$, the identity $\{\gamma^0, \mathscr{C}\} = 0$, and the property $\mathscr{C}^\top = \mathscr{C}^{-1} = -\mathscr{C}$, we obtain

$$(\overline{\psi}_+)_c = \big(\overline{\psi}_c + (\gamma^0)_{ca}(\mathscr{C})_{ab}(\psi)_k(\gamma^0)_{kb}\big)/2,$$

which upon multiplying by $\mathscr{C}_{dc}$, gives $\mathscr{C}\overline{\psi}_+^\top = \big(\mathscr{C}\overline{\psi}^\top + \psi\big)/2 = \psi_+$. Similarly,

$$(\overline{\psi}_-)_c = \mathrm{i}\big(\overline{\psi}_c - (\gamma^0)_{ca}(\mathscr{C})_{ab}(\psi)_k(\gamma^0)_{kb}\big)/2,$$

which upon multiplying by $\mathscr{C}_{dc}$, gives $\mathscr{C}\overline{\psi}_-^\top = \mathrm{i}\big(\mathscr{C}\overline{\psi}^\top - \psi\big)/2 = \psi_-$.

2.7. Using the definition $\theta_b = \mathscr{C}_{bk}\overline{\theta}_k$ gives $(\partial/\partial\overline{\theta}_a)\mathscr{C}_{bk}\overline{\theta}_k = \mathscr{C}_{ba}$. Also $\mathscr{C}^{-1}\gamma^\mu\mathscr{C} = -(\gamma^\mu)^\top$, $\mathscr{C}^\top = -\mathscr{C}$ imply that $(\gamma^\mu\mathscr{C})_{ab} = (\gamma^\mu\mathscr{C})_{ba}$. Hence from the definition of D_a in (2.3.9), the anti-commutator in (2.3.10) follows after carrying one differentiation with respect to $\overline{\theta}_a$ as spelled out above. The anti-commutator in (2.3.11) emerges from the definition $\overline{D}_c = -D_b\mathscr{C}_{bc}^{-1}$, and by multiplying the first one by $-\mathscr{C}_{bc}^{-1}$ giving $\{D_a, \overline{D}_c\} = \mathrm{i}(\gamma^\mu)_{ac}\,\partial_\mu$.

2.8. Using the property $(\mathscr{C}\gamma^\mu)_{ab} = (\mathscr{C}\gamma^\mu)_{ba}$, we have

$$\overline{D}\,\gamma^\mu D = (\mathscr{C}\gamma^\mu)_{ab}\,\{D_a, D_b\}/2 = (\mathrm{i}/2)\,\mathrm{Tr}\,(\gamma^\mu\gamma^\sigma)\partial_\sigma = -2\,\mathrm{i}\,\partial^\mu,$$

where we have used the anti-commutator in (2.3.10), $\mathscr{C} = -\mathscr{C}^{-1}$. Similarly, $\overline{D}\,[\gamma^\mu, \gamma^\nu]D = (\mathscr{C}[\gamma^\mu, \gamma^\nu])_{ab}\,\{D_a, D_b\}/2 = 0$, as it gives rise to the trace of an odd number of gamma matrices.

2.9. From (2.2.14) $\mathscr{C}^{-1}B\,\mathscr{C} = B^\top$, which implies that $(\mathscr{C}B)_{ab} = -(\mathscr{C}B)_{ba}$. The anti-commutator in (2.3.10) implies that $D_aD_b = -D_bD_a - \mathrm{i}(\gamma^\mu\mathscr{C})_{ab}\partial_\mu$. Hence

$$\begin{aligned}\overline{D}BD\,D_a &= D_c\big(\mathscr{C}B\big)_{cd}D_d\,D_a = -D_c\big(\mathscr{C}B\big)_{cd}[D_aD_d + \mathrm{i}(\gamma^\mu\mathscr{C})_{da}\partial_\mu]\\ &= [D_aD_c + \mathrm{i}(\gamma^\mu\mathscr{C})_{ca}\partial_\mu](\mathscr{C}BD)_c - \mathrm{i}\,D_c(\mathscr{C}B\,\gamma^\mu\mathscr{C})_{ca}\partial_\mu,\end{aligned}$$

which is equal to $D_a(\overline{D}BD) - 2\,\mathrm{i}\,(\gamma^\mu BD)_a\partial_\mu$, where we have used the property $(\mathscr{C}B\,\gamma^\mu\mathscr{C})_{ca} = (\gamma^\mu B)_{ac}$ on account that $\mathscr{C} = -\mathscr{C}^{-1}$

2.10.
- (2.3.18): Simply choose $B = I$ in (2.3.13) to obtain this identity.
- (2.3.19): Multiply (2.3.17) by $(\gamma^5)_{ba}$, and choose $A = I$. Add the resulting equation to the one in (2.3.17), with $A = \gamma^5$, b replaced by a, and simplify to obtain this identity.
- (2.3.20): Choose $B = \gamma^5$ in (2.3.13) to obtain

$$\overline{D}\,\gamma^5 D\,D_a = D_a\,\overline{D}\,\gamma^5 D - 2\,\mathrm{i}(\gamma^\mu\gamma^5 D)_a\partial_\mu = D_a\,\overline{D}\,\gamma^5 D + 2\,\mathrm{i}(\gamma^5\gamma^\mu D)_a\partial_\mu,$$

and then use the identity established in (2.3.19) to replace $\overline{D}\gamma^5 D D_a$ by $-\overline{D}D(\gamma^5 D)_a$.

- (2.3.21): Multiply (2.3.20) by $(\overline{D}\gamma^5)_a$ to obtain

$$(\overline{D}\gamma^5 D)^2 = -\overline{D}_b \overline{D}D D_b - 2\,\mathrm{i}\,(\overline{D}\gamma^\mu D)\partial_\mu,$$

and then replace $\overline{D}D D_b$ by the right-hand side of the identity established in (2.3.18), with a replaced by b, and simplify to obtain this identity.

- (2.3.22): Choose $A = \gamma^5\gamma^\sigma$ in (2.3.17) with b replaced by a, and add the resulting equation to the one obtained from (2.3.17) multiplied by $(\gamma^5\gamma^\sigma)_{ab}$, and $A = I$, and simplify to obtain this identity.
- (2.3.23): Choose $A = \gamma^5\gamma^\sigma$ in (2.3.17), then use the result just established in (2.3.22) to replace $(\overline{D}\gamma^5\gamma^\sigma D)\,D_a$ by the expression on the right-hand side of the latter equation and simplify.

2.11. From (2.1.10), $\partial x'/\partial x = \Lambda$, $\partial x'/\partial\theta = -\mathrm{i}\,\overline{\epsilon}\,\gamma K/2$. On the other hand, from (2.4.13), $\partial\theta'/\partial x = 0$, $\partial\theta'/\partial\theta = K$, which lead to the expression of the matrix in question.

2.12. We carry out an expansion of the logarithm as follows:

$$\ln[I-(I-M)] = -\sum_{n\geqslant 1}\begin{pmatrix} I-\Lambda & -T \\ 0 & I-K\end{pmatrix}^n \Big/ n,$$

$$\begin{pmatrix} I-\Lambda & -T \\ 0 & I-K\end{pmatrix}\begin{pmatrix} I-\Lambda & -T \\ 0 & I-K\end{pmatrix} = \begin{pmatrix} (I-\Lambda)^2 & -[(I-\Lambda)T + T(I-K)] \\ 0 & (I-K)^2\end{pmatrix},$$

$$\begin{pmatrix} I-\Lambda & -T \\ 0 & I-K\end{pmatrix}^n = \begin{pmatrix} (I-\Lambda)^n & C_n \\ 0 & (I-K)^n\end{pmatrix},$$

$C_n = -[(I-\Lambda)^{n-1}T + (I-\Lambda)^{n-2}T(I-K) + \ldots + T(I-K)^{n-1}]$, which corresponds to the expression in (2.4.20).

2.13. We may write

$$\begin{pmatrix} C & \eta \\ \xi & D\end{pmatrix} = \begin{pmatrix} C & 0 \\ \xi & I\end{pmatrix}\begin{pmatrix} I & C^{-1}\eta \\ 0 & D-\xi C^{-1}\eta\end{pmatrix},$$

Sdet of the first matrix on the right-hand side of the above equation, is $\det C/\det I$, while for the second one is $\det I/\det\big(D-\xi C^{-1}\eta\big)$, and the stated result follows upon their multiplication.

2.14. Upon adding (2.3.18), (2.3.19), and dividing by 2 gives:

$$\overline{D}D^{\mathrm{R}}D_a = (1/2)D_a\overline{D}D - (1/2)\overline{D}D(\gamma^5 D)_a - \mathrm{i}(\gamma^\mu D)_a\partial_\mu.$$

On the other hand, (2.3.20) gives: $\overline{D}D(\gamma^5 D)_a = -D_a\overline{D}\gamma^5 D - 2\mathrm{i}(\gamma^5\gamma^\mu D)_a\partial_\mu$. From these two equations, the statement of the problem follows upon multiplication by $(1-\gamma^5)/2$ and using the property $\{\gamma^5,\gamma^\mu\} = 0$.

2.15. (i) From (2.2.5), (2.2.6) $\overline{\theta}\xi_{\mathrm{L}} = -\theta^\top\mathscr{C}^{-1}[(1-\gamma^5)/2]\xi$, $\overline{\theta}\xi_{\mathrm{R}} = -\theta^\top\mathscr{C}^{-1}[(1+\gamma^5)/2]\xi$,

$$\mathscr{C}^{-1}[(1-\gamma^5)/2] = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{i}\sigma^2 \end{pmatrix}, \qquad \mathscr{C}^{-1}[(1+\gamma^5)/2] = \begin{pmatrix} \mathrm{i}\sigma^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

The first two results in (i) then immediately follow. From Box 2.1 in the beginning of Sect. 2.6,

$$\overline{\theta}\gamma^\mu\gamma^\nu\theta = -\overline{\theta}\theta\,\eta^{\mu\nu}, \quad \overline{\theta}\gamma^\mu\gamma^\nu\gamma^5\theta = -\overline{\theta}\gamma^5\theta\,\eta^{\mu\nu},$$

which when combined lead to the third result in (i). The last one in (i) follows by adding $0 = \overline{\theta}\gamma^\mu\theta$ to $\overline{\theta}\gamma^5\gamma^\mu\theta$ and using $\overline{\theta} = -\theta^\top\mathscr{C}^{-1}$. Using the fact that $\{\gamma^5,\gamma^0\} = 0$, we have

$$(\overline{\theta}\xi_{\mathrm{L}})^\dagger = \overline{\xi}\theta_{\mathrm{R}} = -\xi\mathscr{C}^{-1}\frac{1}{2}(1+\gamma^5)\theta,$$

and the first statement in (ii) then follows upon using the definition $\mathscr{C}^{-1}\theta = \overline{\theta}^\top$, $[\mathscr{C}^{-1},\gamma^5] = 0$, and the anti-commutativity of the components of $\overline{\theta}$ and of ξ. Similarly, using the facts that $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0(\gamma^\mu)$, $\{\gamma^5,\gamma^0\} = 0$, we have

$$(\overline{\theta}\,\gamma^\mu\,[(1-\gamma^5)/2]\,\xi)^\dagger = \overline{\xi}\gamma^\mu\theta_{\mathrm{L}} = -\xi^\top(\mathscr{C}^{-1}\,\gamma^\mu\,[(1-\gamma^5)/2]\,\mathscr{C})\,\mathscr{C}^{-1}\theta,$$
$$\mathscr{C}^{-1}\gamma^\mu\mathscr{C} = -(\gamma^\mu)^\top.$$

The last equality then follows from the identity $\{\gamma^\mu,\gamma^5\} = 0$, and the anti-commutativity of the spinor field components.

2.16. Using the expression of Λ in (2.6.97), and part (ii) of the previous Problem, together with the identities: $(\overline{\theta}\gamma^5\gamma^\mu\theta)^\dagger = \overline{\theta}\gamma^5\gamma^\mu\theta$, $(\overline{\theta}\gamma^5\theta)^\dagger = -\overline{\theta}\gamma^5\theta$, lead directly to the expression given in the problem.

2.17. It is easily verified that $\mathscr{V}^\rho(x,\theta)$ in (2.6.61), may be rewritten as

$$\mathrm{e}^{\mathrm{i}[\overline{\theta}\gamma^5\gamma^\mu\theta\partial_\mu]/4}\Bigg[V^\rho(x) + \frac{\mathrm{i}}{\sqrt{2}}\overline{\theta}\gamma^\rho\chi(x) + \overline{\theta}\gamma^5\gamma_\lambda\theta\left(A^{\lambda\rho}(x) - \frac{\mathrm{i}}{4}\partial^\lambda V^\rho(x)\right)$$
$$+\,\overline{\theta}\gamma^5\theta\,\overline{\theta}\left(B^\rho(x) + \frac{1}{4\sqrt{2}}\gamma^\mu\gamma^\rho\partial_\mu\chi(x)\right)\Bigg], \quad (*)$$

and note that the quadratic term $(\mathrm{i}\,\overline{\theta}\gamma^5\gamma^\mu\theta\partial_\mu/4)^2/2$ coming from the exponential generates also a term $-\Box\,V^\rho/32$ from the θ-independent term V^ρ

within the square brackets in (*) which combines with the $-(\mathrm{i}/4)(\mathrm{i}/4)\partial^\lambda V^\rho$ coming from the term, next to $A^{\lambda\rho}$, within the round brackets in (*), to give the $-(\mathrm{i}/8)\partial^\lambda V^\rho$ term next to $A^{\lambda\rho}$, within the round brackets in the last term in (2.6.61). Also note that $\overline{\theta}\gamma^\mu\theta\,\overline{\theta}\gamma^\rho = \overline{\theta}\gamma^5\theta\,\overline{\theta}\gamma^\mu\gamma^\rho$, as obtained from the last identity in (2.2.22). On the other hand, since the exponential term represents the translation operator of the argument x^μ of the component fields by $\mathrm{i}\overline{\theta}\gamma^5\gamma^\mu\theta/4$, our expression for $\mathscr{V}^\rho(x,\theta)$, in the Wess-Zumino supergauge, becomes simply as given in (2.6.66).

2.18. Let us work it out for the abelian case first. We have: $\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 4\,\varepsilon^{\mu\nu\alpha\beta}(\partial_\mu V_\nu\partial_\alpha V_\beta) = 4\,\varepsilon^{\mu\nu\alpha\beta}[\partial_\mu(V_\nu\partial_\alpha V_\beta) - V_\nu(\partial_\mu\partial_\alpha)V_\beta]$. The second term within the square brackets gives zero since it is symmetric in $(\mu\alpha)$ while $\varepsilon^{\mu\nu\alpha\beta}$ is anti-symmetric in interchanging these two indices. This establishes the statement of the problem for the abelian case. For the non-abelian case, we have

$$\varepsilon^{\mu\nu\alpha\beta}G_{A\mu\nu}G_{A\alpha\beta} = 4\varepsilon^{\mu\nu\alpha\beta}\Big(\partial_\mu V_{A\nu} + \frac{g}{2}f_{ABC}V_{B\mu}V_{C\nu}\Big)\Big(\partial_\alpha V_{A\beta} + \frac{g}{2}f_{ADE}V_{D\alpha}V_{E\beta}\Big),$$

where note the symmetry in the interchange $(\mu\nu) \leftrightarrow (\alpha\beta)$. Upon using the equalities $\varepsilon^{\mu\nu\alpha\beta}f_{ABC} = \varepsilon^{\mu\beta\alpha\nu}f_{CBA} = \varepsilon^{\mu\nu\beta\alpha}f_{ACB}$, the above equation becomes

$$4\,\varepsilon^{\mu\nu\alpha\beta}\Big(\partial_\mu\big(V_{A\nu}\partial_\alpha V_{A\beta}\big) - V_{A\nu}(\partial_\mu\partial_\alpha)V_{A\beta}\Big)$$
$$+\frac{g}{3}f_{ABC}\partial_\mu\big(V_{A\nu}V_{B\alpha}V_{C\beta}\big) + \frac{g^2}{4}f_{ABC}f_{ADE}V_{B\mu}V_{C\nu}V_{D\alpha}V_{E\beta}\Big).$$

Only the last term needs to be considered. The last factor multiplying g^2, may be rewritten as

$$-\,2\,\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\Big([\,V_\mu, V_\nu\,][\,V_\alpha, V_\beta\,]\Big),$$

where we have used the normalization $\mathrm{Tr}\,(t_A\,t_{A'}) = \delta_{AA'}/2$, and that

$$[\,V_\mu, V_\nu\,] = \mathrm{i}\;t_A f_{ABC}\,V_{B\mu}V_{C\nu}.$$

Using the anti-symmetry property of $\varepsilon^{\mu\nu\alpha\beta}$ under the exchange of its indices, and by simply relabeling them, the last term is then equal to

$$g^2\varepsilon^{\mu\nu\alpha\beta}f_{ABC}f_{ADE}V_{B\mu}V_{C\nu}V_{D\alpha}V_{E\beta} = -8\,g^2\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,[\,V_\mu V_\nu V_\alpha V_\beta\,].$$

The latter is obviously zero, since, for example, the trace factor does not change for $(\mu,\nu,\alpha,\beta) \to (\nu,\alpha,\beta,\mu)$, while $\varepsilon^{\mu\nu\alpha\beta} \to -\varepsilon^{\nu\alpha\beta\mu}$. This establishes the statement of the problem in the non-abelian case as well.

2.19. The part of the Lagrangian density in (2.10.1) depending on the Majorana field ψ may be rewritten as

$$\frac{1}{2}\psi_a\Big[\mathscr{C}^{-1}\Big(\frac{\gamma\partial}{\mathrm{i}}+m\Big)\Big]_{ab}\psi_b+\overline{\eta}_a\psi_a+\sqrt{2}\lambda\big[\varphi_1\,\mathscr{C}^{-1}+\mathrm{i}\,\varphi_2\,(\mathscr{C}^{-1}\gamma^{\,5})\big]_{ab}\big\}\psi_a\psi_b.$$

We recall the anti-symmetry of the charge conjugation matrix $\mathscr{C}$, its commutativity with $\gamma^{\,5}$, the identity $\mathscr{C}^{-1}\gamma^{\mu}\,\mathscr{C}=-(\gamma^{\mu})^{\top}$, and the anti-commutavity of $\psi_a,\ \psi_b$. Integrating by parts, (2.10.4) follows from the action principle by multiplying, in the process, the resulting equation by $(-\mathrm{i}\gamma\partial+m)^{-1}\mathscr{C}$, and finally taking the vacuum expectation value. In particular, in the absence of interaction, we recognize the familiar equation of a spinor field in the presence of an external source $\eta=\mathscr{C}\overline{\eta}^{\top}$.

2.20. From (2.10.4), $\langle\,(\varphi_1(x')\overline{\psi}_a(x'')\psi_a(x))_+\,\rangle\big|_1$ is equal to

$$\begin{aligned}
&(-2\sqrt{2}\lambda)\Big(\frac{\mathrm{i}\gamma\partial+m}{-\Box+m^2}\Big)_{bc}\langle\,\big(\varphi_1(x')\varphi_1(x)\overline{\psi}_a(x'')\psi_c(x)\big)_+\rangle\big|_0\\
&=-2\sqrt{2}\,\lambda\Big(\frac{\mathrm{i}\gamma\partial+m}{-\Box+m^2}\Big)_{ac}\big\{(-\mathrm{i})\Delta_+(x'-x)\,\mathrm{i}\,S_{+ca}(x-x'')\big\}\\
&=-2\sqrt{2}\,\lambda\int(\mathrm{d}z)\Delta_+(x'-z)S_{+ca}(z-x'')\Big(\frac{\mathrm{i}\gamma\partial+m}{-\Box+m^2}\Big)_{ac}\delta^{(4)}(x-z)\\
&=-2\sqrt{2}\,\lambda\int(\mathrm{d}z)\Delta_+(x'-z)S_{+ca}(z-x'')S_{+ac}(x-z).
\end{aligned}$$

By Fourier transform, and taking the trace, the above integral, now applied to $\langle\,(\varphi_1(x')\overline{\psi}_a(x)\psi_a(x))_+\,\rangle\big|_1$, takes the form

$$4\int(\mathrm{d}z)\frac{(\mathrm{d}k)(\mathrm{d}k_1)(\mathrm{d}k_2)}{(2\pi)^{12}}\frac{\mathrm{e}^{\mathrm{i}k(x'-z)}}{k^2+m^2}\frac{\mathrm{e}^{\mathrm{i}k_1(z-x)}}{k_1^2+m^2}\frac{\mathrm{e}^{\mathrm{i}k_2(x-z)}}{k_2^2+m^2}(-k_1k_2+m^2).$$

Upon using the identity

$$(-k_1k_2+m^2)=(1/2)\Big\{[(k_1-k_2)^2+m^2]-[k_1^2+m^2]-[k_2^2+m^2]+3m^2\Big\},$$

integrating over z, and k, we readily obtain the three expressions given in (2.10.16) upon multiplying by $(-2\sqrt{2}\,\lambda)$.

2.21. We use the fact that on may write $\gamma^{\,\alpha}\gamma^{\,5}=(\mathrm{i}/3!)\,\varepsilon^{\,\alpha\rho\omega\lambda}\,\gamma_\rho\gamma_\omega\gamma_\lambda$, and the identity ($\varepsilon^{\,0123}=+1,\ \varepsilon_{\,0123}=-1$)

$$\begin{aligned}
\varepsilon_{\alpha\beta\mu\nu}\,\varepsilon^{\alpha\rho\omega\lambda}=&-\delta^{\rho}{}_{\beta}(\delta^{\omega}{}_{\mu}\,\delta^{\lambda}{}_{\nu}-\delta^{\omega}{}_{\nu}\,\delta^{\lambda}{}_{\mu})-\delta^{\rho}{}_{\mu}(\delta^{\omega}{}_{\nu}\delta^{\lambda}{}_{\beta}-\delta^{\omega}{}_{\beta}\,\delta^{\lambda}{}_{\nu})\\
&-\delta^{\rho}{}_{\nu}\,(\delta^{\omega}{}_{\beta}\,\delta^{\lambda}{}_{\mu}-\delta^{\omega}{}_{\mu}\,\delta^{\lambda}{}_{\beta}).
\end{aligned}$$

Upon substituting the above two equalities in the above expression given in the statement of the problem, and using the elementary anti-commutativity properties of the gamma matrices, we readily recover our earlier expression for the Lagrangian density in (2.15.1) by using now the Latin alphabet for Lorentz indices.

2.22. By definition, $[\mathscr{D}_\mu, \mathscr{D}_\nu]$ follows directly to be given by

$$\frac{1}{8}[\gamma^a, \gamma^b]\Big(\partial_\nu(\omega_\mu)_{ab} - \partial_\mu(\omega_\nu)_{ab}\Big) - \frac{1}{4}(\omega_\nu)_{cd}(\omega_\mu)_{ab}[S^{ab}, S^{cd}],$$

where $S^{ab} = \mathrm{i}[\gamma^a, \gamma^b]/4$ provides a representation of homogeneous Lorentz transformation (spin). That is, from (4.2.10) in Chap. 4 of Vol. I, it satisfies the commutation relation

$$[S^{ab}, S^{cd}] = \mathrm{i}\big(\eta^{ac}S^{bd} - \eta^{bc}S^{ad} + \eta^{bd}S^{ac} - \eta^{ad}S^{bc}\big).$$

Upon replacing this in the previous equation and relabeling some of the indices, the statement of the problem follows.

Chapter 3

3.1. For infinitesimal $\lambda(\tau)$, then $\tau - \tau' = \lambda(\tau)$ implies that $\mathrm{d}\tau'/\mathrm{d}\tau = 1 - \dot{\lambda}$, $\dot{\lambda} \equiv \mathrm{d}\lambda/\mathrm{d}\tau$. Using (3.1.9), we have $a'(\tau') = \big(1 + \dot{\lambda}(\tau)\big)\, a(\tau)$, or $a'(\tau) = a(\tau) + \lambda(\tau)\,\dot{a}(\tau) + a(\tau)\,\dot{\lambda}(\tau)$. From which (3.1.11) follows. Similarly, the relation $X'^\mu(\tau') = X^\mu(\tau)$, in reference to the worldline, leads to (3.1.10).

3.2. From (3.2.16) we note that $(\dot{X} \cdot X')^2 - \dot{X} \cdot \dot{X}\;\; X' \cdot X' = h_{01}\, h_{10} - h_{00}\, h_{11} = -h$, which appears under the square root in (3.2.8). By using this together (3.2.17), (3.2.19), and (3.2.20) immediately follows.

3.3. One may explicitly write

$$\hat{h}_{01} = h_{00}\, ac + h_{01}(bc + ad) + h_{11}\, bd,$$

$$\hat{h}_{00} = h_{00}\, a^2 + 2h_{01}\, ab + h_{11}\, b^2,$$

$$\hat{h}_{11} = h_{00}\, c^2 + 2h_{01} cd + h_{11}\, d^2,$$

For $h_{00} = 0$, and hence $h = -h_{01}^2 \neq 0$, choose $d = b \neq 0$, $a = -(1 + b^2\, h_{11})/2b\, h_{01}$, $c = (1 - b^2\, h_{11})/2b\, h_{01}$ giving $[\,\hat{h}_{\alpha\beta}\,] = \mathrm{diag}[-1, 1]$.

For $h_{00} > 0$, choose $a = h_{01}/\sqrt{-h}$, $b = -h_{00}/\sqrt{-h}$, $c = \pm 1$, $d = 0$. This gives $[\,\hat{h}_{\alpha\beta}\,] = (h_{00})\,\mathrm{diag}[-1, 1]$.

For $h_{00} < 0$, choose $c = h_{01}/\sqrt{-h}$, $d = -h_{00}/\sqrt{-h}$, $a = \pm 1$, $b = 0$. This gives $[\,\hat{h}_{\alpha\beta}\,] = (-h_{00})\,\mathrm{diag}[-1, 1]$.

That is, we may write $[\hat{h}_{\alpha\beta}] = \mathrm{e}^{\phi}\,\mathrm{diag}[-1, 1]$, with e^{ϕ} defining a positive function.

3.4. The chain rule $\partial_\sigma = \partial_\sigma(\tau-\sigma)/\partial_{\tau-\sigma} + \partial_\sigma(\tau+\sigma)/\partial_{\tau+\sigma}$, implies that $\partial_\sigma = -\partial/\partial_{\tau-\sigma} + \partial/\partial_{\tau+\sigma}$. Similarly $\partial_\tau = \partial/\partial_{\tau-\sigma} + \partial/\partial_{\tau+\sigma}$. These give

$$(\partial_\sigma)^2 - (\partial_\tau)^2 = -4\,\partial_{\tau-\sigma}\partial_{\tau+\sigma},$$

from which the wave equation becomes $\partial_{\tau-\sigma}\partial_{\tau+\sigma}X^i(\tau,\sigma) = 0$. Using the facts that

$$\tau = (\tau-\sigma)/2 + (\tau+\sigma)/2, \quad \sigma = -(\tau-\sigma)/2 + (\tau+\sigma)/2,$$

give, from the chain rule,

$$\partial_{\tau-\sigma} = (1/2)(\partial/\partial_\tau - \partial/\partial_\sigma), \quad \partial_{\tau+\sigma} = (1/2)(\partial/\partial_\tau + \partial/\partial_\sigma).$$

Hence $\partial_{\tau\pm\sigma}(\tau \mp \sigma) = 0$, as expected, from which the newly obtained wave equation implies the structure of the solutions mentioned in the problem. We note that for $\sigma \to (\sigma + \triangle\sigma)$, with $\triangle\sigma > 0$, the arguments

$$\big(\tau \mp \sigma\big) \to \big(\tau \mp (\sigma + \triangle\sigma)\big) = \big((\tau \mp \triangle\sigma) \mp \sigma\big),$$

and $\tau > \tau - \triangle\sigma$, $\tau < \tau + \triangle\sigma$, with τ corresponding, respectively, to the future and the past in the evolution process. This justifies the subscripts R/L attached to these solutions as right- and left-movers, respectively.

3.5. Using the integral $\int_0^\pi \mathrm{d}\sigma\,\mathrm{e}^{-2\mathrm{i}N\sigma} = \pi\delta(N,0)$ for integer N, the expression in (3.2.81) gives

$$\int_0^\pi \mathrm{d}\sigma\,\partial_\tau X^- = \ell^2 p^- \pi.$$

On the other hand, the explicit expressions in (3.2.79), (3.2.80) lead to

$$\frac{1}{2}\int_0^\pi \mathrm{d}\sigma[(\partial_\sigma X^i)^2 + (\partial_\tau X^i)^2]$$
$$= \pi\,\ell^2\Big(\frac{1}{2}\,[\alpha^i(0)+\bar{\alpha}^i(0)]^2 + \sum_{m\neq 0}[\alpha^i(-m)\alpha^i(m) + \bar{\alpha}^i(-m)\bar{\alpha}^i(m)]\Big).$$

where recall from (3.2.75) that $\alpha^i(0) = \bar{\alpha}^i(0)$. Upon comparing the latter two integrals with the result obtained by integrating the equality in (3.2.48) over σ from 0 to π gives (3.2.82).

3.6. The identity in (3.2.136) is explicitly given by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = -\begin{pmatrix} A^\top & C^\top \\ B^\top & D^\top \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

which leads to the stated conditions. Since no restrictions are set on the matrix A, and the elements of the matrix D are obtained from those of A, the number of independent components of the generators Λ are:
$(N^2/4)+N(N/2+1)/4+N(N/2+1)/4 = N(N+1)/2$, where $N(N/2+1)/4$ denotes the number of independent elements of the matrix B or of C.

3.7. The expressions in (3.2.146), (3.2.147), may be rewritten as

$$\partial_\tau X = \ell \sum_n \big[\, \bar{\alpha}(n)\, \mathrm{e}^{-2\mathrm{i}n(\tau+\sigma)} + \alpha(n)\, \mathrm{e}^{-2\mathrm{i}n(\tau-\sigma)} \big], \qquad \ell p^{25} = \bar{\alpha}(0) + \alpha(0),$$

$$\partial_\sigma X = \ell \sum_n \big[\, \bar{\alpha}(n)\, \mathrm{e}^{-2\mathrm{i}n(\tau+\sigma)} - \alpha(n)\, \mathrm{e}^{-2\mathrm{i}n(\tau-\sigma)} \big], \;\; 2R\, w = \ell \big(\bar{\alpha}(0) - \alpha(0)\big),$$

where we have, in the process, used (3.2.140), (3.2.141). An elementary integration over σ, as defined below, and the identity $2\,\ell^{\,2}\big(\,\bar{\alpha}^{\,2}(0)+\alpha^{\,2}(0)\,\big) = 4R^{\,2}w^{\,2} + \ell^{\,4}(p^{25})^2$, together give

$$\frac{1}{2}\int_0^\pi \mathrm{d}\sigma[(\partial_\sigma X)^2 + (\partial_\tau X)^2]$$
$$= \pi\frac{1}{2}(4R^{\,2}w^{\,2} + \ell^{\,4}(p^{25})^2) + \ell^{\,2}\pi \sum_{m\neq 0} \big[\, \alpha(-m)\alpha(m) + \bar{\alpha}(-m)\bar{\alpha}(m)\,\big].$$

Now we use (3.2.48), the integral $\int_0^\pi \mathrm{d}\sigma\,\partial_\tau X^- = \ell^{\,2}\pi p^-$, and add the contribution of the $X^{\,i}$ obtained in Problem 3.5, now for $i = 1,\ldots,23$, to the above equation, to obtain (3.2.148).

3.8. In reference to the constraint in (3.2.109), we have from (3.2.146), (3.2.147) for the $X^{25} \equiv X$ contribution,

$$\frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma\,\partial_\tau X \partial_\sigma X = 2R\,\omega\,\ell^{\,2} p^{25} - \ell^{\,2}\sum_{n\neq 0}\big(\,\alpha(-n)\alpha(n) - \bar{\alpha}(-n)\bar{\alpha}(n)\,\big),$$

Adding the contribution of the $X^{\,i}$, which is $-\ell^2$ times the expression on the extreme left-hand side of (3.2.110), now for $i = 1, .., 23$, and setting the sum equal to zero, lead to the constraint in (3.2.152).

3.9. We note that if constraints are imposed on the external source, thus changing the right-hand side of (3.2.173), for example, by imposing a conservation law, the complete expression of a propagator does not follow. Since no constraints were imposed on J^μ, we may vary its components independently. Upon taking the vacuum expectation values $\langle\, 0_+|\,.\,|0_-\rangle$ of (3.2.173) and (3.2.174), setting

$\langle 0_+|A^\mu(x)|0_-\rangle = (-\mathrm{i}\delta/\delta J_\mu(x))\,\langle 0_+|0_-\rangle$, and functionally integrating with respect to the external source, we get

$$\langle 0_+|0_-\rangle = \exp\Big[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)J_\mu(x)D_+^{\mu\nu}(x-x')J_\nu(x')\Big],$$

where $D_+^{\mu\nu}(x-x') = \int(\mathrm{d}p)/(2\pi)^D\,\mathrm{e}^{\mathrm{i}k(x-x')}D_+^{\mu\nu}(k)$ is the propagator,

$$D_+^{ij}(k) = \frac{1}{k^2-\mathrm{i}\epsilon}\left(\delta^{ij}-\frac{k^ik^j}{\mathbf{k}^2}\right),\quad D_+^{00}(k) = -\frac{1}{\mathbf{k}^2},\quad D_+^{0i}=0.$$

Clearly $D_+^{00}(k)$ gives rise to a phase to $\langle 0_+|0_-\rangle$. Upon using the identity

$$\mathrm{i}\Big[\frac{1}{k^2-\mathrm{i}\epsilon}-\frac{1}{k^2+\mathrm{i}\epsilon}\Big] = -\frac{\pi}{|\mathbf{k}|}[\delta(k^0-|\mathbf{k}|)+\delta(k^0+|\mathbf{k}|)],$$

and the expansion $\pi^{ij} = \sum_{\lambda=1}^{D-2}\mathrm{e}^i_\lambda\mathrm{e}^j_\lambda$ [see (3.2.175)], then give

$$|\langle 0_+|0_-\rangle|^2 = \exp\Big[-\int\Big(\mathrm{d}^{D-1}\mathbf{k}/(2|\mathbf{k}|(2\pi)^{D-1}\Big)|J(\lambda,k)|^2\Big] < 1,$$

where $J(\lambda,k) = e^i_\lambda J^i(k)$, $k^0 = |\mathbf{k}|$, establishing the positivity of the formalism.

3.10. We take the vacuum expectation values of (3.2.183), (3.2.184), and make use of the equation

$$\langle 0_+|h^{\mu\nu}(x)|0_-\rangle = (-\mathrm{i}\delta/\delta T_{\mu\nu}(x))\,\langle 0_+|0_-\rangle,$$

where we note that no constraints were imposed on the external source and hence all of its components may be varied independently. Upon integrating with respect to the external source we obtain,

$$\langle 0_+|0_-\rangle = \exp[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')\,T_{\mu\nu}(x)\Delta_+^{\mu\nu,\lambda\sigma}(x-x')T_{\lambda\sigma}(x')],\quad\text{where}$$

$$\Delta_+^{\mu\nu,\lambda\sigma}(x-x') = \int[(\mathrm{d}k)/(2\pi)^D]\,\mathrm{e}^{\mathrm{i}k(x-x')}\Delta_+^{\mu\nu,\lambda\sigma}(k),\ \Delta_+^{00,0i} = \Delta_+^{0i,00} = 0,$$

$$\Delta_+^{00,00}(k) = \frac{D-3}{D-2}\frac{(k^2)}{|\mathbf{k}|^4},\quad \Delta_+^{00,ij}(k) = \frac{2}{D-2}\frac{1}{|\mathbf{k}|^2}\pi^{ij},\quad \Delta_+^{0i,0k}(k) = -\frac{1}{2\,|\mathbf{k}|^2}\pi^{ik},$$

$$\Delta_+^{ij,k\ell}(k) = \frac{1}{k^2-\mathrm{i}\,\epsilon}\frac{1}{D-2}\Big[\frac{D-2}{2}(\pi^{ik}\pi^{j\ell}+\pi^{i\ell}\pi^{jk})-\pi^{ij}\pi^{k\ell}\Big],\epsilon\to+0,$$

$\Delta_+^{ij,00}(k) = (2/[(D-2)|\mathbf{k}|^2])\,\pi^{ij}$, and $\pi^{ij} = \delta^{ij}-k^ik^j/|\mathbf{k}|^2 = \sum_{\lambda=1}^{D-2}e^i_\lambda e^j_\lambda$. Clearly, $\Delta_+^{00,00}$, $\Delta_+^{0i,0k}$, $\Delta_+^{ij,00}$, $\Delta_+^{00,ij}$, provide phase factors to $\langle 0_+|0_-\rangle$. We use the following identity

$$\frac{1}{D-2}\Big[\frac{D-2}{2}(\pi^{ik}\pi^{j\ell}+\pi^{i\ell}\pi^{jk})-\pi^{ij}\pi^{k\ell}\Big]=\sum_{\lambda,\lambda'=1}^{D-2}\epsilon^{ij}(\lambda,\lambda')\,\epsilon^{k\ell}(\lambda,\lambda'),$$

where $\epsilon^{ij}(\lambda,\lambda')$ is defined in (3.2.188). We note that the independent degrees of freedom now is easily obtained from

$$\sum_{\lambda,\lambda'=1}^{D-2}\epsilon^{ij}(\lambda,\lambda')\,\epsilon^{ij}(\lambda,\lambda')=\frac{1}{D-2}\Big[\frac{D-2}{2}(\pi^{ii}\pi^{jj}+\pi^{ij}\pi^{ji})-\pi^{ij}\pi^{ij}\Big]$$

to be simply $D(D-3)/2$ since $\pi^{ii}=D-2$, $\pi^{ij}\,\pi^{ij}=\pi^{ij}\delta^{ij}=D-2$. Finally the vacuum persistence probability is given by

$$|\langle\, 0_+|0_-\rangle|^2=\exp\Big[-\int\sum_{\lambda,\lambda'=1}^{D-2}\frac{\mathrm{d}^{D-1}\mathbf{k}}{2|\mathbf{k}|(2\pi)^{D-1}}\,|T(\lambda,\lambda',k)|^2\Big]<1,$$

where $T(\lambda,\lambda',k)$ is defined in (3.2.187), $k^0=|\mathbf{k}|$, establishing the positivity of the formalism.

3.11. As before no constraints are set on the external source and hence all of its components may be varied independently. We take the vacuum expectation values of (3.2.197), (3.2.198), and setting $\langle\, 0_+|A^{\mu\nu}(x)|0_-\rangle=(-\mathrm{i}\delta/\delta J_{\mu\nu}(x))\,\langle 0_+|0_-\rangle$. Upon integrating with respect to the external source we obtain,

$$\langle 0_+|0_-\rangle=\exp[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')\,T_{\mu\nu}(x)\tilde{\Delta}_+^{\mu\nu,\lambda\sigma}(x-x')T_{\lambda\sigma}(x')],$$

$$\tilde{\Delta}_+^{\mu\nu,\sigma\lambda}(x-x')=\int\frac{(\mathrm{d}k)}{(2\pi)^D}\,\mathrm{e}^{\mathrm{i}k(x-x')}\,\tilde{\Delta}_+^{\mu\nu,\sigma\lambda}(k),$$

$$\tilde{\Delta}_+^{00,00}=0,\ \tilde{\Delta}_+^{00,0i}=\tilde{\Delta}_+^{0i,00}=0,\ \tilde{\Delta}_+^{ij,00}=0,$$

$$\tilde{\Delta}_+^{0i,0j}(k)=-\frac{1}{|\mathbf{k}|^2}\,\pi^{ij},\quad \tilde{\Delta}_+^{ij,k\ell}(k)=\frac{1}{k^2-\mathrm{i}\,\epsilon}\frac{(\pi^{ik}\pi^{j\ell}-\pi^{i\ell}\pi^{jk})}{2}.$$

Clearly, $\tilde{\Delta}_+^{0i,0j}(k)$ gives rise to a phase factor to $\langle 0_+|0_-\rangle$. Upon using the identity

$$\frac{(\pi^{ik}\pi^{j\ell}-\pi^{i\ell}\pi^{jk})}{2}=\sum_{\lambda,\lambda'=1}^{D-2}\varepsilon^{ij}(\lambda,\lambda')\varepsilon^{k\ell}(\lambda,\lambda'),$$

where $\varepsilon^{ij}(\lambda, \lambda')$ is defined in (3.2.200). The number of independent polarization states are given, as before, to be

$$\sum_{\lambda,\lambda'=1}^{D-2} \varepsilon^{ij}(\lambda,\lambda')\,\varepsilon^{ij}(\lambda,\lambda') = \frac{1}{2}\Big[\pi^{ii}\pi^{jj} - \pi^{ij}\pi^{ji}\Big] = \frac{1}{2}(D-2)(D-3).$$

The vacuum persistence probability emerges as

$$|\langle\, 0_+|0_-\rangle|^2 = \exp[-\int \sum_{\lambda,\lambda'=1}^{D-2} \frac{\mathrm{d}^{D-1}\mathbf{k}}{2|\mathbf{k}|(2\pi)^{D-1}}\,|J(\lambda,\lambda',k)|^2] < 1,$$

with $J(\lambda, \lambda', k)$ defined in (3.2.200), $k^0 = |\mathbf{k}|$, thus establishing the positivity of the formalism.

3.12. The Hamiltonian density is given by $\mathscr{H} = P\cdot\dot{X} - \mathscr{L}$, where

$$\mathscr{L} = -(T/2)(\partial_\alpha X\cdot\partial^\alpha X),$$

is the Lagrangian density, and $P^\mu = T\dot{X}^\mu$. These give $\mathscr{H} = (T/2)(\dot{X}^2 + X'^2)$. Upon using $X^\pm = (X^0 \pm X^{D-1})/\sqrt{2}$, we may rewrite $\mathscr{H}$ as

$$\mathscr{H} = \frac{T}{2}(\dot{X}^i\dot{X}^i + X'^iX'^i) - T(\dot{X}^-\dot{X}^+ + X'^-X'^+).$$

From (3.2.62), we obtain,

$$\int_0^\pi \mathrm{d}\sigma(\dot{X}^i\dot{X}^i + X'^iX'^i) = \pi\ell^2\sum_n \alpha^i(-n)\alpha^i(n),$$

where we recall from (3.2.61) that $\alpha^i(0) = \ell p^i$. On the other hand, (3.2.56) implies that $\dot{X}^+ = \ell^2 p^+$, $X'^+ = 0$. From (3.2.64), we then obtain $\int_0^\pi \mathrm{d}\sigma\,(\dot{X}^-\dot{X}^+ + X'^-X'^+) = \pi\ell^4 p^+p^-$. Using the identities in (3.2.46), relating T, ℓ^2, α', the identity $p^2 = p^ip^i - 2p^+p^-$, and the explicit expression of M^2 in (A-3.1), the expression for $H = \int_0^\pi \mathrm{d}\sigma\mathscr{H}$ in (B-3.3) follows.

3.13. The two dimensional Dirac equation in (3.3.10), in the presence of an electromagnetic coupling, reads $[\,\rho^\alpha(\partial_\alpha/\mathrm{i} - \mathrm{e}A_\alpha) + m\,]\psi = 0$. Upon taking the complex conjugate of this equation and using the fact that the Dirac matrices are pure imaginary, give $[\,\rho^\alpha(\partial_\alpha/\mathrm{i} + \mathrm{e}A_\alpha) + m\,]\psi^* = 0$, from which we infer that $\psi_C = \psi^*$, for the charge conjugate spinor.

3.14. A quick way of establishing this is to set on general grounds

$$\delta_{ab}\,\delta_{cd} = B\,\delta_{ad}\delta_{cb} + C\,(\rho^\mu)_{ad}(\rho_\mu)_{cb} + D\,(\rho^5)_{ad}(\rho^5)_{cb}.$$

The coefficients B, C, D, are readily obtained by considering specific matrix elements, e.g., $a = b = c = d = 1$; $a = d = 1, b = c = 2$; $a = b = 1, c = d = 2$.

3.15. The transformation rules in (3.3.37) imply, after straightforward manipulations, that $\delta(\partial^\alpha X^\mu \partial_\alpha X_\mu) = \sqrt{2}\,\partial^\alpha(\bar{\epsilon}\psi^\mu\partial_\alpha X_\mu) - \sqrt{2}\,\bar{\epsilon}\psi^\mu \Box X_\mu$,

$$\frac{1}{\mathrm{i}}\,\delta(\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu) = \frac{1}{\sqrt{2}}\,\partial_\alpha(\bar{\epsilon}\rho^\beta\rho^\alpha\psi_\mu\partial_\beta X^\mu) + \sqrt{2}\,\bar{\epsilon}\psi^\mu\Box X_\mu - \sqrt{2}(\partial_\alpha\bar{\epsilon})(\rho^\beta\rho^\alpha\psi^\mu\partial_\beta X_\mu),$$

from which (3.3.38) follows, up to a total derivative, where, in the process of the derivation, we have used the first identity in (3.3.16), and $\rho^\beta\rho^\alpha\partial_\beta\partial_\alpha X^\mu = -\Box X^\mu$.

3.16. Using the definitions of the ρ^μ, ρ^5 matrices in (3.3.9), the definition of light-cone variables, the light-cone gauge property $X^+ = x^+ + \ell^2 p^+\tau$ in (3.2.56), (3.3.45), and the definitions $\psi_{\mathrm{R/L}} = [(I \pm \rho^5)/2]\psi$, we may write the expression for J^0 in (3.3.46) as

$$\begin{aligned} J^0 =& \frac{1}{2}\,\rho^\beta\rho^0\psi^{\,i}\partial_\beta X^i - \frac{1}{2}\,\ell^2 p^+\psi^- - \frac{1}{2}\,\rho^\beta\rho^0\psi^{\,+}\partial_\beta X^- \\ =& \frac{1}{2}\left[\begin{pmatrix}\psi^{\,i}_{\mathrm{R}}(\partial_0-\partial_1)X^i_{\mathrm{R}}\\ \psi^{\,i}_{\mathrm{L}}(\partial_0+\partial_1)X^i_{\mathrm{L}}\end{pmatrix} - \ell^2 p^+\begin{pmatrix}\psi^-_{\mathrm{R}}\\ \psi^-_{\mathrm{L}}\end{pmatrix} - \begin{pmatrix}\psi^+_{\mathrm{R}}(\partial_0-\partial_1)X^-_{\mathrm{R}}\\ \psi^+_{\mathrm{L}}(\partial_0+\partial_1)X^-_{\mathrm{L}}\end{pmatrix}\right]. \end{aligned}$$

The constraints then follow by using the definitions $\partial_\pm = (\partial_0 \pm \partial_1)/2$, and setting the above equation equal to zero. Also recall that $\partial_\pm X_{\mathrm{R/L}}(\tau \mp \sigma) = 0$.

3.17. Using the explicit expressions of the matrices ρ^0, ρ^1 in (3.3.9), it easily follows that for the fermionic parts

$$\begin{aligned} T^{\mathrm{F}}_{00} &= -\frac{\mathrm{i}}{2}\,\psi^{\,\mu}_{\mathrm{R}}\,\partial_-\psi_{\mu\mathrm{R}} - \frac{\mathrm{i}}{2}\,\psi^{\,\mu}_{\mathrm{L}}\,\partial_-\psi_{\mu\mathrm{L}}, \\ T^{\mathrm{F}}_{01} &= \frac{\mathrm{i}}{2}\,\psi^{\,\mu}_{\mathrm{R}}\,\partial_-\psi_{\mu\mathrm{R}} + \frac{\mathrm{i}}{2}\,\psi^{\,\mu}_{\mathrm{L}}\,\partial_-\psi_{\mu\mathrm{L}}. \end{aligned}$$

By invoking the boundary conditions satisfied by the spinor ψ^μ in (3.3.52), the above equations give

$$\begin{aligned} T^{\mathrm{F}}_{00} + T^{\mathrm{F}}_{01} &= \mathrm{i}\left(\psi^{\,i}_{\mathrm{L}}\partial_+\psi^{\,i}_{\mathrm{L}} - \psi^{\,+}_{\mathrm{L}}\partial_+\psi^{\,-}_{\mathrm{L}}\right), \\ T^{\mathrm{F}}_{00} - T^{\mathrm{F}}_{01} &= \mathrm{i}\left(\psi^{\,i}_{\mathrm{R}}\partial_-\psi^{\,i}_{\mathrm{R}} - \psi^{\,+}_{\mathrm{R}}\partial_+\psi^{\,-}_{\mathrm{R}}\right), \\ T^{\mathrm{B}}_{00} + T^{\mathrm{B}}_{01} &= 2\left(\partial_+X^i_{\mathrm{L}}\,\partial_+X^i_{\mathrm{L}} - \ell^2 p^+\,\partial_+X^-_{\mathrm{L}}\right) \\ T^{\mathrm{B}}_{00} - T^{\mathrm{B}}_{01} &= 2\left(\partial_-X^i_{\mathrm{R}}\,\partial_-X^i_{\mathrm{R}} - \ell^2 p^+\,\partial_-X^-_{\mathrm{R}}\right). \end{aligned}$$

For the bosonic part we have $T^{B}_{00} = [\partial_0 X^{\mu}\partial_0 X_{\mu} + \partial_1 X^{\mu}\partial_1 X_{\mu}]/2$, $T^{B}_{01} = \partial_0 X^{\mu}\partial_1 X_{\mu}$. The three constraints mentioned in the problem immediately follow upon setting $[T_{00} + T_{01}] = 0$, $[T_{00} - T_{01}] = 0$, $T_{01} = \big([T_{00} + T_{01}] - [T_{00} - T_{01}]\big)/2 = 0$.

3.18. For the R boundary condition, $\partial_+ X^-_{\mathrm{L}}$, $\partial_+ \psi^-_{\mathrm{L}}$ have the general structures

$$\partial_+ X^-_{\mathrm{L}} = \frac{1}{2}\,\ell^2 p^- + \frac{\ell}{2}\sum_{n\neq 0} A^-(n)\,\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)},$$

$$\partial_+ \psi^-_{\mathrm{L}} = -\frac{\mathrm{i}}{\sqrt{2}}\,\ell \sum_{n} \chi^-(n)\, n\,\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}.$$

The zero mode part, i.e., the $\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}$-independent part, of the right-hand side of the first equation is $\ell^2 p^-/2$, while for the second one, it is zero. The zero mode part of $\partial_+ X^i_{\mathrm{L}}\partial_+ X^i_{\mathrm{L}} + \mathrm{i}\,\psi^i_{\mathrm{L}}\partial_+\psi^i_{\mathrm{L}}/2$, is clearly as given on the right-hand of (3.3.66). The statement in the problem then follows from the application of (3.3.54). The demonstration for the NS boundary condition is almost identical.

3.19. Using the facts that $\partial_\alpha \psi^+ = 0$ and $\partial_1 X^+ = 0$ [see (3.3.51), (3.3.52)], we have, with $T_{01} = T^{F}_{01} + T^{B}_{01}$, the following explicit expressions: (see also Problem 3.17)

$$T^{F}_{01} = -\frac{\mathrm{i}}{2}\,(\psi^+_{\mathrm{L}}\partial_+\psi^-_{\mathrm{L}} - \psi^+_{\mathrm{R}}\partial_-\psi^-_{\mathrm{R}}) + \frac{\mathrm{i}}{2}\,(\psi^i_{\mathrm{L}}\partial_+\psi^i_{\mathrm{L}} - \psi^i_{\mathrm{R}}\partial_-\psi^i_{\mathrm{R}}),$$
$$T^{B}_{01} = -\,\ell^2 p^+\partial_1 X^- + \partial_0 X^i\partial_1 X^i.$$

consistent with (3.3.56). For the R boundary condition, obviously the σ-integrals of $\partial_+\psi^-_{\mathrm{L}}$ and $\partial_1 X^-$ are both zero on account that $\int_0^{\pi}\mathrm{d}\sigma\ \mathrm{e}^{-2\mathrm{i}n\sigma} = 0$, for non-zero integer n. On the other-hand, for the NS boundary condition, the same reason gives zero for the σ- integral of $\partial_1 X^-$ and, $\psi^+ = 0$, in this case. On the other hand as a consequence of the orthogonality relation

$$\int_0^{\pi}\mathrm{d}\sigma\ \mathrm{e}^{-2\mathrm{i}(n-m)\sigma} = \pi\,\delta_{n,m},$$

the remaining terms in T_{01} readily give the constraints in (3.3.77), and (3.3.78), by finally invoking the boundary condition $T_{01} = 0$ holding true as a special case of (3.3.35).

3.20. Since no constraints were imposed on the external sources $\overline{K}^{\mu}, K^{\mu}$, we may vary each of their components independently. Upon taking the vacuum expectation values $\langle\, 0_- | \,.\, | 0_+\rangle$ of (3.3.177), (3.3.178), and setting $\langle\, 0_+ | \psi^{\mu}_a(x) | 0_-\rangle = (-\mathrm{i}\delta/\delta\overline{K}_{\mu a}(x))\langle\, 0_+ | 0_-\rangle$, and integrating with respect to the sources, we obtain

$$\langle\, 0_+|0_-\rangle = \exp\big[\; \mathrm{i}\int (dx)(dx\,')\,\overline{K}_{\mu a}(x)\Delta_{+\,ab}^{\;\mu\nu}(x-x\,')K_{\nu b}(x\,')\big],$$

where the Rarita-Schwinger propagator $\Delta_{+\,ab}^{\;\mu\nu}(x-x')$ in 10 dimensions is given by $\Delta_{+\,ab}^{\;\mu\nu}(x-x\,') = \int[\,(\mathrm{d}p)/(2\pi)^{10}\,]\,\mathrm{e}^{\mathrm{i}p\,(x-x\,')}\,\Delta_{+\,ab}^{\;\mu\nu}(p)$, and $\Delta_{+\,ab}^{\;\mu\nu}(p)$ is explicitly worked out to be $(\gamma\cdot p = \gamma^\mu p_\mu)$

$$\Delta_+^{ij}(p) = \frac{(-\gamma\cdot p)}{p^2-\mathrm{i}\epsilon}\alpha^{ij}(p), \quad \Delta_+^{00}(p) = \frac{7}{8}\frac{(\gamma\cdot p)}{\mathbf{p}^2},$$

$$\Delta_+^{0i}(p) = +\frac{1}{8\mathbf{p}^2}\big(p^{\,i}+\gamma^j p^j\,\gamma^{\,i}\big)\,\gamma^{\,0}, \quad \Delta_+^{i0}(p) = -\frac{1}{8\mathbf{p}^2}\big(p^{\,i}+\gamma^j p^j\,\gamma^{\,i}\big)\,\gamma^{\,0},$$

$$\alpha^{ij} = \Big(\delta^{ij}-\frac{p^{\,i}p^{\,j}}{\mathbf{p}^2}\Big) + \frac{1}{8}\Big(\delta^{\,ik}-\frac{p^{\,i}p^{\,k}}{\mathbf{p}^2}\Big)\gamma^{\,k}\gamma^{\,\ell}\Big(\delta^{\ell j}-\frac{p^{\,\ell}p^{\,j}}{\mathbf{p}^2}\Big).$$

Clearly only $\Delta_+^{ij}(p)$ propagates.

3.21. Upon setting $\rho = \mathrm{e}^\tau\,\mathrm{e}^{\mathrm{i}\sigma}$, with $W = W_{\mathrm{R}} + \mathrm{i}\,W_{\mathrm{Im}}$, $ad - bc = 1$, and $0 \leq \sigma \leq \pi$, we obtain

$$\begin{aligned} W_{\mathrm{Re}} &= [\,(ac\,\mathrm{e}^{2\tau}+bd) + \mathrm{e}^\tau(ad+bc)\cos\sigma\,]/D, \\ W_{\mathrm{Im}} &= [\,\mathrm{e}^\tau\sin\sigma\,]/D \;\geq 0 \\ D &= (c^2\,\mathrm{e}^{2\tau}+d^2) + 2\,cd\,\mathrm{e}^\tau\cos\sigma \;\geq (c\,\mathrm{e}^\tau - d)^2, \end{aligned}$$

which establish the facts that W maps the upper complex ρ-plane into itself and the real line into itself. Since any three of the real variables $z_1, z_2, \ldots, z_{n-1}, z_n$, encountered in Sect. 3.5.1, may be chosen at will, on account that three of real parameters a, b, c, d, such that $ad - bc = 1$, are arbitrary, a natural choice, for a scattering process, is $\tau_1 \to -\infty$, $\tau_n \to +\infty$, which correspond to $z_1 = 0$, $z_n = \infty$. Finally, z_{n-1} was chosen to be 1, corresponding to $\tau = 0$, obtaining the following restriction on the variables

$$0 = z_1 < z_2 < \cdots < z_{n-3} < z_{n-2} < 1 = z_{n-1} < z_n = \infty,$$

as appearing in (3.5.13).

3.22. This involves three terms:

$$\begin{aligned} &e_2\cdot k_1\,\langle\,0;k_3\,|\,e_3\cdot\alpha(1)\;\;\mathrm{e}^{k_2\cdot\alpha(1)^\dagger}\,\mathrm{e}^{-k_2\cdot\alpha(1)}\;e_1\cdot\alpha(1)^\dagger\,|\,0;k_1+k_2\,\rangle \\ &\quad = (e_2\cdot k_1\;e_1\cdot e_3 + e_2\cdot k_1\;e_3\cdot k_2\;e_1\cdot k_3)\,\langle\,0;k_3\,|\,0;k_1+k_2\,\rangle, \end{aligned}$$

$$\langle\, 0; k_3 \,|\, e_3 \cdot \alpha(1)\; e_2 \cdot \alpha(1)\; \mathrm{e}^{k_2\cdot\alpha(1)^\dagger}\, \mathrm{e}^{-k_2\cdot\alpha(1)}\; e_1 \cdot \alpha(1)^\dagger \,|\, 0; k_1 + k_2 \,\rangle$$
$$= e_3 \cdot k_2\; e_2 \cdot e_1 \,\langle\, 0; k_3 | 0; k_1 + k_2 \,\rangle,$$

$$\langle\, 0; k_3 \,|\, e_3 \cdot \alpha(1)\; e_2 \cdot \alpha(1)^\dagger\; \mathrm{e}^{k_2\cdot\alpha(1)^\dagger}\, \mathrm{e}^{-k_2\cdot\alpha(1)}\; e_1 \cdot \alpha(1)^\dagger \,|\, 0; k_1 + k_2 \rangle$$
$$= e_1 \cdot k_3\; e_3 \cdot e_2 \;\langle\, 0; k_3 \,|\, 0; k_1 + k_2 \rangle,$$

where we have used, in the process, the fact that $p^\mu |0;k\rangle = k^\mu |0;k\rangle$, $k_i^2 = 0$, that x generates momentum translation, and that $k_1 + k_2 + k_3 = 0$, and, in the process of derivation, the directions of momenta were finally chosen such that the conservation of momenta reads as just given. Adding these three terms lead immediately to the expression in (3.5.26).

3.23. The three-point function in question, up to an overall coupling parameter, may be rewritten as

$$e_{1\mu}\, e_{2\nu}\, e_{3\varrho} \left(\eta^{\mu\varrho} k_1^\nu + \eta^{\mu\nu} k_2^\varrho + \eta^{\nu\varrho} k_3^\mu \right). \qquad (^*)$$

It is sufficient to establish the anti-symmetry property of, say, under the exchange $(e_1, k_1) \leftrightarrow (e_2, k_2)$. Momentum conservation allows us to write, respectively, in reference to these corresponding momenta: $k_1 = -k_2 - k_3$, $k_2 = -k_1 - k_3$, $k_3 = -k_1 - k_2$. Using the properties $\mathrm{e}_i \cdot k_i = 0$, $i = 1, 2, 3$, the above three-point function may be rewritten as

$$-\; e_{1\mu}\, e_{2\nu}\, e_{3\varrho} \left(\eta^{\mu\varrho} k_3^\nu + \eta^{\mu\nu} k_1^\varrho + \eta^{\nu\varrho} k_2^\mu \right), \qquad (^{**})$$

which upon the exchange $(e_1, k_1) \leftrightarrow (e_2, k_2)$ it is transformed to

$$-\; e_{2\mu}\, e_{1\nu}\, e_{3\varrho} \left(\eta^{\mu\varrho} k_3^\nu + \eta^{\mu\nu} k_2^\varrho + \eta^{\nu\varrho} k_1^\mu \right).$$

By a mere relabeling of the Lorentz indices, note that this is just the initial three-point function of opposite sign.

3.24. The three-point function, in question, in bosonic string theory, with $\ell \to 0$, up to an overall coupling parameter, as in the previous problem, is given by

$$e_{1\mu}\, e_{2\nu}\, e_{3\varrho} \left(\eta^{\mu\varrho} k_1^\nu + \eta^{\mu\nu} k_2^\varrho + \eta^{\nu\varrho} k_3^\mu \right).$$

Using momentum conservation: $k_1 + k_2 + k_3 = 0$, we may rewrite k_1, k_2, k_3, respectively, within the brackets above as

$$k_1 = -\frac{1}{2}(k_2 + k_3 - k_1), \quad k_2 = -\frac{1}{2}(k_1 + k_3 - k_2), \quad k_3 = -\frac{1}{2}(k_1 + k_2 - k_3).$$

Upon using the properties

$$e_i \cdot k_i = 0,\ i = 1, 2, 3,$$

the above three-point function simply becomes

$$-\frac{1}{2}\, e_{1\mu}\, e_{2\nu}\, e_{3\varrho} \Big(\eta^{\mu\varrho}(k_3 - k_1)^{\nu} + \eta^{\mu\nu}(k_1 - k_2)^{\varrho} + \eta^{\nu\varrho}(k_2 - k_3)^{\mu}\Big),$$

and the proportionality of the latter to the superstring expression in (3.5.74) is now evident. If the reader has gone through solution of the previous problem, then note that the equality of the two equations (*), (**) in it imply they are also both equal to [(*) + (**)]/2, which is the equation derived above.

3.25. The equality follows by using, in the process, the basic equalities:

$$\begin{aligned}
\xi^{\mu\nu}\, \partial_\rho \xi_{\mu\lambda}\, \partial^\rho \xi^\lambda{}_\nu &= \frac{1}{2}\, \partial_\rho (\xi^{\mu\nu}\, \xi_{\mu\lambda} \partial^\rho\, \xi^\lambda{}_\nu),\\
\xi^{\mu\nu}\, \partial_\mu \xi^{\lambda\rho}\, \partial_\nu \xi_{\lambda\rho} &= \partial_\mu (\xi^{\mu\nu}\, \xi^{\lambda\rho}\, \partial_\nu \xi_{\lambda\rho}) - \xi^{\mu\nu}\, \xi^{\lambda\rho}\, \partial_\mu \partial_\nu \xi_{\lambda\rho},\\
\xi^{\mu\nu}\, \partial_\lambda \xi_\mu{}^\rho\, \partial_\rho \xi_\nu{}^\lambda &= \partial_\lambda (\xi^{\mu\nu}\, \xi_\mu{}^\rho\, \partial_\rho \xi_\nu{}^\lambda) - \xi^{\mu\nu}\, \partial_\nu \xi_{\rho\lambda}\, \partial^\lambda \xi_\mu{}^\rho,
\end{aligned}$$

where, for convenience, we have *relabeled* some of the indices in writing down these equations. The last two equalities easily follow. The first one in detail is given by

$$\begin{aligned}
\xi^{\mu\nu}\, \partial_\rho \xi_{\mu\lambda}\, \partial^\rho \xi^\lambda{}_\nu &= \partial_\rho \Big(\xi^{\mu\nu} \xi_{\mu\lambda} \partial^\rho \xi^\lambda{}_\nu\Big) - \partial_\rho \xi^{\mu\nu} \xi_{\mu\lambda} \partial^\rho \xi^\lambda{}_\nu\\
&= \partial_\rho \Big(\xi^{\mu\nu} \xi_{\mu\lambda} \partial^\rho \xi^\lambda{}_\nu\Big) - \xi^{\mu\nu}\, \partial_\rho \xi_{\mu\lambda}\, \partial^\rho \xi^\lambda{}_\nu
\end{aligned}$$

where in writing the last term, on the extreme right-hand side, we have exchanged the indices $\lambda \leftrightarrow \nu$, and when this last term is brought to the left-hand side of the equality, the result follows.

3.26. The functional derivatives in (3.6.15) as applied to the integral in (3.6.14) is explicitly given by

$$(2\pi)^D \delta^{(D)}(k_1 + k_2 + k_3)\, \frac{\kappa}{4}\, \mathrm{Sym}[\,\cdot\,],$$

where $[\,\cdot\,]$ is equal to

$$\begin{aligned}
&\Big[k_{3\mu_1} k_{3\mu_2} \eta_{\nu_1\varrho_1} \eta_{\nu_2\varrho_2} + k_{3\nu_1} k_{3\nu_2} \eta_{\mu_1\varrho_1} \eta_{\mu_2\varrho_2}\Big]\\
+\ &\Big[k_{2\varrho_1} k_{2\varrho_2} \eta_{\mu_1\nu_1} \eta_{\mu_2\nu_2} + k_{2\mu_1} k_{2\mu_2} \eta_{\nu_1\varrho_1} \eta_{\nu_2\varrho_2}\Big]\\
+\ &\Big[k_{1\nu_1} k_{1\nu_2} \eta_{\mu_1\varrho_1} \eta_{\mu_2\varrho_2} + k_{1\varrho_1} k_{1\varrho_2} \eta_{\mu_1\nu_1} \eta_{\mu_2\nu_2}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\left[k_{3\mu_1}k_{2\varrho_2}\eta_{\nu_1\varrho_1}\eta_{\mu_2\nu_2}+k_{2\varrho_1}k_{3\mu_2}\eta_{\mu_1\nu_1}\eta_{\nu_2\varrho_2}\right]\\
&+\left[k_{3\nu_1}k_{2\mu_2}\eta_{\mu_1\varrho_1}\eta_{\nu_2\varrho_2}+k_{2\mu_1}k_{3\nu_2}\eta_{\nu_1\varrho_1}\eta_{\mu_2\varrho_2}\right]\\
&+\left[k_{1\nu_1}k_{2\varrho_2}\eta_{\mu_1\varrho_1}\eta_{\mu_2\nu_2}+k_{2\varrho_1}k_{1\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_2\varrho_2}\right]\\
&+\left[k_{1\varrho_1}k_{2\mu_2}\eta_{\mu_1\nu_1}\eta_{\nu_2\varrho_2}+k_{2\mu_1}k_{1\varrho_2}\eta_{\nu_1\varrho_1}\eta_{\mu_2\nu_2}\right]\\
&+\left[k_{1\nu_1}k_{3\mu_2}\eta_{\mu_1\varrho_1}\eta_{\nu_2\varrho_2}+k_{3\mu_1}k_{1\nu_2}\eta_{\nu_1\varrho_1}\eta_{\mu_2\varrho_2}\right]\\
&+\left[k_{1\varrho_1}k_{3\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_2\varrho_2}+k_{3\nu_1}k_{1\varrho_2}\eta_{\mu_1\varrho_1}\eta_{\mu_2\nu_2}\right],
\end{aligned}
$$

where recall that "Sym" stands for the symmetrization operation to be applied to the expression following it over $\mu_1 \leftrightarrow \mu_2$, $\nu_1 \leftrightarrow \nu_2$, $\varrho_1 \leftrightarrow \varrho_2$. This gives precisely the vertex part in (3.6.10) with κ identified with the string coupling parameter through $\kappa = g_c/2$.

Index

α' providing a fundamental mass scale, 215
$\mathscr{F}$-type supersymmetry breaking, 160
(NS,NS) boundary condition, 244
(NS,R) boundary condition, 244
(R,NS) boundary condition, 244
(R,R) boundary condition, 244
3+1 formalism of general relativity, 67

action and worldsheet supersymmetry, 239
ADM formalism, 67
AdS/CFT correspondence, 191
anti-symmetric field, 234
Ashtekar variables, 66
asymptotically safe, 61

Barbero-Immirzi constant, 73
Bekenstein-Hawking Entropy formula, 3, 87
Bianchi identity, 18
black holes, 2
bosonic massless fields of superstrings, 272
bosonic strings, 198
  action in conformal gauge, 206
  action of, 200, 203
  boundary conditions, 208
  closed strings boundary conditions, 208
  open strings boundary conditions, 208
boundary conditions
  closed superstrings, 244
  open superstrings, 243
brane and the bulk, 279
branes, 224

Calabi-Yau compactification, 268
Chan-Paton degrees of freedom, 217, 222, 223
Chan-Paton states, 281
charge conjugation matrix, 103
  properties of, 106
chirality, 249
closed bosonic strings, 211
compactification, 224
  closed strings, 227
  into a circle, 225
  open strings, 228
compactified dimension, 224
conformal gauge, 204, 206
connection, the, 8, 11, 15
criterion of renormalizability, 97, 159
critical dimension of spacetime
  bosonic strings, 216, 221
  superstrings, 245, 247, 256
cylindrical functionals, 77

D p-branes, 209
D branes, 209, 230, 278, 279
  a Higgs-like mechanism, 283
  and anti-symmetric fields, 284
  massless versus massive vector particles, 283
  more than one, 281
D branes and massless particles, 279
D branes: spacetime filling, 209
densitized triad, 66, 70
DeWitt coefficients, 49
dilaton, 221
Dirac equation
  in 10 dimensions, 269
  in two dimensions, 235, 236
Dirichlet boundary conditions, 208

E.B. Manoukian, *Quantum Field Theory II*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-33852-1

duality of superstrings, 266
duality transformation between various superstrings theories, 267

Einstein tensor, 20
Einstein's equation of general relativity, 26
Einstein's theory of Gravitation and string theory, 304
Einstein-Hilbert action, 23
energy-momentum tensor, definition of, 26
Euler-Poincaré characteristic, 56, 81
expansions of products of anti-commuting components of a Majorana spinor, 109

fermion massless fields of superstrings, 269
fermionic vertex in string theory, 301
Fierz identities involving the charge conjugation matrix, 180
Fierz identity, the classical, 177, 181
fine tuning, 169

gamma matrices
  chiral representation, 326
  Dirac representation, 326
  in 10 dimensions, 327
  in 2 dimensions, 327
  Majorana representation, 327
  properties in 10 dimensions, 327
gauge transformation of a holonomy, 72
gauges
  DeWitt gauge, 42
  conformal gauge, 204
  harmonic , 29
  light-cone gauge, 204
gaugino, 126
Gauss-Bonnet Theorem in four dimensions, 56, 81
geodesic coordinate system, 23
gluinos, 164
Goldstino, 161
Goldstone, 161
Grassmann variables, 98
gravitational "electric" flux, 75
gravitational quadrupole radiation formula, 39
gravitino, 126
  in 10 dimensions, 258
graviton
  in 10 dimensions, 232
graviton vertex in string theory, 297
group $USp(N)$, 224
GSO projection method, 252

harmonic gauge, 29
Hawking radiation, 3, 87
heat kernel, 48
Heterotic superstrings, 264
hierarchy problem, 100, 101, 168, 169
Higgs bosons, 164
higgsino, 165
holographic principle, 193
holonomy, 66, 71
Hurwitz function, 247

intertwiners, 76, 77

Kalb-Raymond field, 221, 222
Kaluza-Klein excitation number, 227, 229
Kronecker delta, symmetric, 44

lapse, 67
left-chiral superfields
  left-chiral , 130
light-cone gauge, 204, 206, 241
light-cone variables, 204
links, 75
little hierarchy problem, 169
Loop Quantum Gravity, 66
  and a three dimensional quantum field theory, 66
Lorentz transformations, 329

Majorana condition of a spinor
  in two dimensions, 236
Majorana spinor, 103, 106
  definition of, 107
mass spectrum: closed bosonic strings, 219
mass spectrum: open bosonic strings, 215
massless fields of bosonic strings, 230
massless vector field
  in 10 dimensions, 230
metric
  Minkowski, xv, 329
Minimal Supersymmetric Model (MSSM), 165
Minkowski metric, xv, 329

Nambu-Goto action, 201
Neumann boundary conditions, 208

Neveu-Schwarz boundary condition, 243
nodes, 75

open bosonic strings, 209
open strings and D branes, 279
orbifold, 268
oriented closed superstrings, 263
oriented open superstrings, 263

parametrization of a string, 198
partial integration in curved spacetime, 91
photino, 126
polarization of gravitons, 32
principle of equivalence, 2, 8

quantum states of geometry of space, 77

Rarita-Schwinger field, 170, 174
Rarita-Schwinger massless field
  in 10 dimensions, 258, 271
Rarita-Schwinger particle, 126
Raymond boundary condition, 243
relativistic particle, 194
  action of, 195
relativistic superparticle, 196
  action of, 197
Ricci tensor, 19
Riemann curvature tensor, 17, 19
Riemann zeta function, 215

scalar curvature, 19
scalar superfield, 127
scattering theory for strings, 287
Schwinger-Dewitt technique, 45
shift vector, 67
sparticles, 97
spin connection, 171, 180
spin network, 76
spin-network state, 76
spontaneous symmetry breaking, 157
squark, 164
statistics, 99
string diagrams, 288, 295
string interactions, 287
string sigma model action, 201, 203
string tension, 201
string theory and Einstein's theory of gravitation, 304
string theory and Yang-Mills field theory, 308
super-covariant derivative in superspace, 111
super-Inhomogeneous Lorentz Transformation, 104, 105
super-Poincaré algebra, 99
super-Poincaré Transformations, 104, 118
  group properties, 105
supercurrent, 240
superderivative, 111
  left-handed, 131
superderivative, right-handed, 132
superderivatives
  products of, and summations formulae, 111
superdeterminant Sdet, 115
  product of supermatrices, 116
superfield
  right-chiral, 130
superfields, 126
  (scalar-) vector, 133, 144
  pure vector, 135, 146
  right-chiral, 143
  scalar, 139
  spinor, 138, 148
  left-chiral, 143
supergravity, 102, 170
  action of, 176
supermatrices, 115
supermultiplet, 98, 118
supermultiplets, 106
superpartners, 98
superpotential, 157, 164
superspace, 103
supersymmetric actions, 126
supersymmetric field theories, 126
  gauge theories, 163
  Maxwell field, 150
  supergravity, 176
  supersymmetrization of the standard model, 163
  Wess and Zumino, 153
  Yang-Mills field, 151
supersymmetric transformations in superspace, 104
supersymmetry, 97
  local version, 170
supersymmetry transformations
  group properties, 104
supertrace STr, 115
  definition of, 116
symplectic group $USp(N)$, 224

T-duality, 224, 230
tetrad, 10, 171
tetrad fields, 176

three-point function of massless vector particles in superstring theory, 303
three-point functions in string theory, 291, 293
torsion, 172
Type I superstrings, 262
Type IIA superstrings, 261, 263
Type IIB superstrings, 261, 263

unification of gauge couplings, 165
unoriented string theories, 222
unoriented strings, 222

valence of a node, 75
Veneziano amplitude, 189, 295
Veneziano amplitude in string theory, 294
vertices in string theory, 287, 292, 297
vierbein, 10, 171
volume element in superspace, invariance of, 117

Wess-Zumino gauge, 134
Weyl scale transformation, 204
winding number, 225
worldsheet, 198
  supersymmetry, 237

Yang & Mills-field theory and string theory, 308