Appendix A
Bi-tensors

In this appendix we define the notion of bi-tensor, the mathematical structure behind generally covariant Green's functions, and discuss some properties that are going to be useful for our purposes.

A.1 Definition

Just as higher-rank tensors are constructed using the tensor products (in the sense of fibre bundle theory) of vectors and covectors, bi-tensors can be constructed through some other type of tensor product of ordinary tensors. In order to formalize this construction, it is convenient to first remind some properties of ordinary tensors and of the corresponding tensor product.

A.1.1 Tensors

Manifold and Scalars

We start with a D-dimensional real differentiable manifold $\mathcal{M}$ with atlas $A_{\mathcal{M}}$, i.e. a set of pairs (U_i, f_i) of open sets $U_i \subset \mathcal{M}$ and homeomorphisms

$$\begin{aligned} f_i \quad &: \quad U_i \to \mathbb{R}^D \\ & \quad\; p \mapsto x_i^\mu, \quad \mu = 0, 1, \ldots, d, \end{aligned} \tag{A.1.1}$$

such that the U_i cover all of $\mathcal{M}$ and the transition functions from $\mathbb{R}^D$ to $\mathbb{R}^D$

$$f_{ij} \equiv f_i \circ f_j^{-1} : f_j\left(U_i \cap U_j\right) \to f_i\left(U_i \cap U_j\right), \tag{A.1.2}$$

E. Mitsou, *Infrared Non-local Modifications of General Relativity*,
Springer Theses, DOI 10.1007/978-3-319-31729-8

are smooth. Any continuous map $\phi : \mathcal{M} \to \mathbb{R}$ can then be represented by functions ϕ_i from $\mathbb{R}^D$ to $\mathbb{R}$ by pulling it back along some f_i^{-1}

$$\phi_i \equiv \phi \circ f_i^{-1} : f_i(U_i) \to \mathbb{R}. \tag{A.1.3}$$

A scalar field is then defined as such a map for which all ϕ_i are smooth. Inverting we get $\phi = \phi_i \circ f_i$, so on $U_i \cap U_j$ we have

$$\phi_i \circ f_i = \phi_j \circ f_j, \qquad \Rightarrow \qquad \phi_j = \phi_i \circ f_{ij}, \tag{A.1.4}$$

which is nothing but the transformation rule for a scalar function

$$\phi_i(x_i) = \phi_j(x_j), \tag{A.1.5}$$

under the coordinate transformation $x_i = f_{ij}(x_j)$. Since the U_i cover $\mathcal{M}$ and the f_i are homeomorphisms, we have that the ϕ_i functions fully determine ϕ. Finally, we note that the set of scalar fields, denoted by $C^\infty(\mathcal{M})$, forms an algebra whose addition and multiplication operations are the ordinary point-wise addition and multiplication in the target space $\mathbb{R}$.

Tangent Bundle

We then consider the tangent bundle $T^1\mathcal{M}$. This is a $2D$-dimensional differentiable manifold along with a continuous surjective map $\pi : T^1\mathcal{M} \to \mathcal{M}$, such that $\pi^{-1}(p) \simeq \mathbb{R}^D$ for all $p \in \mathcal{M}$. In fibre bundle language, $T^1\mathcal{M}$ is the total space, $\mathcal{M}$ is the base and $\mathbb{R}^D$ is the fibre. This structure means that $T^1\mathcal{M}$ *locally* looks like $\mathcal{M} \times \mathbb{R}^D$, i.e. every point of $\mathcal{M}$ has a neighbourhood $U_i \subset \mathcal{M}$ such that $\pi^{-1}(U_i) \simeq U_i \times \mathbb{R}^D$. As a matter of fact, once π is given, we restrict the atlas of $\mathcal{M}$ to the charts whose open set U_i is small enough to satisfy this condition, i.e. to the sets which "trivialize" the fibre bundle. The atlas of the tangent bundle $A_{T^1\mathcal{M}}$ is then constructed out of $A_\mathcal{M}$ as follows. For every chart $(U_i, f_i) \in A_\mathcal{M}$ we pick an open set $V_i \in T^1\mathcal{M}$ and an homeomorphism

$$\begin{aligned} g_i \quad : \quad & V_i \to \mathbb{R}^{2D} \\ & q \mapsto \left(x_i^\mu, k_i^\nu\right), \end{aligned} \tag{A.1.6}$$

such that

$$\pi(V_i) = U_i, \qquad \left(f_i \circ \pi \circ g_i^{-1}\right)(x_i, k_i) = x_i, \qquad \bigcup_i V_i = T^1\mathcal{M}, \tag{A.1.7}$$

i.e. g_i is such that the function associated to the projection map is the trivial projection onto the base coordinates.[1] Moreover, the set of charts (V_i, g_i) must be such that the corresponding transition functions

[1]The fact that we can cover $T^1\mathcal{M}$ with as many V_i as there are U_i is possible because we have demanded that $\pi^{-1}(U_i) \simeq U_i \times \mathbb{R}^D$.

$$g_{ij} \equiv g_i \circ g_j^{-1} : g_j \left(V_i \cap V_j\right) \to g_i \left(V_i \cap V_j\right), \tag{A.1.8}$$

read

$$g_{ij}(x_j, k_j) = \left(f_{ij}^{\mu}(x_j), \frac{\partial f_{ij}^{\nu}}{\partial x_j^{\rho}}(x_j)\, k_j^{\rho} \right). \tag{A.1.9}$$

A set of such pairs (V_i, g_i) constitutes an atlas $A_{T^1\mathcal{M}}$ for $T^1\mathcal{M}$. The appearance of the transition functions of $\mathcal{M}$ in the transformation of the fibre coordinates in (A.1.9) shows that the structure of $T^1\mathcal{M}$ is naturally induced by the one of $\mathcal{M}$.

A vector field X is a section of this bundle, i.e. a continuous map $X : \mathcal{M} \to T^1\mathcal{M}$ that is a right-inverse of the projection $\pi \circ X = \mathrm{id}_{\mathcal{M}}$. It can be expressed through local $\mathbb{R}^D \to \mathbb{R}^D$ functions, i.e. on U_i we define its pullback $X_i \equiv X \circ g_i^{-1}$, which by the property $\pi \circ X = \mathrm{id}_{\mathcal{M}}$ has the form

$$\begin{aligned} X_i \quad &: \quad f_i(U_i) \to g_i(V_i) \\ & \quad x_i^{\mu} \mapsto \left(x_i^{\mu}, X_i^{\nu}(x_i)\right), \end{aligned} \tag{A.1.10}$$

and the $X_i^{\mu}(x_i)$ are required to be smooth. As for the scalars, the full set of X_i functions fully determines X. Since the projection map is trivial, the relevant information ultimately lies in the functions $X_i^{\mu}(x)$ that are what one usually refers to as "the local components of the vector field" on U_i.[2] As in the case of scalar fields, we can invert $X = X_i \circ g_i$ and have that on $U_i \cap U_j$

$$X_i \circ g_i = X_j \circ g_j, \quad \Rightarrow \quad X_j = X_i \circ g_{ij}, \tag{A.1.11}$$

which, given (A.1.9), translates into the well-known rule

$$X_i^{\mu}(x_i) = \frac{\partial x_i^{\mu}}{\partial x_j^{\nu}}(x_j)\, X_j^{\nu}(x_j), \tag{A.1.12}$$

under the coordinate transformation $x_i = f_{ij}(x_j)$. The set of sections, denoted by $\Gamma(T^1\mathcal{M})$, forms an $C^{\infty}(\mathcal{M})$-vector space, whose addition and multiplication by a $\phi \in C^{\infty}(\mathcal{M})$ operations are defined on each U_i through the functions X_i^{μ}, which then determine the resulting vector field.[3] We have that if X_i^{μ}, Y_i^{μ} and ϕ_i are the local functions associated to X, Y and ϕ, respectively, then the local functions of $X + Y$ and ϕX are given by $X_i^{\mu} + Y_i^{\mu}$ and $\phi_i X_i^{\mu}$.[4]

---

[2]The advantage of the section representation is that it is global and thus unique, while the $X_i^{\mu}(x)$ information is local and contains as many functions as the number of U_i that are needed to cover $\mathcal{M}$.

[3]Indeed, we cannot define these operations using directly the maps X and Y because their target space is not a space of numbers.

[4]Note that scalar fields can also be expressed in this fibre bundle language as sections of $T^0\mathcal{M}$. The base is still $\mathcal{M}$, the fibre is just $\mathbb{R}$, the transition maps are trivial $\xi_{ij}(x_i, k_j) = (f_{ij}^{\mu}(x_i), k_j)$ and the scalar fields are sections which in local coordinates are given by functions $x_i^{\mu} \mapsto \left(x_i^{\mu}, \phi_i(x_i)\right)$.

At this point we can make contact with the alternative definition of a vector field which is as a derivation on $C^\infty(\mathcal{M})$, i.e. an $\mathbb{R}$-linear operator $D_X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ obeying the Leibniz rule

$$D_X\left(\alpha\phi + \beta\phi'\right) = \alpha D_X\phi + \beta D_X\phi', \quad D_X\left(\phi\phi'\right) = \left(D_X\phi\right)\phi' + \phi D_X\phi', \tag{A.1.13}$$

where α, β are real constants. Indeed, these properties fully determine D_X: if ϕ_i denotes the local functions of ϕ then the local functions of $D_X\phi$ are $X_i^\mu \partial_\mu \phi_i$, for some functions X_i^μ which we can identify with the fibre components of a section (A.1.10). Indeed, the fact that $D_X\phi \in C^\infty(\mathcal{M})$ implies

$$X_i^\mu(x_i)\frac{\partial}{\partial x_i^\mu} = X_j^\nu(x_j)\frac{\partial}{\partial x_j^\nu}, \tag{A.1.14}$$

which is precisely (A.1.12). It is a common abuse of terminology to call this derivation the "vector field", in which case the ∂_μ form a basis of vector fields.

Finally, anticipating the generalization to tensors, we must look for yet another operator interpretation of vector fields. To that end we can define the cotangent bundle $T_1\mathcal{M}$ following the same steps as we did for $T^1\mathcal{M}$, only this time with coordinates $\left(x_i^\mu, k_{i\nu}\right)$ and with transition functions obeying

$$g_{ij}(x_j, k_j) = \left(f_{ij}^\mu(x_j), \frac{\partial f_{ji}^\rho}{\partial x_i^\nu}(f_{ij}(x_j))\, k_\rho \right). \tag{A.1.15}$$

A covector α is then a section of $T_1\mathcal{M}$, and has a natural action as a linear functional $\alpha : \Gamma(T^1\mathcal{M}) \to C^\infty(\mathcal{M})$. Indeed, its associated local functions $\alpha_i \equiv \alpha \circ \psi_i^{-1}$

$$\begin{aligned} \alpha_i \quad : \quad & f_i(U_i) \to g_i(V_i) \\ & x_i^\mu \mapsto \left(x_i^\mu, \alpha_{i\nu}(x_i)\right), \end{aligned} \tag{A.1.16}$$

transform as

$$\alpha_{i\mu}(x_i) = \frac{\partial x_j^\nu}{\partial x_i^\mu}(x_i(x_j))\, \alpha_{j\nu}(x_j), \tag{A.1.17}$$

under the coordinate transformation $x_i = f_{ij}(x_j)$, and thus

$$\phi_i(x_i) \equiv \alpha_{i\mu}(x_i) X_i^\mu(x_i) = \alpha_{j\mu}(x_j) X_j^\mu(x_j) \equiv \phi_j(x_j) \tag{A.1.18}$$

(Footnote 4 continued)
The addition and multiplication operations on $\Gamma(T^0\mathcal{M})$ must then be defined through the local functions.

transforms as the local function on U_i of a scalar. This defines the interior product $X \cdot \alpha \in C^\infty(\mathcal{M})$. Just as ∂_μ provides a basis for vector fields, because of its transformation properties, so does the differential $\mathrm{d}x^\mu$ provide a basis for covectors

$$\mathrm{d}x_i^\mu = \frac{\partial x_i^\mu}{\partial x_j^\nu}\,\mathrm{d}x_j^\nu, \tag{A.1.19}$$

and we have the analogue of (A.1.14)

$$\alpha_{i\mu}(x_i)\,\mathrm{d}x_i^\mu = \alpha_{j\nu}(x_j)\,\mathrm{d}x_j^\nu. \tag{A.1.20}$$

Alternatively, the vectors can also be interpreted as linear functionals on $\Gamma(T_1(\mathcal{M}))$. It is this dual linear operator interpretation that generalizes straightforwardly to the case of higher-rank tensors.

Tensor Bundle

Having defined $T^1\mathcal{M}$ and $T_1\mathcal{M}$ we can construct the tensor product bundle

$$T_m^n\mathcal{M} \equiv T_1\mathcal{M} \underbrace{\otimes \cdots \otimes}_{m \text{ times}} T_1\mathcal{M} \otimes T^1\mathcal{M} \underbrace{\otimes \cdots \otimes}_{n \text{ times}} T^1\mathcal{M}. \tag{A.1.21}$$

The $\otimes$ operation means that one takes the tensor product of the fibres at each point $p \in \mathcal{M}$, but keeps the same base manifold $\mathcal{M}$. The fibre coordinates will therefore take values in the vector space generated by $k_{\mu_1}^1 \ldots k_{\mu_m}^m k_1^{\nu_1} \ldots k_n^{\nu_n}$, thus corresponding to $k_{\mu_1\ldots\mu_m}^{\nu_1\ldots\nu_n}$ coordinates. So $T_m^n\mathcal{M}$ is a $(D + D^{n+m})$-dimensional differentiable manifold with a projection map $\pi : T_m^n\mathcal{M} \to \mathcal{M}$ and fibre $\pi^{-1}(p) \simeq \mathbb{R}^{D^{n+m}}$. The set of charts (g_i, V_i)

$$\begin{aligned} g_i \quad &: \quad V_i \to \mathbb{R}^{D+D^{n+m}} \\ & \qquad q \mapsto \left(x_i^\mu, k_{i\mu_1\ldots\mu_m}^{\nu_1\ldots\nu_n}\right), \end{aligned} \tag{A.1.22}$$

is such that

$$\pi(V_i) = U_i\,, \quad \left(f_i \circ \pi \circ g_i^{-1}\right)(x_i, k_i) = x_i, \quad \bigcup_i V_i = T_m^n\mathcal{M}, \tag{A.1.23}$$

and the transition functions $g_{ij} \equiv g_i \circ g_j^{-1}$ are of the form

$$g_{ij}(x_j, k_j) = \left(f_{ij}^\mu(x), \frac{\partial f_{ji}^{\alpha_1}}{\partial x_i^{\mu_1}}(f_{ij}(x_j)) \ldots \frac{\partial f_{ji}^{\alpha_m}}{\partial x_i^{\mu_m}}(f_{ij}(x_j))\, \frac{\partial f_{ij}^{\nu_1}}{\partial x_j^{\beta_n}}(x_j) \ldots \frac{\partial f_{ij}^{\nu_n}}{\partial x_j^{\beta_n}}(x_j)\, k_{\alpha_1\ldots\alpha_m}^{\beta_1\ldots\beta_n} \right). \tag{A.1.24}$$

A tensor of rank (n, m) is then a section of $T^n_m\mathcal{M}$, i.e. a map $T : \mathcal{M} \to T^n_m\mathcal{M}$ that is a right-inverse of the projection $\pi \circ T = \mathrm{id}_{\mathcal{M}}$. Thus, defining the local functions $T_i \equiv T \circ f_i^{-1}$ we have

$$\begin{aligned} T_i \quad : \quad & f_i(U_i) \to g_i(V_i) \\ & x_i^\mu \mapsto \left(x_i^\mu, T_{i\mu_1\ldots\mu_m}^{\ \nu_1\ldots\nu_n}(x_i)\right), \end{aligned} \tag{A.1.25}$$

and the fibre components $T_{i\mu_1\ldots\mu_m}^{\ \nu_1\ldots\nu_n}(x_i)$, given (A.1.24), transform as

$$T_{i\mu_1\ldots\mu_m}^{\ \nu_1\ldots\nu_n}(x_i) = \frac{\partial x_j^{\alpha_1}}{\partial x_i^{\mu_1}}(x_i(x_j))\ldots\frac{\partial x_j^{\alpha_m}}{\partial x_i^{\mu_m}}(x_i(x_j))\frac{\partial x_i^{\nu_1}}{\partial x_j^{\beta_n}}(x_j)\ldots\frac{\partial x_i^{\nu_n}}{\partial x_j^{\beta_n}}(x_j)\, T_{j\alpha_1\ldots\alpha_m}^{\ \beta_1\ldots\beta_n}(x_j), \tag{A.1.26}$$

under the coordinate transformation $x_i = f_{ij}(x_j)$. As in the case of (co-)vectors, by a slight abuse of language, one usually calls $T_{i\mu_1\ldots\mu_m}^{\ \nu_1\ldots\nu_n}(x_i)$ the components of the tensor field. The addition and multiplication by a scalar operations are defined through the local functions just as in the case of vectors. We can now include the tensor product among the operations of interest, which is also defined through the local functions. If $T \in \Gamma(T^n_m\mathcal{M})$ and $S \in \Gamma(T^s_r\mathcal{M})$, then $T \otimes S \in \Gamma(T^{n+s}_{m+r}\mathcal{M})$ is given by $(T \otimes S)_i \equiv (T \otimes S) \circ f_i^{-1}$

$$(T \otimes S)_i(x_i) = \left(x_i^\mu, T_{i\mu_1\ldots\mu_m}^{\ \nu_1\ldots\nu_n}(x_i) S_{i\mu_{m+1}\ldots\mu_{m+r}}^{\ \nu_{n+1}\ldots\nu_{n+s}}(x_i)\right). \tag{A.1.27}$$

Finally, using the $X = X^\mu\partial_\mu$ and $\alpha = \alpha_\mu \mathrm{d}x^\mu$ interpretation of (co-)vectors, the "basis of $\Gamma(T^n_m\mathcal{M})$" in this case is the tensor product

$$\mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_m} \otimes \partial_{\nu_1} \otimes \cdots \otimes \partial_{\nu_n}, \tag{A.1.28}$$

where here $\otimes$ means "multiplication and evaluation at the same point of $\mathcal{M}$", so that

$$\begin{aligned} T_{i\mu_1\ldots\mu_m}^{\ \nu_1\ldots\nu_n}(x_i)\, \mathrm{d}x_i^{\mu_1} \otimes \cdots \otimes \mathrm{d}x_i^{\mu_m} \otimes \frac{\partial}{\partial x_i^{\nu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_i^{\nu_n}} = T_{j\mu_1\ldots\mu_m}^{\ \nu_1\ldots\nu_n}(x_j)\, \mathrm{d}x_j^{\mu_1} \\ \otimes \cdots \otimes \mathrm{d}x_j^{\mu_m} \otimes \frac{\partial}{\partial x_j^{\nu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_j^{\nu_n}}. \end{aligned} \tag{A.1.29}$$

A.1.2 Bi-tensors

Bi-manifold and Bi-scalars

We now wish to construct tensor-like fields that depend on two points of $\mathcal{M}$. We therefore begin by defining the Cartesian product $\mathcal{M}^2 \equiv \mathcal{M}_{\mathrm{L}} \times \mathcal{M}_{\mathrm{R}}$, where these are two copies of $\mathcal{M}$ that we will call the "left" and "right" ones. In the above product it is understood that $\mathcal{M}^2$ has the product topology and atlas $A_{\mathcal{M}_{\rm L}} \times A_{\mathcal{M}_{\rm R}}$, i.e. the one made of the pairs

$$\left(U_{i|j}, f_{i|j}\right) \equiv \left(U_i \times U_j, f_i \times f_j\right). \tag{A.1.30}$$

Thus, a chart on $\mathcal{M}^2$ is a pair of open sets, one on $\mathcal{M}_{\rm L}$ and one on $\mathcal{M}_{\rm R}$, followed by a pair of functions that coordinatize each open set independently. The product topology gives

$$U_{i|j} \cap U_{k|l} \equiv \left(U_i \times U_j\right) \cap (U_k \times U_l) = (U_i \cap U_k) \times \left(U_j \cap U_l\right), \tag{A.1.31}$$

and the same for the union operation, and the transition functions decompose

$$f_{ik|jl} \equiv f_{i|k} \circ f_{j|l}^{-1} = \left(f_i \circ f_k^{-1}\right) \times \left(f_j \circ f_l^{-1}\right), \tag{A.1.32}$$

so that these two manifolds do not "see" each other, i.e. one can perform coordinate transformations on each one of them independently. A bi-scalar field is a map $\phi : \mathcal{M}^2 \to \mathbb{R}$ such that the functions

$$\begin{aligned} \phi_{i|j} \equiv \phi \circ f_{i|j}^{-1} \quad : \quad & f_i(U_i) \times f_j(U_j) \to \mathbb{R} \\ & \left(x_i^\mu, y_j^\nu\right) \mapsto \phi_{i|j}\left(x_i, y_j\right), \end{aligned} \tag{A.1.33}$$

are smooth in both arguments. Following the same steps as for the ordinary scalar field, its transformation under *independent* coordinate transformations $x_i = f_{ik}(x_k)$ and $y_j = f_{jl}(y_l)$ is thus

$$\phi_{i|j}(x_i, y_j) = \phi_{k|l}(x_k, y_l). \tag{A.1.34}$$

Bi-tensor Bundle

We can now define the bi-tensor bundle $B_m^n|_r^s\mathcal{M}$ as follows. It is a differentiable fibre bundle of dimension $2D + D^{n+m+r+s}$, based on $\mathcal{M}^2$, with projection map $\pi_B : B_m^n|_r^s\mathcal{M} \to \mathcal{M}^2$ and fibre $\pi_B^{-1}(p_{\rm L}, p_{\rm R}) \simeq \mathbb{R}^{D^{n+m+r+s}}$. Its atlas $A_{B_m^n|_r^s\mathcal{M}}$ is constructed as follows. For every pair $\left(U_{i|j}, f_{i|j}\right) \in A_{\mathcal{M}^2}$, we pick an open set $V_{i|j} \subset B_m^n|_r^s\mathcal{M}$ and a homeomorphism

$$\begin{aligned} g_{i|j} \quad : \quad & V_{i|j} \to \mathbb{R}^{2D+D^{n+m+r+s}} \\ & q \mapsto \left(x_i^\mu, y_j^\nu, k_{i\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\,|_{j\rho_1\ldots\rho_r}^{\;\;\sigma_1\ldots\sigma_s}\right), \end{aligned} \tag{A.1.35}$$

such that

$$\pi_B(V_{i|j}) = U_{i|j}, \quad \left(f_{i|j} \circ \pi_B \circ g_{i|j}^{-1}\right)\left(x_i, y_j, k_{i|j}\right) = (x_i, y_j), \quad \bigcup_{i,j} V_{i|j} = B_m^n|_r^s\mathcal{M}, \tag{A.1.36}$$

and the transition functions $g_{ik|jl} \equiv g_{i|j} \circ g_{k|l}^{-1}$ are of the form

$$g_{ik|jl}\left(x_k, y_l, k_{k|l}\right) = \Bigg(f_{ik}^{\mu}(x_k), f_{jl}^{\nu}(y_l), \frac{\partial f_{ki}^{\alpha_1}}{\partial x_i^{\mu_1}}(f_{ik}(x_k)) \dots \frac{\partial f_{ki}^{\alpha_m}}{\partial x_i^{\mu_m}}(f_{ik}(x_k)) \\ \frac{\partial f_{ik}^{\nu_1}}{\partial x_k^{\beta_n}}(x_k) \dots \frac{\partial f_{ik}^{\nu_n}}{\partial x_k^{\beta_n}}(x_k) \frac{\partial f_{lj}^{\gamma_1}}{\partial y_j^{\rho_1}}(f_{jl}(y_l)) \dots \frac{\partial f_{lj}^{\gamma_m}}{\partial y_j^{\rho_m}}(f_{jl}(y_l)) \\ \frac{\partial f_{jl}^{\sigma_1}}{\partial y_l^{\delta_n}}(y_l) \dots \frac{\partial f_{jl}^{\sigma_n}}{\partial y_l^{\delta_n}}(y_l)\, k_{k\alpha_1\dots\alpha_m}^{\ \beta_1\dots\beta_n}\big|_{l\gamma_1\dots\gamma_r}^{\ \delta_1\dots\delta_s} \Bigg). \tag{A.1.37}$$

Note that we have used a column to distinguish between the two types of indices, i.e. the "left" ones mixing with Jacobians evaluated at the left point x, and the "right" ones mixing with Jacobians evaluated at the right point y. A bi-tensor G would then be a section of $B_m^n|_r^s\mathcal{M}$, i.e. a continuous map $G : \mathcal{M}^2 \to B_m^n|_r^s\mathcal{M}$ that is a right-inverse for the projection map $\pi_B \circ G = \mathrm{id}_{\mathcal{M}^2}$. Thus, defining the functions $G_{i|j} \equiv G \circ f_{i|j}^{-1}$, in local coordinates

$$\begin{aligned} G_{i|j} \quad : \quad & f_{i|j}(U_{i|j}) \to g_{i|j}(V_{i|j}) \\ & \left(x_i^{\mu}, y_j^{\nu}\right) \mapsto \left(x_i^{\mu}, y_j^{\nu}, G_{i\mu_1\dots\mu_m}^{\ \nu_1\dots\nu_n}\big|_{j\rho_1\dots\rho_r}^{\ \sigma_1\dots\sigma_s}(x_i, y_j)\right), \end{aligned} \tag{A.1.38}$$

the local components $G_{i\mu_1\dots\mu_m}^{\ \nu_1\dots\nu_n}|_{j\rho_1\dots\rho_r}^{\ \sigma_1\dots\sigma_s}(x_i, y_j)$, given (A.1.37), transform as

$$G_{i\mu_1\dots\mu_m}^{\ \nu_1\dots\nu_r}\big|_{j\rho_1\dots\rho_r}^{\ \sigma_1\dots\sigma_s}(x_i, y_j) = \frac{\partial x_k^{\alpha_1}}{\partial x_i^{\mu_1}}(x_i(x_k)) \dots \frac{\partial x_k^{\alpha_m}}{\partial x_i^{\mu_m}}(x_i(x_k)) \frac{\partial x_i^{\nu_1}}{\partial x_k^{\beta_1}}(x_k) \dots \frac{\partial x_i^{\nu_n}}{\partial x_k^{\beta_n}}(x_k) \\ \frac{\partial y_l^{\gamma_1}}{\partial y_j^{\rho_1}}(y_j(y_l)) \dots \frac{\partial y_l^{\gamma_r}}{\partial y_j^{\rho_r}}(y_j(y_l)) \frac{\partial y_j^{\sigma_1}}{\partial y_l^{\delta_1}}(y_l) \dots \frac{\partial y_j^{\sigma_s}}{\partial y_l^{\delta_s}}(y) \\ \times G_{k\alpha_1\dots\alpha_m}^{\ \beta_1\dots\beta_r}\big|_{l\gamma_1\dots\gamma_r}^{\ \delta_1\dots\delta_s}(x_k, y_l), \tag{A.1.39}$$

under the independent coordinate transformations $x_i = f_{ik}(x_k)$ and $y_j = f_{jl}(y_l)$. In order to express such an object in the notation (A.1.28) we need to define a new kind of product. We thus use the notation

$$G_{\mu_1\dots\mu_m}^{\nu_1\dots\nu_r}\big|_{\rho_1\dots\rho_r}^{\sigma_1\dots\sigma_s}(x, y) \left(\mathrm{d}x^{\mu_1} \otimes \dots \otimes \mathrm{d}x^{\mu_m} \otimes \frac{\partial}{\partial x^{\nu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\nu_n}} \right) \\ \otimes_B \left(\mathrm{d}y^{\rho_1} \otimes \dots \otimes \mathrm{d}y^{\rho_m} \otimes \frac{\partial}{\partial y^{\sigma_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{\sigma_s}} \right), \tag{A.1.40}$$

and dub $\otimes_B$ the "bi-tensor" product, which means that the tensors on each side are evaluated on independent points of $\mathcal{M}$. This notation is again consistent with the transformation rule (A.1.39) given the way the basis transforms. It is then straightforward to generalize the concept to bi-tensor densities and also to "tri-tensors", "quadri-tensors" etc., by taking more and more bi-tensor products.

A.1.3 Bi-tensor Calculus

Differentiation

Since a bi-tensor basically "lives" on two points of the manifold, and their corresponding tangent tensor spaces, it can be covariantly differentiated at each point separately. Indeed, the transformation (A.1.39) implies that one can apply covariant derivatives at each point separately, and with respect to the corresponding indices only, because x and y are independent. One must simply let the notation reflect the choice of point, so we will use $\nabla_{\rm L}$ and $\nabla_{\rm R}$ for the operators on bi-tensors, while we will use $\nabla_\mu|$ and $\nabla|_\mu$ for their representation on the bi-tensor components. For example, given $G \in \Gamma(B^1_0|^1_1\mathcal{M})$,

$$\nabla_\mu|\, G^\nu|^\rho_\sigma(x,y) \equiv \frac{\partial}{\partial x^\mu}\, G^\nu|^\rho_\sigma(x,y) + \Gamma^\nu{}_{\alpha\mu}(x)\, G^\alpha|^\rho_\sigma(x,y), \tag{A.1.41}$$

are the local components of $\nabla_{\rm L} G \in \Gamma(B^1_1|^1_1\mathcal{M})$, while

$$\nabla|_\mu\, G^\nu|^\rho_\sigma(x,y) \equiv \frac{\partial}{\partial y^\mu}\, G^\nu|^\rho_\sigma(x,y) + \Gamma^\rho{}_{\alpha\mu}(y)\, G^\nu|^\alpha_\sigma(x,y) - G^\nu|^\rho_\alpha(x,y)\Gamma^\alpha{}_{\sigma\mu}(y), \tag{A.1.42}$$

are the local components of $\nabla_{\rm R} G \in \Gamma(B^1_0|^1_2\mathcal{M})$. Pay attention to the various dependencies and index contractions. With this additional information the commutator of covariant derivatives generalizes accordingly. We have for instance

$$\left[\nabla_\mu|, \nabla|_\nu\right] G^\rho|^\sigma_\tau(x,y) = 0, \tag{A.1.43}$$

$$\left[\nabla_\mu|, \nabla_\nu|\right] G^\rho|^\sigma_\tau(x,y) = R^\rho{}_{\alpha\mu\nu}(x)\, G^\alpha|^\sigma_\tau(x,y), \tag{A.1.44}$$

$$\left[\nabla|_\mu, \nabla|_\nu\right] G^\rho|^\sigma_\tau(x,y) = R^\sigma{}_{\alpha\mu\nu}(y)\, G^\rho|^\alpha_\tau(x,y) - G^\rho|^\sigma_\alpha(x,y) R^\alpha{}_{\tau\mu\nu}(y). \tag{A.1.45}$$

Integration on $\mathcal{M}$

Remember that integration is defined on manifolds by splitting the integral through a partition of unity subordinate to the open cover U_i and evaluating the integral on each U_i using the local functions. More precisely, let us denote by I the set of indices indexing the open sets U_i of $A_\mathcal{M}$. We can then pick a locally finite covering $I' \subset I$, i.e. a subset $\{U_i\}_{i\in I'}$ that still covers $\mathcal{M}$ but such that for every $p \in \mathcal{M}$ there exists only a finite number of U_i for which $p \in U_i$. Smooth manifolds which admit such locally finite refinements are called "paracompact". Then, a partition of unity subordinate to $\{U_i\}_{i\in I'}$ is the attribution of a scalar ρ_i to each U_i with $i \in I'$ such that

- $\mathrm{supp}(\rho_i) \subset U_i$,
- $\sum_{i \in I'} \rho_i = 1$.

The integral of a scalar field ϕ over $\mathcal{M}$ is then defined as follows. One first needs to define a measure, i.e. a D-form ω such that the local density functions

$$\omega_{i\,\mu_1 \ldots \mu_D}(x_i) = \omega_i(x_i)\varepsilon_{\mu_1 \ldots \mu_D}, \tag{A.1.46}$$

have positive definite sign $\omega_i(x_i) > 0$. Given a metric tensor g, the physically sensible choice is $\omega_i(x_i) = \sqrt{-g_i(x_i)}$ where $g_i(x_i)$ is the determinant of $g_{i\mu\nu}(x_i)$. We then have that the integral of ϕ is given by

$$\int_{\mathcal{M}} \omega\, \phi \equiv \sum_{i \in I'} \int \mathrm{d}^D x_i\, \rho_i(x_i)\omega_i(x_i)\phi_i(x_i). \tag{A.1.47}$$

where ρ_i, ω_i and ϕ_i are the local functions of ρ, ω and ϕ on U_i, respectively. The sum in the right-hand side is well defined because for each $i \in I'$ only but a finite number of elements are non-zero.

The generalization to bi-tensors is straightforward. It relies on the fact that if ρ_i^{L} and ρ_i^{R} form partitions of unity of $\mathcal{M}_{\mathrm{L}}$ and $\mathcal{M}_{\mathrm{R}}$ subordinate to their respective atlases, then $\rho_i^{\mathrm{L}} \rho_j^{\mathrm{R}}$ forms a partition of unity of $\mathcal{M}^2$ subordinate to $\left\{U_{i|j}\right\}_{(i,j) \in I'^2}$. As for differentiation, one can then define the integration on $\mathcal{M}_{\mathrm{L}}$ and $\mathcal{M}_{\mathrm{R}}$ independently. For example, given $G \in \Gamma(B^n_m|^0_0\mathcal{M})$, which is a scalar on $\mathcal{M}_{\mathrm{R}}$, one can define

$$\int_{\mathcal{M}_{\mathrm{R}}} \omega\, G, \tag{A.1.48}$$

by specifying the local functions on U_i

$$\left(\int_{\mathcal{M}_{\mathrm{R}}} \omega\, G\right)^{\nu_1 \ldots \nu_n}_{i\,\mu_1 \ldots \mu_m}(x_i) \equiv \sum_{j \in I'} \int \mathrm{d}^D y_j\, \rho_j^{\mathrm{R}}(y_j)\, \omega(y_j)\, G^{\nu_1 \ldots \nu_n}_{i\,\mu_1 \ldots \mu_m}|_j(x_i, y_j). \tag{A.1.49}$$

Given the independence of the two space-time points, the above object is clearly an element of $\Gamma(T^n_m \mathcal{M}_{\mathrm{L}})$. As is usual in the literature, we will use a slightly less rigorous notation to describe such integrals, i.e. one that does not care about how the integral is partitioned or about the fact that usually several coordinate charts are needed. In this case for instance we can write

$$\left(\int_{\mathcal{M}_{\mathrm{R}}} \omega\, G\right)^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m}(x) \equiv \int_{\mathcal{M}} \mathrm{d}^D y\, \omega(y)\, G^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m}|(x, y), \tag{A.1.50}$$

so that one can see with respect to which manifold we are integrating. Finally, we define the following notations. For $T \in \Gamma(T^n_m \mathcal{M})$ and $T' \in \Gamma(T^m_n \mathcal{M})$,

$$(G \cdot_\omega T)^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}(x) = \int_{\mathcal{M}} \mathrm{d}^D y \, \omega(y) \, G^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_m}|^{\sigma_1 \dots \sigma_m}_{\rho_1 \dots \rho_n}(x, y) \, T^{\rho_1 \dots \rho_n}_{\sigma_1 \dots \sigma_m}(y), \qquad (A.1.51)$$

$$\left(T' \cdot_\omega G\right)^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}(y) = \int_{\mathcal{M}} \mathrm{d}^D x \, \omega(x) \, T'^{\sigma_1 \dots \sigma_m}_{\rho_1 \dots \rho_n}(x) \, G^{\rho_1 \dots \rho_n}_{\sigma_1 \dots \sigma_m}|^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}(x, y), \qquad (A.1.52)$$

are also elements of $\Gamma(T^n_m \mathcal{M})$ and $\Gamma(T^m_n \mathcal{M})$, respectively. We thus have that, for any measure ω, the elements of $\Gamma(B^n_m|^m_n \mathcal{M})$ can be thought of as left-$\mathbb{R}$-linear endomorphisms of $\Gamma(T^n_m \mathcal{M})$ and right-$\mathbb{R}$-linear endomorphisms of $\Gamma(T^m_n \mathcal{M})$. Finally, since in the physically relevant cases $\omega(x) = \sqrt{-g(x)}$, a dot without argument means $\cdot_{\sqrt{-g}}$.

A.2 Bi-tensor Distributions

The notion of bi-tensor combined with the notion distribution, ultimately allows to define the notion of functional analysis on manifolds. The most interesting cases for us are the Dirac delta bi-tensor and the Green's bi-tensor. Disclaimer: here we will only focus on the aspects of the generalization of these notions to curved space-time. We will not concern ourselves with the functional analysis side of the field, i.e. we will not care about domains, continuity and convergence issues that are nevertheless crucial aspects of the theory of distributions.

A.2.1 The Dirac Delta Bi-tensor

The Dirac delta bi-tensor is defined, as the ordinary Dirac delta, by its distributional properties. The $\binom{n}{m}$-Dirac delta bi-tensor associated to the measure ω is the bi-tensor $\Delta \in \Gamma(B^n_m|^m_n \mathcal{M})$ satisfying

$$\Delta \cdot_\omega T = T, \quad T' \cdot_\omega \Delta = T', \qquad (A.2.1)$$

for all $T \in \Gamma\left(T^n_m \mathcal{M}\right)$ and $T' \in \Gamma\left(T^m_n \mathcal{M}\right)$. This uniquely determines its associated local functions, which are of course going to be related to the Dirac delta function. The latter transforms as a scalar density of weight -1 under a diffeomorphism $x' = f(x)$. Indeed,

$$1 \equiv \int \mathrm{d}^D x' \, \delta^{(D)}(x') = \int \mathrm{d}^D x \, \det\left[\frac{\partial f}{\partial x}(x)\right] \delta^{(D)}(f(x)) = \int \mathrm{d}^D x \, \delta^{(D)}(x), \qquad (A.2.2)$$

so

$$\delta^{(D)}(f(x)) = \det\left[\frac{\partial f}{\partial x}(x)\right]^{-1} \delta^{(D)}(x). \qquad (A.2.3)$$

Thus the combination $\delta^{(D)}(x)/\omega(x)$ is a scalar. Repeating with the shifted diffeomorphism $f(x) \to f(x) - f(y)$, we get

$$\delta^{(D)}(f(x) - f(y)) = \det \left[\frac{\partial f}{\partial x}(x) \right]^{-1} \delta^{(D)}(x - y), \tag{A.2.4}$$

so that

$$\frac{\delta^{(D)}(x - y)}{\omega(x)} = \frac{\delta^{(D)}(x - y)}{\omega(y)} = \frac{\delta^{(D)}(x - y)}{\sqrt{\omega(x)}\sqrt{\omega(y)}}, \tag{A.2.5}$$

are all the same scalar *when x and y are coordinates of the same chart*, and thus transform together under the same transition functions. We can now make the link with the Dirac delta bi-tensor. To fully determine the latter it suffices to determine its local functions $\Delta_{i|j} \equiv \Delta \circ f^{-1}_{i|j}$ on the open sets $U_{i|j}$. We then have that

$$\Delta_{i\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n}|_{j\rho_1 \dots \rho_n}^{\sigma_1 \dots \sigma_m}(x_i, y_j) = 0, \quad U_i \cap U_j = \emptyset \in \mathcal{M}, \tag{A.2.6}$$

while, if $U_i \cap U_j$ is non-empty in $\mathcal{M}$,

$$\begin{aligned} &\Delta_{i\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n}|_{j\rho_1 \dots \rho_n}^{\sigma_1 \dots \sigma_m}(x_i, y_j) \\ &= \frac{\partial f_{ji}^{\sigma_1}}{\partial x_i^{\mu_1}}(f_{ij}(y_j)) \dots \frac{\partial f_{ji}^{\sigma_m}}{\partial x_i^{\mu_m}}(f_{ij}(y_j)) \frac{\partial f_{ij}^{\nu_1}}{\partial y_j^{\rho_1}}(y_j) \dots \frac{\partial f_{ij}^{\nu_n}}{\partial y_j^{\rho_n}}(y_j) \frac{\delta^{(D)}(x_i - f_{ij}(y_j))}{\omega_i(x_i)}. \end{aligned} \tag{A.2.7}$$

The latter is obtained by considering the case $i = j$

$$\Delta_{i\mu_1 \dots \mu_m}^{\nu_1 \dots \nu_n}|_{i\rho_1 \dots \rho_n}^{\sigma_1 \dots \sigma_m}(x_i, y_i) = \delta_{\mu_1}^{\sigma_1} \dots \delta_{\mu_m}^{\sigma_m} \delta_{\rho_1}^{\nu_1} \dots \delta_{\rho_n}^{\nu_n} \frac{\delta^{(D)}(x_i - y_i)}{\omega_i(x_i)}, \tag{A.2.8}$$

and transforming the right coordinate y_i to y_j using the transition function f_{ij}. An important property for what follows is the one involving the left and right-differentiations

$$\nabla_{\rm L} \Delta = -\nabla_{\rm R} \Delta, \quad \Delta \cdot_\omega \nabla T = -(\nabla T) \cdot_\omega \Delta, \tag{A.2.9}$$

which is proved by convolution with test tensors and integration by parts.

A.2.2 Bi-tensor Green's Functions

Let $L[\nabla]$ denote a covariant differential operator acting on $\Gamma(T^n_m \mathcal{M})$, i.e. the space of $\binom{n}{m}$-tensors. In terms of local components we thus have[5]

[5]The bi-tensor notation here might appear misleading since L is made of differential operators acting on a single space-time point, but since it is an endomorphism on $\Gamma(T^n_m \mathcal{M})$, we can express

$$(LT)^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m} \equiv L^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n} T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}. \tag{A.2.11}$$

Note that because of the derivatives the kernel Ker$[L]$ is non-zero, i.e. there exists T such that $LT = 0$. There are therefore, roughly speaking, as many inverses of L as there are elements in Ker$[L]$. A Green's function for L is a bi-tensor $G \in \Gamma(B^n_m|^m_n \mathcal{M})$ such that its local functions satisfy

$$L^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}[\nabla_{\rm L}](x)\, G^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x,y) = \Delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x,y), \tag{A.2.12}$$

where here it is the Dirac delta bi-tensor associated with $\sqrt{-g}$ that is being used. Given such a bi-tensor G, we have that the operator

$$L^{-1}_G T \equiv G \cdot T, \tag{A.2.13}$$

is a *right*-inverse of L, i.e.

$$LL^{-1}_G = {\rm id}_{\Gamma(T^n_m\mathcal{M})}. \tag{A.2.14}$$

In this thesis, we will only focus on right-inverses that are $\mathbb{R}$-linear operators

$$L^{-1}\left(\alpha T + \alpha' T'\right) = \alpha L^{-1} T + \alpha' L^{-1} T', \quad \alpha, \alpha' \in \mathbb{R}, \text{constant} \tag{A.2.15}$$

and which can therefore be expressed as the convolution with a Green's bi-tensor.[6] On flat space-time we have that the bi-tensor structure of G simplifies considerably. For the local functions corresponding to the same charts on $\mathcal{M}_{\rm L}$ and $\mathcal{M}_{\rm R}$, i.e. when x and y are in the same coordinate chart, the converse property (A.3.1) along with Poincaré covariance imply that all Green's bi-tensors can be expressed in terms of a Green's function

$$G^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x,y) = \delta^{\mu'_1}_{\mu_1}\ldots\delta^{\mu'_m}_{\mu_m}\delta^{\nu_1}_{\nu'_1}\ldots\delta^{\nu_n}_{\nu'_n} G(x-y), \tag{A.2.17}$$

where

$$(LG)(x) = \delta^{(D)}(x). \tag{A.2.18}$$

(Footnote 5 continued)
it as the convolution with a bi-tensor indeed. We just need to rewrite

$$(LT)^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}(x) = \int {\rm d}^D y \sqrt{-g(y)}\, \Delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x,y) L^{\nu'_1\ldots\nu'_n}_{\mu'_1\ldots\mu'_m}|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n} T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y), \tag{A.2.10}$$

and then integrate by parts the covariant derivatives in L so that they act on Δ. The boundary terms drop because of the Dirac delta in Δ and the result is the convolution of T with a bi-tensor.

[6]These must be contrasted with the more general case where the right-inverse is given by an affine operator

$$L^{-1}_{h,G}(T) \equiv h + L^{-1}_G T, \tag{A.2.16}$$

with $h \in$ Ker$[L]$ a homogeneous solution of L that is independent of T.

We will use the "r" subscript when referring to retarded Green's functions, i.e. those that obey

$$G_{\mathrm{r},\dots}{}^{\dots}|_{\dots}^{\dots}(x, y) = 0, \quad \text{unless } y \text{ is in the past light-cone of } x. \tag{A.2.19}$$

On flat-space time this condition uniquely determines G because it totally determines the initial conditions

$$\lim_{x^0 \to -\infty} \partial_{x^0}^n G(x) = 0, \tag{A.2.20}$$

where n goes from 0 to the degree of L. For instance, the retarded Green's function of $L = \Box - m^2$ reads

$$G_\mathrm{r}(x) \equiv \lim_{\epsilon \to 0^+} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\exp\left[i\eta_{\mu\nu}k^\mu x^\nu\right]}{\left(k^0 + i\epsilon\right)^2 - \vec{k}^2 - m^2}, \tag{A.2.21}$$

and in $D = 4$ takes the simple form

$$\begin{aligned} G_\mathrm{r}(x) &= -\frac{1}{2\pi}\,\theta(x^0)\left[\delta(|x|^2) - \theta(|x|^2)\frac{m J_1\,(m|x|)}{2|x|}\right] \\ &= -\frac{1}{4\pi}\left[\frac{\delta(x^0 - |\vec{x}|)}{|\vec{x}|} - \theta(x^0)\,\theta(|x|^2)\frac{m J_1\,(m|x|)}{|x|}\right], \end{aligned} \tag{A.2.22}$$

where

$$|x| \equiv \sqrt{-\eta_{\mu\nu}x^\mu x^\nu}, \quad |\vec{x}| \equiv \sqrt{\delta_{ij}x^i x^j}, \tag{A.2.23}$$

and J_1 is a Bessel function of the first kind. We see that $G_\mathrm{r}(x - y)$, seen as a function of y, has a singular part which is supported only *on* the past light-cone of x and a non-singular part which is supported on the *inside* of the cone. The latter vanishes in the $m \to 0$ limit, consistent with the fact that the information then propagates only at the speed of light and its trajectory is thus stuck on the cone. Finally, note that the domain of definition of L_r^{-1} are the tensors that vanish sufficiently fast at past infinity for $m \neq 0$ and past null infinity for $m = 0$.

The generalization to curved space-time presents the following subtleties. First of all, the retarded Green's bi-tensor of $\Box - m^2$ is still supported inside the past light-cone, it is just that the latter is now non-trivial. Indeed, there might be more than one geodesic linking a given pair of points, the most striking example being the gravitational lensing effect. Second, one needs to impose global hyperbolicity on the pair $(\mathcal{M}, g)$ in order to have a causal space-time with a past that extends to infinity, and in which case the past light-cone would also extend to the infinite past. In that case, the domain of definition of L_r^{-1} are the tensors that vanish sufficiently fast at past infinity. More precisely, since $\Box - m^2$ is second-order, taking t to denote the global time coordinate (Geroch's theorem), we need

$$\lim_{t\to-\infty} T = 0, \quad \lim_{t\to-\infty} \nabla_N T = 0, \tag{A.2.24}$$

for any past-pointing time-like N (light-like for $m = 0$). Since in practical calculations one may have other differential operators acting on T before $L_{\rm r}^{-1}$, imposing the above condition will not suffice in general, so we will need to be more conservative. If $\mathcal{C}_x$ denotes the interior of the past light-cone of x, then we will demand that $\overline{\text{supp}(T) \cap \mathcal{C}_x}$ is compact for all x and will refer to such tensors as tensors with "finite past".

A.3 Green's Bi-tensor Properties

A.3.1 Converse of (A.2.12)

Here we show that (A.2.12) holds also when one acts on the point y instead of x

$$L^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}\big|^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}[\nabla_{\rm R}](y)\, G^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}\big|^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(x, y) = \Delta^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}\big|^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}(x, y). \tag{A.3.1}$$

Indeed, acting with $L[\nabla_{\rm R}](y)$ on (A.2.12) and using (A.2.9) we get

$$\begin{aligned}
&L^{\nu'_1\ldots\nu'_n}_{\mu'_1\ldots\mu'_m}\big|^{\lambda_1\ldots\lambda_m}_{\kappa_1\ldots\kappa_n}[\nabla_{\rm R}](y)\, L^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}[\nabla_{\rm L}](x)\, G^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}\big|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x, y)\\
&\qquad = L^{\nu'_1\ldots\nu'_n}_{\mu'_1\ldots\mu'_m}\big|^{\lambda_1\ldots\lambda_m}_{\kappa_1\ldots\kappa_n}[\nabla_{\rm R}](y)\Delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\big|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x, y)\\
&\qquad = L^{\nu'_1\ldots\nu'_n}_{\mu'_1\ldots\mu'_m}\big|^{\lambda_1\ldots\lambda_m}_{\kappa_1\ldots\kappa_n}[-\nabla_{\rm L}](x)\Delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\big|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x, y).
\end{aligned} \tag{A.3.2}$$

Thus, the convolution with a test tensor on $\mathcal{M}_{\rm L}$, using $[\nabla_{\rm L}, \nabla_{\rm R}] = 0$, gives

$$\begin{aligned}
&\int {\rm d}^D x\, \sqrt{-g(x)}\, L^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}[\nabla_{\rm L}](x)\left[L^{\nu'_1\ldots\nu'_n}_{\mu'_1\ldots\mu'_m}\big|^{\lambda_1\ldots\lambda_m}_{\kappa_1\ldots\kappa_n}[\nabla_{\rm R}](y)\, G^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}\big|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x, y)\, T^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}(x)\right]\\
&\overset{(A.3.2)}{=} \int {\rm d}^D x\, \sqrt{-g(x)}\, L^{\nu'_1\ldots\nu'_n}_{\mu'_1\ldots\mu'_m}\big|^{\lambda_1\ldots\lambda_m}_{\kappa_1\ldots\kappa_n}[-\nabla_{\rm L}](x)\Delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\big|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x, y)\, T^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}(x)\\
&\overset{\text{i.b.p.}}{=} \int {\rm d}^D x\, \sqrt{-g(x)}\, \Delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\big|^{\mu'_1\ldots\mu'_m}_{\nu'_1\ldots\nu'_n}(x, y) L^{\nu'_1\ldots\nu'_n}_{\mu'_1\ldots\mu'_m}\big|^{\lambda_1\ldots\lambda_m}_{\kappa_1\ldots\kappa_n}[\nabla_{\rm L}](x)\, T^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}(x)\\
&= (LT)^{\lambda_1\ldots\lambda_m}_{\kappa_1\ldots\kappa_n}(y),
\end{aligned} \tag{A.3.3}$$

where in the second step we have integrated by parts and the boundary terms have dropped because of the Dirac delta. Comparing the first line with the last we get that the term in square brackets obeys the distributional definition of Δ, i.e. (A.3.1).

A.3.2 Conditions for also Being a Left-Inverse

In general L^{-1} is not a left-inverse $L^{-1}L \neq \text{id}$, as it is most obvious when acting on $h \in \text{Ker}[L]$

$$L^{-1}Lh = 0. \tag{A.3.4}$$

The most general statement is rather

$$\left(L^{-1}L - \text{id}\right) T \in \text{Ker}[L], \quad \forall T \in \Gamma(T^n_m\mathcal{M}), \tag{A.3.5}$$

since applying L from the left will give zero. Note that in general the resulting element of Ker$[L]$ will depend on g, because L does, and is obviously also $\mathbb{R}$-linear in T. Indeed, because of the very existence of non-zero elements in Ker$[L]$, left-inverses generically do not exist. To understand this intuitively consider for instance the operator ∂_t^2 in one dimension and the following acausal Green's function

$$G(t,t') = \theta(t-t')\theta(t'-t_0)(t-t') - \theta(t_0-t')\theta(t'-t)(t'-t), \tag{A.3.6}$$

so that the inverse operation is

$$(\partial^{-2} f)(t) \equiv \int_{-\infty}^{\infty} \mathrm{d}t'\, G(t,t') f(t') = \int_{t_0}^{t} \mathrm{d}t' (t-t') f(t'), \tag{A.3.7}$$

and we get

$$(\partial^{-2}\partial^2 f)(t) - f(t) = -f(t_0) - f'(t_0)(t-t_0) \in \text{Ker}\left[\partial^2\right]. \tag{A.3.8}$$

It is clear that with this definition, ∂^{-2} is a left inverse $\partial^{-2}\partial^2 = \text{id}$ only on the subspace of functions obeying $f(t_0) = f'(t_0) = 0$. Moreover, we see that the resulting element of the kernel is determined by the boundaries of the convolution, i.e. the support of the Green's function with respect to the second argument. In the retarded case where $t_0 \to -\infty$ the integral makes sense only for functions that decrease sufficiently fast at infinity, i.e.

$$\lim_{t\to-\infty} f(t) = 0, \quad \lim_{t\to-\infty} \dot{f}(t) = 0, \tag{A.3.9}$$

and then ∂^{-2} is a left-inverse. This is actually the case in any dimension and on arbitrary geometries, i.e. the obstruction to being a left-inverse is generated by non-trivial boundaries of the support of G. Now that we have understood this using the simplest example, let us consider the case $L = \Box$ which is the one of interest in this thesis, for arbitrary dimension and for globally hyperbolic $(\mathcal{M}, g)$ so that the past light-cones extend to past infinity. We have

$$
\begin{aligned}
\Box_{\mathrm{r}}^{-1}\Box T^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}(x) &= \int_{\mathcal{M}} \mathrm{d}^D y\, \sqrt{-g(y)}\, G_{\mathrm{r}\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\Box T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y) \\
&= \int_{\mathcal{U}} \mathrm{d}^d y\, \sqrt{-g(y)}\, G_{\mathrm{r}\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\, N^\mu(y)\nabla_\mu T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y) \\
&\quad - \int_{\mathcal{M}} \mathrm{d}^d y\, \sqrt{-g(y)}\, \nabla|^\mu G_{\mathrm{r}\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\, \nabla_\mu T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y) \\
&= \int_{\mathcal{U}} \mathrm{d}^d y\, \sqrt{-g(y)}\, G_{\mathrm{r}\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\, N^\mu(y)\nabla_\mu T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y) \\
&\quad - \int_{\mathcal{U}} \mathrm{d}^d y\, \sqrt{-g(y)}\, N^\mu(y)\nabla|_\mu G_{\mathrm{r}\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\, T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y) \\
&\quad + \int_{\mathcal{M}} \mathrm{d}^D y\, \sqrt{-g(y)}\, \Box_y G_{\mathrm{r}\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\, T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y) \\
&\overset{(A.3.1)}{=} \int_{\mathcal{U}} \mathrm{d}^d y\, \sqrt{-g(y)}\, W_{x,\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}(y) + T^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}(x),
\end{aligned} \tag{A.3.10}
$$

where N is the normal vector to $\mathcal{U}$ and

$$
W_{x,\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}(y) \equiv G_{\mathrm{r}\mu_1\ldots\mu_m}^{\;\;\nu_1\ldots\nu_n}\big|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\, \overleftrightarrow{\nabla}_N\, T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y), \tag{A.3.11}
$$

is the Wronskian of $G(x, y)$ and $T(y)$ with respect to the derivative operator ∇_N acting on y. The question now is: what is $\mathcal{U}$? If the integrand we started with was smooth, then by Stokes' theorem we would have that $\mathcal{U}$ is the boundary of the support of the integrand. However, since $G_{\mathrm{r}}(x, y)$ is non-zero only when y is on the past light-cone of x, we have that it is actually a distribution, just like in the flat space-time case (A.2.22). Thus, the integration by parts has to be understood in the way it is used for distributions: the boundary term is supported on the boundary of the support of the distribution. Since in our case the integrand is supported on the past light-cone $\mathcal{L}_x$ of x, the integral of the Wronskian is actually supported on $\partial\mathcal{L}_x$ which lies at past (null) infinity. Thus, we have that the conditions that one must impose on T for $\Box_{\mathrm{r}}^{-1}$ to be a left inverse are (A.2.24), i.e. precisely the ones for which $\Box_{\mathrm{r}}^{-1}$ is defined anyway. We thus have that $\Box_{\mathrm{r}}^{-1}$ is also a left inverse on the domain of $\Gamma(T^n_m\mathcal{M})$ where it is defined.

It is quite interesting to see how this computation goes through in the massive case $L = \Box - m^2$ since then the support of the Green's function is inside the past light-cone so that $\partial\mathcal{U} = \mathcal{L}_x$. For simplicity let us work on flat space-time, since in that case we have an explicit result (A.2.22). We then see that we have the singular part of $\Box^{-1}$, which is treated as before and thus gives an integral supported at past infinity. The smooth part which is supported on the inside of the cone however has a non-zero limit $|x| \to 0$ from the inside of the cone

$$
\lim_{|x|\to 0^+} G_{\mathrm{r}}(x) = -\frac{1}{4\pi}\left[\frac{\delta(x^0 - |\vec{x}|)}{|\vec{x}|} - \frac{m^2}{2}\,\theta(x^0)\right], \tag{A.3.12}
$$

so the corresponding Wronskian boundary term is not zero and lies on $\mathcal{L}_x$, not on $\partial\mathcal{L}_x$. Therefore, if we wanted this to be zero for all x then we would need to impose $T = 0$. However, what we see is that the smooth part of $G_r(x)$ is actually constant on the light-cone, so that the Wronskian (A.3.11) is a total derivative. Thus, by Stokes' theorem it also amounts to an integral that is supported on $\partial\mathcal{L}_x$ at past infinity. For generic space-times we would need to know the limiting behaviour of the Green's function on the light-cone to answer the question of left-inversion.

A.3.3 Commutation Relations of L^{-1}

We are now interested in understanding the commutator $\left[M, L^{-1}\right]$ where $M[\nabla]$ is some differential operator. To do so we can simply act with the derivation $[M, \cdot\,]$ on the equation $LL^{-1} = \text{id}$ to get

$$[M, L]\, L^{-1} + L\left[M, L^{-1}\right] = 0. \tag{A.3.13}$$

Isolating $\left[M, L^{-1}\right]$ would require the use of a left-inverse which, as we have seen in the previous section, does not exist when acting on generic functions. We can make use of the weaker equation (A.3.5) to get the most conservative statement

$$\left[M, L^{-1}\right] T = -L^{-1}\,[M, L]\, L^{-1}T + X, \quad X \in \text{Ker}[L]. \tag{A.3.14}$$

where X is $\mathbb{R}$-linear in $[M, L]\, L^{-1}T$. For instance, in the case $L = \Box$ and $M = \nabla_\mu$, we get the following rule for the retarded inverse on a scalar field of finite past

$$\left[\nabla_\mu, \Box_r^{-1}\right]\phi = \Box_r^{-1}\left(R_\mu^\nu \nabla_\nu \Box_r^{-1}\phi\right), \tag{A.3.15}$$

i.e. there is no X part precisely because then $\Box_r^{-1}$ is also a left inverse. Isolating $\Box_r^{-1}\nabla_\mu$ and restricting to an Einstein space-time $R_{\mu\nu} = \kappa\, g_{\mu\nu}$, where κ is constant, we get

$$\Box_r^{-1}\nabla_\mu = \left(1 - \kappa\Box_r^{-1}\right)\nabla_\mu\Box_r^{-1}. \tag{A.3.16}$$

Inverting the operator in the bracket in a causal way, we get

$$\nabla_\mu\Box_r^{-1} = (\Box - \kappa)_r^{-1}\,\nabla_\mu. \tag{A.3.17}$$

A.3.4 Displacing the Indices of Green's Bi-tensors

Since we only use metric compatible covariant derivatives $[\nabla, g] = 0$, we have that $[g, L] = 0$ for any differential operator L. At the level of the Green's bi-tensors we have that the isomorphism g between $\Gamma(T_m^n\mathcal{M})$ and $\Gamma(T_{m+1}^{n-1}\mathcal{M})$

$$T^{\nu_1\ldots\nu_{n-1}}_{\mu_1\ldots\mu_{m+1}} = g_{\mu_{m+1}\nu_n} T^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}, \tag{A.3.18}$$

induces an isomorphism between the Green's functions of L in $\Gamma(B^n_m|^m_n\mathcal{M})$ and the ones in $\Gamma(B^{n-1}_{m+1}|^{m+1}_{n-1}\mathcal{M})$ which is found through

$$\begin{aligned}(G\cdot T)^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}(x) &\equiv \int \mathrm{d}^D y\,\sqrt{-g(y)}\,G^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\,T^{\rho_1\ldots\rho_n}_{\sigma_1\ldots\sigma_m}(y)\\ &= \int \mathrm{d}^D y\,\sqrt{-g(y)}\,G^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\,g^{\rho_n\sigma_{m+1}}(y)\,T^{\rho_1\ldots\rho_{n-1}}_{\sigma_1\ldots\sigma_{m+1}}(y)\\ &\equiv g^{\mu_{m+1}\nu_n}(x)\int \mathrm{d}^D y\,\sqrt{-g(y)}\,G^{\nu_1\ldots\nu_{n-1}}_{\mu_1\ldots\mu_{m+1}}|^{\sigma_1\ldots\sigma_{m+1}}_{\rho_1\ldots\rho_{n-1}}(x,y)\,T^{\rho_1\ldots\rho_{n-1}}_{\sigma_1\ldots\sigma_{m+1}}(y),\end{aligned} \tag{A.3.19}$$

so that

$$G^{\nu_1\ldots\nu_{n-1}}_{\mu_1\ldots\mu_{m+1}}|^{\sigma_1\ldots\sigma_{m+1}}_{\rho_1\ldots\rho_{n-1}}(x,y) = g_{\mu_{m+1}\nu_n}(x)\,G^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_m}|^{\sigma_1\ldots\sigma_m}_{\rho_1\ldots\rho_n}(x,y)\,g^{\rho_n\sigma_{m+1}}(y). \tag{A.3.20}$$

Indeed, the latter trivially obeys $L[\nabla_{\mathrm{L}}]G = \Delta$ since $[g, L] = 0$.