Appendix A
Rigid Cohomology

In this appendix, we give a brief review of the classical theory of rigid cohomology, as developed by Berthelot using the theory of Tate's rigid spaces. We therefore assume that the reader is familiar with this theory, as well as with the theory of formal schemes. In the likely event of the reader's dissatisfaction with this brief and incomplete overview, we recommend consulting Le Stum's comprehensive textbook on the subject [4], as well as Berthelot's original preprint [1].

Let k be a field of characteristic p, and $\mathscr{V}$ a complete discrete valuation ring with residue field k and fraction field K of characteristic 0. Let π denote a uniformiser for $\mathscr{V}$. A *variety* X over k will mean a separated scheme of finite type, and a formal scheme over $\mathscr{V}$ will mean a π-adic formal scheme, separated and topologically of finite type over $\mathscr{V}$. More concretely then, formal scheme X we be one locally of the form Spf(A) for A a topological $\mathscr{V}$-algebra, isomorphic to a quotient $\mathscr{V}\langle x_1,\ldots,x_n\rangle/I$ of a Tate algebra over $\mathscr{V}$. We may therefore view the category $\mathbf{Sch}_k$ of k-varieties as a full subcategory of the category $\mathbf{FSch}_{\mathscr{V}}$ of formal schemes over $\mathscr{V}$, consisting of those formal schemes on which $\pi=0$. There is a natural 'reduction mod π' functor

$$\begin{aligned} \mathbf{FSch}_{\mathscr{V}} &\to \mathbf{Sch}_k \\ \mathfrak{X} &\mapsto \mathfrak{X}_k =: X \end{aligned}$$

from the category of formal schemes over $\mathscr{V}$ to varieties over k.

A rigid analytic variety will be meant in the sense of Tate, that is a locally G-ringed space $\mathscr{X}$, locally isomorphic to Sp(A_K) for A_K an affinoid K-algebra, that is a topological K-algebra isomorphic to a quotient $K\langle x_1,\ldots,x_n\rangle/I$ of a Tate

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algebra over K. The category of rigid analytic varieties spaces will be denoted $\mathbf{An}_K$. The functor $A \mapsto A_K := A \otimes_{\mathscr{V}} K$ induces a 'generic fibre functor'

$$\begin{aligned} \mathbf{FSch}_{\mathscr{V}} &\to \mathbf{An}_K \\ \mathfrak{X} &\mapsto \mathfrak{X}_K =: \mathscr{X} \end{aligned}$$

from the category of formal schemes over $\mathscr{V}$ to rigid analytic varieties over K.

As a general rule, we will denote schemes by roman letters (e.g. X, Y, Z), formal schemes by gothic letters (e.g. $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$) and rigid analytic varieties by script letters (e.g. $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$).

A.1 Specialisation and Tubes

Let A be a topologically finite type $\mathscr{V}$-algebra, and write $A_K := A \otimes_{\mathscr{V}} K$. Then points of the rigid analytic variety $\mathrm{Sp}(A_K)$ are just maximal ideals of A_K, and given such a point $\mathfrak{m}$, the intersection $\mathfrak{m} \cap A$ (or rather the inverse image of $\mathfrak{m}$ under the natural map $A \to A_K$, if A has π-torsion) is an open prime ideal of A, hence a point of $\mathrm{Spf}(A)$. This construction extends to give a specialisation map

$$\mathrm{sp} : \mathfrak{P}_K \to \mathfrak{P}$$

for any formal scheme $\mathfrak{P}$, which by extending $A \to A_K$ to a map

$$\mathscr{O}_{\mathfrak{P}} \to \mathrm{sp}_* \mathscr{O}_{\mathfrak{P}_K}$$

makes the specialisation map a morphism of (G-)ringed spaces. Since the topological space of $\mathfrak{P}$ and its special fibre $P := \mathfrak{P}_k$ coincide, we can also view the specialisation map as a continuous map

$$\mathrm{sp} : \mathfrak{P}_K \to P,$$

and the first key notion in the theory of rigid cohomology is that of a tube.

Definition A.1 For any locally closed subscheme $X \subset P$, we define the tube of X to be

$$]X[_{\mathfrak{P}} := \mathrm{sp}^{-1}(X) \subset \mathfrak{P}_K.$$

A priori, this is simply a subset of $\mathfrak{P}_K$, however, the following result shows that it is in fact a rigid analytic variety over K in its own right.

Proposition A.2 ([1], Proposition 1.1.1) *Let $f_i, g_j \in \Gamma(P, \mathscr{O}_P)$ be such that*

$$X = V(f_1, \ldots, f_n) \cap D(g_1, \ldots, g_m) \subset P,$$

and let $\tilde{f}_i, \tilde{g}_j \in \Gamma(\mathfrak{P}, \mathscr{O}_{\mathfrak{P}})$ *be functions lifting the* f_i *and* g_j *respectively. Then*

$$]X[_{\mathfrak{P}}= \left\{\alpha \in \mathfrak{P}_K \,\middle|\, \left|\tilde{f}_i(\alpha)\right| < 1 \;\forall i,\; \left|\tilde{g}_j(\alpha)\right| \geq 1 \text{ for some } j\right\}.$$

Example A.3 Let $\mathfrak{P} = \widehat{\mathbb{A}}^n_{\mathscr{V}} = \mathrm{Spf}(\mathscr{V}\langle x_1, \ldots, x_n\rangle)$ be formal affine space over $\mathscr{V}$, and $\beta = \mathrm{Spec}(k) \subset P = \mathbb{A}^n_k$ the point given by some $\beta = (\beta_1, \ldots, \beta_n) \in k$. Then if $\tilde{\beta} = (\tilde{\beta}_n, \ldots, \tilde{\beta}_n)$ is any lift of β to $\mathscr{V}$, the tube $]\beta[_{\mathfrak{P}}$ is simply the 'residue disc'

$$\left\{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{D}^n_K \,\middle|\, \left|\alpha_i - \tilde{\beta}_i\right| < 1 \;\forall i\right\}$$

inside the n-dimensional closed unit ball $\mathbb{D}^n_K$ over K.

Example A.4 Let $\mathfrak{P} = \widehat{\mathbb{P}}^1_{\mathscr{V}}$ be the formal projective line over $\mathscr{V}$, and $X = \mathbb{A}^1_k \subset P = \mathbb{P}^1_k$. Then

$$]X[_{\mathfrak{P}}= \left\{\alpha \in \mathbb{P}^{1,\mathrm{an}}_K \,\middle|\, |\alpha| \leq 1\right\} = \mathbb{D}^1_K$$

is the usual closed unit disc over K.

These 'tubes of radius 1' are entirely canonical and naturally defined, however, also useful are more general tubes that are constructed on a more ad-hoc basis, and only for certain subschemes of the special fibre P.

Definition A.5 Let $Y \subset P$ be a closed subscheme, cut out by regular functions $f_i \in \mathscr{O}_P$ lifting to $\tilde{f}_i \in \mathscr{O}_{\mathfrak{P}}$. Then for $\eta < 1$ inside the value group $|K^*| \otimes \mathbb{Q}$ we define the closed tube of radius η to be

$$[Y]_\eta := \left\{\alpha \in \mathfrak{P}_K \,\middle|\, \left|\tilde{f}_i(\alpha)\right| \leq \eta \;\forall i\right\}.$$

Note that in general this depends on the choice of the $\tilde{f}_i$, however, one can show that if $\eta > |\pi|$ then these are well-defined, and therefore glue to give tubes $[Y]_\eta$ when Y is not necessarily cut out by global functions. One clearly has the equality

$$\bigcup_{\eta<1} [Y]_\eta =]Y[_{\mathfrak{P}}$$

and in fact one can check that this covering is admissible.

Example A.6 Let $\mathfrak{P} = \widehat{\mathbb{A}}^1_{\mathscr{V}} = \mathrm{Spf}(\mathscr{V}\langle x\rangle)$ and $Y = V(x) \subset P = \mathbb{A}^1_k$. Then

$$]Y[_{\mathfrak{P}}= \left\{\alpha \in \mathbb{D}^1_K \,\middle|\, |\alpha| < 1\right\} = \mathbb{D}^{1,\circ}_K$$

is the open unit disc over K, while

$$[Y]_\eta = \left\{\alpha \in \mathbb{D}^1_K \middle| \, |\alpha| \le \eta\right\} = \mathbb{D}^1_{K,\eta}$$

is the closed disc of radius η over K. The covering

$$\bigcup_{\eta<1} [Y]_\eta =]Y[_{\mathfrak{P}}$$

is the 'standard' admissible covering of $\mathbb{D}^{1,\circ}_K$ by the $\mathbb{D}^1_{K,\eta}$.

Definition A.7 Let $U \subset P$ be an open subscheme, and let $\tilde{g}_j \in \mathscr{O}_{\mathfrak{P}}$ be regular functions such that $U = P \cap (\cup_j D(\tilde{g}_j))$. Then define, for $\lambda < 1$, the closed tube of radius $1/\lambda$ to be

$$[U]_{1/\lambda} = \left\{\alpha \in \mathfrak{P}_K \, \middle| \left|\tilde{g}_j(\alpha)\right| \ge \lambda \text{ for some } j\right\}.$$

Again, these depend on the $\tilde{g}_j$ in general, but are well-defined when $\lambda > |\pi|$, and glue over an open covering of $\mathfrak{P}$. We also have the equality of rigid analytic varieties

$$\bigcap_{\lambda<1} [U]_{1/\lambda} =]U[_{\mathfrak{P}}.$$

Example A.8 Let $\mathfrak{P} = \widehat{\mathbb{P}}^1_{\mathscr{V}}$ and $U = \mathbb{A}^1_k \subset P = \mathbb{P}^1 k$. Then

$$[U]_{1/\lambda} = \left\{\alpha \in \mathbb{P}^{1,\mathrm{an}}_K \middle| \, |\alpha| \le 1/\lambda\right\} = \mathbb{D}^1_{K,1/\lambda}$$

is the usual closed disc of radius $1/\lambda$ over K. We also have that

$$]U[_{\mathfrak{P}} = \left\{\alpha \in \mathbb{P}^{1,\mathrm{an}}_K \middle| \, |\alpha| \le 1\right\} = \mathbb{D}^1_K$$

is just the closed unit disc over K, so we do indeed have $\bigcap_{\lambda<1}[U]_{1/\lambda} =]U[_{\mathfrak{P}}$.

A.2 Pairs and Frames

Definition A.9 A *pair* over k is an open immersion $j : X \hookrightarrow Y$ of k-varieties. A *frame* over $\mathscr{V}$ is a triple $(X, Y, \mathfrak{P})$ where (X, Y) is a pair over k and $i : Y \hookrightarrow \mathfrak{P}$ is a closed immersion of formal $\mathscr{V}$-schemes.

A morphism of frames is simply a commutative diagram

$$\begin{CD} X' @>>> Y' @>>> \mathfrak{P}' \\ @VVV @VVV @VVV \\ X @>>> Y @>>> \mathfrak{P} \end{CD}$$

and a morphism of pairs is defined similarly.

Definition A.10 A morphism of frames

$$\begin{CD} X' @>>> Y' @>>> \mathfrak{P}' \\ @VVV @VVV @VVV \\ X @>>> Y @>>> \mathfrak{P} \end{CD}$$

is said to be smooth (resp. étale, resp. flat) if there exists an open subscheme $\mathfrak{U} \subset \mathfrak{P}'$ containing X such that $\mathfrak{U} \to \mathfrak{P}$ is smooth (resp. étale, resp. flat). A morphism of pairs

$$\begin{CD} X' @>>> Y' \\ @VVV @VVV \\ X @>>> Y \end{CD}$$

is said to be proper (resp. finite) if $Y' \to Y$ is, and a morphism of frames is said to be proper (resp. finite) if the corresponding morphism of pairs is. A frame is said to be smooth, proper, flat, étale, finite if the morphism $(X, Y, \mathfrak{P}) \to (\mathrm{Spec}(k), \mathrm{Spec}(k), \mathrm{Spf}(\mathscr{V})$ is.

One of the general principles in p-adic cohomology is that the p-adic cohomology of a scheme X in characteristic p is closely related to the de Rham cohomology of a lift of X to characteristic 0. In the theory of rigid cohomology this principle feeds directly into the definition, with $]X[_{\mathfrak{P}}$ playing the role of a lift of X to characteristic 0. In order for de Rham cohomology to be well behaved, we will need these tubes to satisfy smoothness conditions, and the next result shows how to guarantee this on the level of frames.

Proposition A.11 *Let*

$$\begin{CD} X' @>>> Y' @>>> \mathfrak{P}' \\ @VVV @VVV @VVV \\ X @>>> Y @>>> \mathfrak{P} \end{CD}$$

be a smooth (resp. étale, resp. flat) morphism of frames. Then the induced morphism of tubes

$$]X'[_{\mathfrak{P}'} \to]X[_{\mathfrak{P}}$$

is smooth (resp. étale, resp. flat).

Proof Since the tube $]X[_{\mathfrak{P}}$ in unchanged when replacing $\mathfrak{P}$ by an open subscheme $\mathfrak{U}$ containing X, this follows from the fact that the generic fibre of a smooth (resp. étale, resp. flat) morphism of formal schemes is smooth (resp. étale, resp. flat). □

Example A.12 As a crucial example of this, if we take a smooth formal scheme $\mathfrak{P}$ over $\mathscr{V}$ of dimension n, and a k-rational point $\beta \in P(k)$ of its generic fibre, then the tube $]\beta[_{\mathfrak{P}}$ is isomorphic to the n-dimensional open unit ball over K.

Of course, if we are to define the p-adic cohomology of an object in characteristic p to be the de Rham cohomology of a suitable lift, then one of the first tasks will be to verify that this is independent of the choice of lift. The starting point for doing to is the following result.

Proposition A.13 (Weak Fibration Theorem, [1], Proposition 1.3.1) *Let*

$$\begin{array}{ccccc} & \nearrow & Y' & \longrightarrow & \mathfrak{P}' \\ & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & \mathfrak{P} \end{array}$$

be an étale morphism of frames. Then the induced map

$$]X[_{\mathfrak{P}'} \to]X[_{\mathfrak{P}}$$

is an isomorphism.

Actually, if we want a well-behaved theory using de Rham cohomology of a lift, working on the tubes $]X[_{\mathfrak{P}}$ will not be enough, as the following example shows.

Example A.14 Let $X = \mathbb{A}^1_k$ and $\mathfrak{P} = \widehat{\mathbb{P}}^1_{\mathscr{V}}$. Then $]X[_{\mathfrak{P}} \cong \mathbb{D}^1_K$, and the de Rham cohomology of $\mathbb{D}^1_K$ is infinite dimensional.

The way to 'fix' this example, which provides the clue to how to define rigid cohomology in general, is as follows.

Example A.15 Let $K\langle x\rangle^\dagger$ denote the ring of functions $f = \sum_i a_i x^i$ such that $|a_i|\,\rho^i \to 0$ for some $\rho > 1$ (depending on f). In other words, these are the functions converging on some closed disc of radius strictly greater than 1. Then the 'de Rham cohomology' of $K\langle x\rangle^\dagger$, i.e. the cohomology of the complex

$$0 \to K\langle x\rangle^\dagger \xrightarrow{f \mapsto df} K\langle x\rangle^\dagger \cdot dx \to 0$$

is K in degree 0 and 0 in degree 1.

We will therefore want to work with differential forms with domain of convergence slightly bigger that $]X[_{\mathfrak{P}}$, and this leads to the notion of a 'strict neighbourhood' of a tube, introduced in the next section.

A.3 Strict Neighbourhoods and Overconvergence

If $(X, Y, \mathfrak{P})$ is a frame, and $Z = Y \setminus X$, then it follows easily from the definitions that

$$]Y[_{\mathfrak{P}} =]X[_{\mathfrak{P}} \cup]Z[_{\mathfrak{P}},$$

however, this covering is not admissible in general.

Example A.16 Let $\mathfrak{P} = \widehat{\mathbb{A}}^1_{\mathscr{V}} = \mathrm{Spf}(\mathscr{V}\langle x \rangle)$, $Y = P = \mathbb{A}^1_k$ and $X = \mathbb{A}^1_K \setminus \{0\}$. Then $]Y[_{\mathfrak{P}} = \mathbb{D}^1_K$ is the closed unit disc, $]Z[_{\mathfrak{P}} = \mathbb{D}^{1,\circ}_K$ is the open unit disc, and $]X[_{\mathfrak{P}}$ is the annulus

$$\left\{ \alpha \in \mathbb{D}^1_K \,\middle|\, |\alpha| = 1 \right\}$$

of inner and outer radius 1. Hence the covering $]Y[_{\mathfrak{P}} =]X[_{\mathfrak{P}} \cup]Z[_{\mathfrak{P}}$ is not admissible.

Definition A.17 An subspace $]X[_{\mathfrak{P}} \subset V \subset]Y[_{\mathfrak{P}}$ is said to be a strict neighbourhood of $]X[_{\mathfrak{P}}$ if the covering

$$]Y[_{\mathfrak{P}} = V \cup]Z[_{\mathfrak{P}}$$

is admissible.

Of course, the same definition applies if the decomposition $]Y[_{\mathfrak{P}} =]X[_{\mathfrak{P}} \cup]Z[_{\mathfrak{P}}$ is replaced by any disjoint, possibly non-admissible covering $V = U_1 \cup U_2$ of a rigid space V. For example, we may speak of a strict neighbourhood of $]X[_{\mathfrak{P}} \cap [Y]_\eta$ inside $[Y]_\eta$.

Example A.18 Let $X \subset \mathbb{A}^n_k$ be an affine scheme defined by polynomials $f_1, \dots, f_r \in k[x_1, \dots x_n]$, lift these to $\tilde{f}_i \in \mathscr{V}[x_1, \dots, x_n]$ and let $\mathfrak{P} \subset \mathbb{P}^n_{\mathscr{V}}$ denote the completion of the projective closure of $V(\tilde{f}_1, \dots, \tilde{f}_r) \subset \mathbb{A}^n_{\mathscr{V}}$, note we have a natural locally closed embedding $X \hookrightarrow \mathfrak{P}$, the closure of X being equal to the special fibre P of $\mathfrak{P}$. Let $Z \subset P$ denote a closed complement for X.

Then the rigid analytic variety $\mathfrak{P}_K$ naturally embeds inside analytic projective space $\mathbb{P}^{n,\mathrm{an}}_K$, the tube $]X[_{\mathfrak{P}}$ is simply the intersection of $\mathfrak{P}_K$ with the closed unit ball $\mathbb{D}^n_K \subset \mathbb{P}^{n,\mathrm{an}}_K$, and the tube $]Z[_{\mathfrak{P}}$ is the intersection of $\mathfrak{P}_K$ with the 'hypertube at ∞'

$$]\mathbb{P}^{n-1}_k[_{\widehat{\mathbb{P}}^n_{\mathscr{V}}} = \left\{ \alpha = (\alpha_0 : \dots : \alpha_n) \in \mathbb{P}^{n,\mathrm{an}} \,\middle|\, |\alpha_0| < \sup_{i \neq 0} |\alpha_i| \right\}.$$

Strict neighbourhoods of $]X[_{\mathfrak{P}}$ inside $\mathfrak{P}_K =]P[_{\mathfrak{P}}$ are given by intersecting $\mathfrak{P}_K$ with closed n-dimensional balls $\mathbb{D}^n_{K,\rho}$ of radius $\rho > 1$, and in fact any strict neighbourhood can be shown to lie between $\mathbb{D}^n_{K,\rho} \cap \mathfrak{P}_K$ and $\mathbb{D}^n_{K,\rho'} \cap \mathfrak{P}_K$ for some ρ, ρ'.

More generally, if W is an open subcheme of P such that $X = Y \cap W$, then the open subspaces

$$V^\lambda := [W]_{1/\lambda} \cap]Y[_{\mathfrak{P}}$$

are all strict neighbourhoods of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$. If the tube $]Y[_{\mathfrak{P}}$ is quasi-compact, then one can use the maximum principle to show that in fact these V^λ form a cofinal system of such strict neighbourhoods, i.e. for any strict neighbourhood V of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$ there exists some λ such that $V^\lambda \subset V$. If $]Y[_{\mathfrak{P}}$ is not quasi-compact, however, then this is no longer true. Recall that we have

$$]Y[_{\mathfrak{P}} = \bigcup_{\eta<1} [Y]_\eta$$

and even in $]Y[$ is not, each $[Y]_\eta$ will be quasi-compact. Hence for fixed η, the $V^\lambda \cap [Y]_\eta$ form a cofinal system of strict neighbourhoods of $]X[_{\mathfrak{P}} \cap [Y]_\eta$ inside $[Y]_\eta$. To obtain a cofinal family in $]Y[_{\mathfrak{P}}$, then, we need to allow λ to vary with η. This motivates the following definition.

Definition A.19 For $\eta, \lambda < 1$ we set $V_\eta^\lambda := [Y]_\eta \cap [W]_{1/\lambda}$. For any sequence $(\underline{\eta}, \underline{\lambda}) = \{(\eta_n, \lambda_n)\}$ of pairs with $\eta_n \to 1$ we set

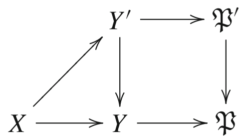
$$V_{\underline{\eta}}^{\underline{\lambda}} = \bigcup_n V_{\eta_n}^{\lambda_n},$$

this is a strict neighbourhood of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$.

Proposition A.20 ([1], 1.2.4) *The collection of* $V_{\underline{\eta}}^{\underline{\lambda}}$ *as* $(\underline{\eta}, \underline{\lambda})$ *ranges over all such sequences as above form a cofinal system of strict neighbourhoods of* $]X[_{\mathfrak{P}}$ *inside* $]Y[_{\mathfrak{P}}$.

One of the most important results in setting up the theory of rigid cohomology is the Strong Fibration Theorem, which says that the isomorphism of the Weak Fibration Theorem extends to strict neighbourhoods.

Theorem A.21 (Strong Fibration Theorem, [1], Théorème 1.3.5) *Let*



be a proper, étale morphism of frames. Then there exist strict neighbourhoods V' *of* $]X[_{\mathfrak{P}'}$ *in* $]Y'[_{\mathfrak{P}'}$ *and* V *of* $]X[_{\mathfrak{P}}$ *in* $]Y[_{\mathfrak{P}}$ *such that* $V' \to V$ *is an isomorphism.*

Note that the result immediately implies that we can find arbitrarily small (i.e. cofinal systems of strict neighbourhoods) V', V satisfying the conclusions of the theorem.

Definition A.22 Let $\mathscr{F}$ be a sheaf on $]Y[_{\mathfrak{P}}$. Then we define

$$j_X^\dagger \mathscr{F} := \mathrm{colim}_V\, j_{V*} j_V^{-1} \mathscr{F}$$

where the limit is taken over all strict neighbourhoods V of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$, and $j_V : V \to]Y[_{\mathfrak{P}}$ denotes the inclusion.

Example A.23 Consider the frame $(X, Y, \mathfrak{P})$ from Example A.18. Then a cofinal system of neighbourhoods is given by the intersection of $V(\tilde{f}_1, \ldots, \tilde{f}_r)$ with the closed disc $\mathbb{D}^n_{K,1/\lambda}$ of radius $1/\lambda$ for $\lambda \to 1$. If we let

$$K\langle x_1, \ldots, x_n \rangle^\dagger = \mathrm{colim}_{\lambda<1} K\langle \lambda x_1, \ldots, \lambda x_n \rangle$$

denote the 'Monsky–Washnitzer' algebra of power series which converge on some $\mathbb{D}^n_{K,1/\lambda}$, then we can therefore calculate

$$\Gamma(]Y[_{\mathfrak{P}}, j_X^\dagger \mathscr{O}_{]Y[_{\mathfrak{P}}}) = \frac{K\langle x_1, \ldots, x_n \rangle^\dagger}{(\tilde{f}_1, \ldots, \tilde{f}_r)}.$$

One can moreover show that the analogues of Theorem A and B hold in this situation, i.e. the higher cohomologies of coherent $j_X^\dagger \mathscr{O}_{]Y[_{\mathfrak{P}}}$-modules vanish.

More generally, if we take a frame $(X, Y, \mathfrak{P})$, the the restriction $j_X^\dagger \mathscr{O}_{[Y]_\eta}$ of $j_X^\dagger \mathscr{O}_{]Y[}$ to $[Y]_\eta$ (for η close enough to 1) is given by $\mathrm{colim}_{\lambda<1} j_{\lambda*} \mathscr{O}_{V_\eta^\lambda}$, where $j_\lambda : V_\eta^\lambda \to [Y]_\eta$ is the natural inclusion. We are now ready to define rigid cohomology.

Definition A.24 We say that a pair (X, Y) over k is *realisable* if there exists a smooth frame $(X, Y, \mathfrak{P})$. For (X, Y) a realisable pair, we define its rigid cohomology of X to be

$$H^i_{\mathrm{rig}}((X, Y)/K) := H^i(]Y[_{\mathfrak{P}}, j_X^\dagger \Omega^*_{]Y[_{\mathfrak{P}}/K})$$

for any smooth frame $(X, Y, \mathfrak{P})$.

Using the Strong Fibration Theorem, one can show that this is independent of the choice of $\mathfrak{P}$.

Theorem A.25 ([4], Corollary 7.4.3) $H^i_{\mathrm{rig}}((X, Y)/K)$ *only depends on the pair* (X, Y) *and not the choice of formal scheme* $\mathfrak{P}$.

While not every pair (X, Y) is realisable, it is *locally* realisable, in that there exists an open cover $Y = \cup_i Y_i$ such that, setting $X_i = X \cap Y_i$, the pairs (X_i, Y_i) are all realisable. This allows one to define rigid cohomology for arbitrary pairs using Čech methods, c.f. [2]. A slightly more subtle application of the Strong Fibration Theorem is the following.

Theorem A.26 ([4], Corollary 8.2.2) *Let* (X, Y) *be a realisable pair. If* Y *is proper, then the rigid cohomology* $H^i_{\mathrm{rig}}((X, Y)/K)$ *of the pair* (X, Y) *only depends on* X *and not on the choice of compactification* Y.

We may therefore write $H^i_{\mathrm{rig}}(X/K)$ in place of $H^i_{\mathrm{rig}}((X,Y)/K)$, a marginally trickier use of descent theory again allows us to define $H^i_{\mathrm{rig}}(X/K)$ when X does not admit a realisable compactification (X,Y) (we will in general call a variety X realisable if there exists a smooth and proper frame $(X,Y,\mathfrak{P})$, i.e. it admits a realisable compactification).

A.4 Coefficients

As a 'de Rham type' cohomology theory, the coefficients of the theory of rigid cohomology will be certain coherent sheaves with integrable connection. The only slightly subtlety is in the fact that the 'overconvergence' property introduced in the previous section needs to be applied not just to the sheaves themselves, but to the connection on them as well.

So let $(X,Y,\mathfrak{P})$ be a smooth frame, and $\mathscr{E}$ a coherent $j_X^\dagger \mathscr{O}_{]Y[_{\mathfrak{P}}}$ module with integrable connection $\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{]Y[_{\mathfrak{P}}/K}$. Then via the natural map $\mathscr{O}_{]Y[_{\mathfrak{P}}} \to j_X^\dagger \mathscr{O}_{]Y[_{\mathfrak{P}}}$ we may consider $\mathscr{E}$ as a (not necessarily coherent) $\mathscr{O}_{]Y[_{\mathfrak{P}}}$-module with integrable connection. Hence it has a 'formal Taylor series', constructed as follows. For all $n \geq 0$, let $]Y[^{(n)}_{\mathfrak{P}}$ denote the nth infinitesimal neighbourhood of $]Y[_{\mathfrak{P}}$ inside $]Y[^2_{\mathfrak{P}}$, and let $p_i^{(n)} :]Y[^{(n)}_{\mathfrak{P}} \to]Y[_{\mathfrak{P}}$ denote the two projections.

Definition A.27 A *stratification* on a $\mathscr{O}_{]Y[_{\mathfrak{P}}}$-module $\mathscr{E}$ is a collection of isomorphisms

$$\theta^{(n)} : p_2^{(n)*}\mathscr{E} \to p_1^{(n)*}\mathscr{E}$$

for each n, such that:

1. $\theta^{(1)} = \mathrm{id}$;
2. if $m \leq n$ then the natural diagram

$$\begin{array}{ccc} p_2^{(n)*}\mathscr{E} & \longrightarrow & p_1^{(n)*}\mathscr{E} \\ \downarrow & & \downarrow \\ p_2^{(m)*}\mathscr{E} & \longrightarrow & p_1^{(m)*}\mathscr{E} \end{array}$$

   commutes;
3. (the cocyle condition) if $]Y[^{2,(n)}_{\mathfrak{P}}$ denotes the nth infinitesimal neighbourhood of $]Y[_{\mathfrak{P}}$ inside $]Y[^3_{\mathfrak{P}}$, and $p_{ij}^{(n)} :]Y[^{2,(n)}_{\mathfrak{P}} \to]Y[^{(n)}_{\mathfrak{P}}$ the projections, then

$$p_{12}^*(\theta^{(n)}) \circ p_{23}^*(\theta^{(n)}) = p_{13}^*(\theta^{(n)}).$$

If we let $\mathscr{P}^1_{]Y[_{\mathfrak{P}}/K}$ denote the structure sheaf of $]Y[^{(2)}_{\mathfrak{P}}$, then the 'second level' of a stratification on a $\mathscr{O}_{]Y[_{\mathfrak{P}}}$-module $\mathscr{E}$ can be viewed as an isomorphism

$$\theta^{(2)} : \mathscr{P}^1_{]Y[_{\mathfrak{P}}/K} \otimes \mathscr{E} \xrightarrow{\sim} \mathscr{E} \otimes \mathscr{P}^1_{]Y[_{\mathfrak{P}}/K}.$$

By definition of a stratification, the induced map

$$(\theta^{(2)} \circ 1 \otimes \mathrm{id}) - \mathrm{id} \otimes 1 : \mathscr{E} \to \mathscr{E} \otimes \mathscr{P}^1_{]Y[_{\mathfrak{P}}/K}$$

lands in the kernel of the natural map $\mathscr{E} \otimes \mathscr{P}^1_{]Y[_{\mathfrak{P}}/K} \to \mathscr{E}$, in other words inside $\mathscr{E} \otimes \Omega^1_{]Y[_{\mathfrak{P}}/K}$. Hence we obtain a K linear map

$$\mathscr{E} \to \mathscr{E} \otimes \Omega^1_{]Y[_{\mathfrak{P}}/K}$$

and in fact one can show that this map is an integrable connection on $\mathscr{E}$.

Proposition A.28 ([4], Sect. 4.1) *This induces an equivalence of categories between stratified $\mathscr{O}_{]Y[_{\mathfrak{P}}}$-modules and $\mathscr{O}_{]Y[_{\mathfrak{P}}}$-modules with integrable connection.*

Note that there are no coherence hypotheses in the proposition. Hence to any $\mathscr{O}_{]Y[_{\mathfrak{P}}}$-module $\mathscr{E}$ with integrable connection, we get a collection of isomorphisms

$$\theta^{(n)} : p_2^{(n)*}\mathscr{E} \to p_1^{(n)*}\mathscr{E}$$

which fit together to give a map

$$\hat{\theta} : \varprojlim_n p_2^{(n)*}\mathscr{E} \to \varprojlim_n p_1^{(n)*}\mathscr{E}$$

called the formal Taylor series of $\mathscr{E}$. If we consider the frame $(X, Y, \mathfrak{P}^2)$ where Y is embedded in $\mathfrak{P}^2$ via the diagonal, then for all $n \geq 0$ the natural map $(]Y[_{\mathfrak{P}})^2 \to]Y[_{\mathfrak{P}^2}$ induces a map $]Y[^{(n)}_{\mathfrak{P}} \to]Y[_{\mathfrak{P}^2}$, and hence the following definition makes sense.

Definition A.29 We say that an integrable connection ∇ on a coherent $j_X^\dagger \mathscr{O}_{]Y[_{\mathfrak{P}}}$ module $\mathscr{E}$ is *overconvergent* if its formal Taylor series $\hat{\theta}$ converges on some strict neighbourhood of the diagonal. In other words $\mathscr{E}$ is overconvergent if there exists an isomorphism

$$\theta : p_2^*\mathscr{E} \to p_1^*\mathscr{E}$$

of $j_X^\dagger \mathscr{O}_{]Y[_{\mathfrak{P}^2}}$-modules such that for all n the Taylor isomorphism $\theta^{(n)}$ is obtained by pulling back θ via $]Y[^{(n)}_{\mathfrak{P}} \to]Y[_{\mathfrak{P}^2}$.

Definition A.30 An overconvergent isocrystal on $(X, Y)/K$ is a coherent $j_X^\dagger \mathscr{O}_{]Y[_{\mathfrak{P}}}$ module $\mathscr{E}$ together with an overconvergent connection ∇. The category of such objects is denoted $\mathrm{Isoc}^\dagger((X, Y)/K)$.

Again, applying the Strong Fibration Theorem in varying degrees of subtlety gives the following.

Theorem A.31 ([1], Théorème 2.3.1, 2.3.5) *The category* $\mathrm{Isoc}^\dagger((X,Y)/K)$ *only depends on the pair* (X,Y) *and not on the choice of formal scheme* $\mathfrak{P}$. *When* Y *is proper, it further only depends on* X*, and we will therefore write* $\mathrm{Isoc}^\dagger(X/K)$.

Again, while this gives a definition only for 'realisable' varieties, we may use descent to define $\mathrm{Isoc}^\dagger((X,Y)/K)$ and $\mathrm{Isoc}^\dagger(X/K)$ in the non-realisable case. If $(X,Y,\mathfrak{P})$ is a smooth frame and $\mathscr{E} \in \mathrm{Isoc}^\dagger((X,Y)/K)$, then we define

$$H^i_{\mathrm{rig}}((X,Y)/K,\mathscr{E}) := H^i(]Y[_{\mathfrak{P}}, \mathscr{E} \otimes j_X^\dagger \Omega^*_{]Y_{\mathfrak{P}}}),$$

again, this only depends on (X,Y), and moreover only on X when Y is proper, in which case we will write $H^i_{\mathrm{rig}}(X/K,\mathscr{E})$.

The category $\mathrm{Isoc}^\dagger((X,Y)/K)$ is functorial in both (X,Y) and K. Concretely, if $(X',Y') \to (X,Y)$ is a morphism of pairs, then we get a pullback functor $\mathrm{Isoc}^\dagger((X,Y)/K) \to \mathrm{Isoc}^\dagger((X',Y')/K)$, and if $K \to K'$ is an isometric extension of complete, discretely valued fields, with $k \to k'$ the induced extension of residue fields, then we also get a base extension functor $\mathrm{Isoc}^\dagger((X,Y)/K) \to \mathrm{Isoc}^\dagger((X_{k'},Y_{k'})/K')$.

In particular, once we have chosen a lift of Frobenius σ on K, we have a canonical Frobenius pullback functor

$$F^* : \mathrm{Isoc}^\dagger((X,Y)/K) \to \mathrm{Isoc}^\dagger((X,Y)/K).$$

Definition A.32 A Frobenius structure on an overconvergent isocrystal $\mathscr{E}$ is an isomorphism $\varphi : F^*\mathscr{E} \to \mathscr{E}$, and an overconvergent isocrystal with Frobenius structure is called an overconvergent F-isocrystal. The category of such objects is denoted by $F\text{-}\mathrm{Isoc}^\dagger((X,Y)/K)$.

A Frobenius structure $\varphi : F^*\mathscr{E} \to \mathscr{E}$ induces a Frobenius structure on $H^i_{\mathrm{rig}}((X,Y)/K)$, that is an isomorphism

$$\varphi : H^i_{\mathrm{rig}}((X,Y)/K,\mathscr{E}) \otimes_{K,\sigma} K \to H^i_{\mathrm{rig}}((X,Y)/K,\mathscr{E}).$$

The main structural result concerning the cohomology of overconvergent isocrystals is then the following.

Theorem A.33 ([3], Theorem 1.2.1) *Let* X *be a* k*-variety, and* $\mathscr{E} \in F\text{-}\mathrm{Isoc}^\dagger(X/K)$ *an overconvergent* F*-isocrystal. Then the cohomology groups* $H^i_{\mathrm{rig}}(X/K,\mathscr{E})$ *are finite dimensional over* K.

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Appendix B
Adic Spaces and Rigid Spaces

One of the key insights of this book is that by venturing outside the world of 'classical' rigid analytic spaces, one can interpret the bounded Robba ring $\mathscr{E}_K^\dagger$ entirely within the framework of Berthelot's theory of frames and overconvergent structure sheaves. It therefore becomes necessary to work systematically with Huber's theory of adic spaces, or equivalently Fujiwara–Kato's theory of rigid spaces, and while these approaches to rigid analytic geometry are becoming better and better known, it is still not a language that enjoys universal familiarity. The purpose of this appendix, therefore, is to give a brief overview of these two approaches to rigid analytic geometry, aimed at those who have a reasonable understanding of:

- Tate's theory of rigid analytic spaces over a complete, non-archimedean field K;
- the theory of formal schemes over its valuation ring $\mathscr{V}$.

The reader interested in leaning more about the subject can consult the foundational references [2, 3, 4], or for a slightly more gentle introduction to adic spaces, Sect. 2 of [7]. Much of the material in this appendix can be found in either the introduction to [2] or Sect. 2 of [7]. Although both theories work in much greater generality, we will only need the case where K is discretely valued, so we will therefore assume that this is so and let $\pi \in \mathscr{V}$ denote a uniformiser. If A is an affinoid algebra over K, we will let $\mathrm{Sp}\,(A)$ denote the corresponding affinoid rigid analytic space over K, in the sense of Tate.

B.1 Huber's Adic Spaces

Huber's approach to analytic geometry via adic spaces is essentially concerned with finding the correct notion of a 'point' in rigid geometry, just as the development of scheme theory hinged upon the introduction of 'generic' points extending the classical notion of a point on an variety over an algebraically closed field. Classically, points of Tate rigid spaces $\mathrm{Sp}(A)$ can be viewed as certain kinds of discrete valuations on the affinoid algebra A, and it has been known ever since these spaces were first

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introduced that there was a sense in which they were 'missing' many points. The most blatant example of this can be seen in the need to define a G-topology on rigid spaces using admissible coverings, rather than having an honest topological space with honest open coverings, and very closely related to this is the fact that there exist non-zero sheaves on rigid spaces whose stalks at every point are zero.

What this last phenomenon in particular tells is is that the actual points of a rigid space X do not account for all the points of the associated topos X_{rig}, and what Huber's theory of adic spaces does is provide a genuine topological space underlying the topos X_{rig} (since X_{rig} can be shown to be sober, we know that there exists a unique such space). Although this space can actually be described entirely within the terms of Tate's theory, using the notion of prime filters as in [6], Huber also extends the scope of rigid geometry far beyond that of spaces locally of finite type over a complete, non-archimedean field by replacing the discrete valuations appearing in Tate's theory by more general, not necessarily discrete, not necessarily rank 1, valuations on more general kinds of 'affinoid algebras'.

Throughout this Appendix, and indeed the whole book, we will write all totally ordered abelian groups multiplicatively. If Γ is a totally ordered abelian group, then we can put a total order and a multiplication law on $\Gamma \cup \{0\}$ by decreeing that $0 < \gamma$ and $0 \cdot \gamma = 0$ for all $\gamma \in \Gamma$, and that $0 \cdot 0 = 0$.

Definition B.1 1. A valuation on a ring A with values in a totally ordered abelian group Γ is a multiplicative map $v : A \to \Gamma \cup \{0\}$ such that $v(x + y) \leq \max\{v(x), v(y)\}$ for all $x, y \in A$.

2. Two valuations $v : A \to \Gamma \cup \{0\}$ and $v' : A \to \Gamma' \cup \{0\}$ are considered to be equivalent if there exists an ordered homomorphism $v(A) \cong v'(A)$ such that the diagram

$$\begin{array}{ccccc} A & \longrightarrow & v(A) & \longrightarrow & \Gamma \cup \{0\} \\ & \searrow & \downarrow & & \\ & & v'(A) & \longrightarrow & \Gamma' \cup \{0\} \end{array}$$

commutes.

3. If A has a topology, then we say a valuation $v : A \to \Gamma \cup \{0\}$ is continuous if for all $\gamma \in \Gamma$, the set $A_\gamma = \{a \in A \mid v(a) < \gamma\}$ is open in A.

Example B.2 We will give examples of three kinds of valuation on the Tate algebra $K\langle x\rangle = \left\{ \sum_{i\geq 0} a_i x^i \,\middle|\, a_i \to 0 \right\}$.

1. Let L/K be a finite extension, $\alpha \in \mathscr{O}_L$ and $\mathrm{ev}_\alpha : K\langle x\rangle \to L$ the map given by evaluation at α. Then there exists a unique norm $|\cdot| : L \to \mathbb{R}_{\geq 0} = \mathbb{R}_{>0} \cup \{0\}$ extending the fixed norm on K, and we obtain a valuation $v_\alpha : K\langle x\rangle \to \mathbb{R}_{\geq 0}$ by setting $v_\alpha(f) = |f(\alpha)|$. Note that this valuation is discrete, in that $v(K\langle x\rangle) = r^{\mathbb{Z}} \cup \{0\}$ for some $r \in \mathbb{R}_{>0}$.
2. Let $\widehat{\overline{K}}$ denote the completion of the algebraic closure of K, and choose a sequence of closed discs (of some centre and radius) $D_1 \supset D_2 \supset \ldots$ such that $\cap_i D_i = \emptyset$.

Then the valuation $v : K\langle x\rangle \to \mathbb{R}_{\geq 0}$, $v(f) = \inf_i \sup_{\alpha \in D_i} |f(\alpha)|$ is a valuation on $K\langle x\rangle$. It is no longer discrete, but it is still rank 1.

3. Consider the totally ordered abelian group $\Gamma = \mathbb{R}_{>0} \oplus \gamma^{\mathbb{Z}}$ where the ordering is defined by decreeing that $r < \gamma < 1$ for all $r \in \mathbb{R}$ with $0 < r < 1$. Then there exists a valuation $v : K\langle x\rangle \to \Gamma \cup \{0\}$ defined by $v(\sum_i a_i x^i) = \max_i |a_i|\gamma^i$. This valuation is no longer rank 1.

Definition B.3 1. A Tate algebra is a complete topological K-algebra R such that there exists a subring $R_0 \subset R$ such that aR_0 for $a \in K$ forms a neighbourhood basis of 0 in the topology on R (such a subring R_0 is called a *ring of definition* for R). We let R° denote the subring of power bounded elements, i.e. those elements $r \in R$ for which there exists $a \in K$ such that $\{r^n\}_{n\geq 0} \subset aR_0$.

2. An affinoid algebra (in the sense of Huber) is a pair (R, R^+) where R is a Tate algebra over K, and $R^+ \subset R^\circ$ is an open and integrally closed subring of R.

Remark B.4 Actually, Huber considers much more general kinds of affinoid algebras, however, for the purposes of this book the more restrictive kind just introduced will suffice.

Example B.5 Of course, the fundamental example of an affinoid algebra (in the sense of Huber) is (A, A°) where A is an affinoid algebra in the sense of Tate, and $A^\circ \subset A$ is the subring of power bounded elements. By definition, these are exactly the affinoid algebras which are considered to be topologically of finite type over K.

Example B.6 An affinoid algebra that doesn't come under the scope of Tate's theory (and is the key example in this whole book) is the pair $(S_K, \mathscr{V}[\![t]\!])$ where $S_K = K \otimes \mathscr{V}[\![t]\!]$ is endowed with the p-adic topology, i.e. given the topology induced by the norm $\left\|\sum_i a_i t^i\right\| = \sup_i |a_i|$.

Example B.7 If (R, R^+) is an affinoid algebra, then we can consider the Tate algebra

$$(R, R^+)\langle x_1, \ldots, x_n\rangle = \left(R\langle x_1, \ldots, x_n\rangle, R^+\langle x_1, \ldots x_n\rangle\right)$$

where

$$R\langle x_1, \ldots, x_n\rangle = \left\{ \sum_I a_I x^I \,\middle|\, a_I \to 0 \text{ as } |I| \to \infty \right\}$$

and $R^+\langle x_1, \ldots, x_n\rangle = R^+[\![x_1, \ldots, x_n]\!] \cap R\langle x_1, \ldots, x_n\rangle$. We will also refer to $R\langle x_1, \ldots, x_n\rangle$ and $R^+\langle x_1, \ldots, x_n\rangle$ as the Tate algebras over R and R^+ respectively. A ring of definition for $R\langle x_1, \ldots, x_n\rangle$ is given by $R_0\langle x_1, \ldots, x_n\rangle$, where R_0 is a ring of definition for R.

Affinoid algebras provide the building blocks for the theory of adic spaces, and the 'affine' spaces in this theory arise as the adic spectra of these affinoid algebras.

Definition B.8 Let (R, R^+) be an affinoid algebra over K. Then we define the adic spectrum $\mathrm{Spa}\left(R, R^+\right)$ to be the set of equivalence classes of continuous valuations $v : R \to \Gamma \cup \{0\}$ such that $v(R^+) \leq 1$.

We can put a topology on $\mathrm{Spa}\left(R, R^+\right)$ as that generated by the open subsets

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) = \left\{v \in \mathrm{Spa}\left(R, R^+\right) \middle| \, v(f_i) \le v(g) \ \forall i\right\}$$

for any $f_1, \ldots, f_n, g \in R$ such that $(f_1, \ldots, f_n) = R$. Such subsets of $\mathrm{Spa}\left(R, R^+\right)$ are called *rational* subsets. For f_i, g as above we may consider the localised affinoid algebra

$$(R, R^+)\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle = \left(R\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle, R^+\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle\right)$$

where

$$R\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle = \frac{R\left\langle x_1, \ldots, x_n\right\rangle}{(gx_i - f_i)}$$

and $R^+\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle$ is the integral closure of the completion of the image of $R^+\left[\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right]$ inside $R\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle$.

Proposition B.9 ([3], Proposition 1.3) *The natural map*

$$\mathrm{Spa}\left((R, R^+)\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle\right) \to \mathrm{Spa}\left(R, R^+\right)$$

is injective, with image the rational subset $U\left(\frac{f_1, \ldots, f_n}{g}\right)$*. Moreover, the affinoid algebra* $(R, R^+)\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle$ *is universal with this property.*

We may extend the associations

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) \mapsto R\left\langle \frac{f_1, \ldots, f_n}{g}\right\rangle$$
$$U\left(\frac{f_1, \ldots, f_n}{g}\right) \mapsto R^+\left\langle \frac{f_1, \ldots, f_n}{g}\right\rangle$$

to presheaves of rings $\mathscr{O}_X$ and $\mathscr{O}_X^+$ on the topological space $X = \mathrm{Spa}\left(R, R^+\right)$. Unfortunately, it is not necessarily true that $\mathscr{O}_X$ and $\mathscr{O}_X^+$ are sheaves.

Proposition B.10 ([3], Theorem 2.2) *Let* (R, R^+) *be an affinoid algebra over* K*, and assume that one of the following is true:*

1. *R is strongly Noetherian, that is $R\left\langle x_1, \ldots, x_n\right\rangle$ is Noetherian for all $n \ge 0$;*
2. *there exists a Noetherian ring of definition $R_0 \subset R$ such that R is a finitely generated R_0 algebra.*

Then $\mathscr{O}_X$ and $\mathscr{O}_X^+$ are sheaves, and moreover the stalks

$$\mathscr{O}_{X,x} = \mathrm{colim}_{U \supset x} \mathscr{O}_X(U), \quad \mathscr{O}_{X,x}^+ = \mathrm{colim}_{U \supset x} \mathscr{O}_X^+(U)$$

at any point $x \in X$ are local rings.

Definition B.11 An adic space over K is a triple $(X, \mathscr{O}_X, \mathscr{O}_X^+)$ where $\mathscr{O}_X$ and $\mathscr{O}_X^+$ are sheaves of complete topological rings on X, locally isomorphic to $\mathrm{Spa}\left(R, R^+\right)$ for (R, R^+) an affinoid algebra over K.

Example B.12 If X is a rigid analytic variety over K in the sense of Tate, then covering X by affinoids $\mathrm{Sp}\,(A)$ we may glue the adic spaces $\mathrm{Spa}\,(A, A^\circ)$ together to produce an adic space X^{ad} over K.

Example B.13 To illustrate how this functor works, consider the open unit disc $X = \cup_{m \geq 0} \mathrm{Sp}\left(K\left\langle p^{1/m}x\right\rangle\right)$ where

$$K\left\langle p^{1/m}x\right\rangle = \left\{ \sum_i a_i x^i \,\middle|\, |a_i|\, p^{-i/m} \to 0 \right\}$$

is the ring of functions on the closed disc of radius $p^{-1/m}$. Writing

$$K\left\langle p^{1/m}x\right\rangle = \frac{K\langle x, y\rangle}{(py - x^m)}$$

we can see that the adic space X^{ad} can be identified with the subset of the closed unit disc $\mathbb{D}_K^1 = \mathrm{Spa}\,(K\langle x\rangle, \mathscr{V}\langle x\rangle)$ consisting of valuations v such that there exists an $m \geq 0$ with $v(p^{-1}x^m) \leq 1$. Note that this is *not* the same thing as the 'naïve' open unit disc

$$\left\{\alpha \in \mathbb{D}_K^1 \,\middle|\, v_\alpha(x) < 1\right\}$$

which is in fact *not* an open subset of $\mathbb{D}_K^1$.

Example B.14 Since $\mathscr{V}[[t]]$ is Noetherian the 'bounded open unit disc' $\mathbb{D}_K^b := \mathrm{Spa}\,(S_K, \mathscr{V}[[t]])$ is an adic space over K, which can be described as follows. For any $m \geq 0$ we have

$$S_K\left\langle p^{1/m}t\right\rangle \cong K\left\langle p^{1/m}t\right\rangle$$

and hence $\mathbb{D}_K^b$ contains the open unit disc $\mathbb{D}_K^{1,\circ}$ as an open subset. We also have extra 'boundary' points $\{\xi, \xi_-\}$ where ξ is the open point

$$\{\xi\} = \left\{\alpha \in \mathbb{D}_K^b \mid v_\alpha(t) = 1\right\}$$

corresponding to the natural p-adic valuation on $S_K \subset \mathscr{E}_K$, and ξ_- is the rank 2 valuation from Example B.2.3.

Example B.15 To see an example of how the introduction of adic spaces solves the problem of rigid spaces not having enough points, consider again the adic closed unit disc $\mathbb{D}^1_K := \mathrm{Spa}\,(K\langle x\rangle, \mathscr{V}\langle x\rangle)$. Then we have open subsets U, V of $\mathbb{D}^1_K$

$$U = \mathbb{D}^{1,\circ}_K = \left\{\alpha \in \mathbb{D}^1_K \,\middle|\, \exists m \geq 0 \text{ s.t. } v_\alpha(p^{-1}x^m) \leq 1\right\}$$
$$V = \left\{\alpha \in \mathbb{D}^1_K \,\middle|\, v_\alpha(x) = 1\right\}$$

i.e. the open unit disc and the closed annulus of radius 1, but it is now longer true that $\mathbb{D}^1_K = U \cup V$, since the valuation considered in Example B.2.3 satisfies $v(x) < 1$ but $v(p^{-1}x^m) > 1$ for all $m \geq 0$. While it is trivially true that

$$\mathbb{D}^1_K = \left\{\alpha \in \mathbb{D}^1_K \,\middle|\, v_\alpha(x) = 1\right\} \cup \left\{\alpha \in \mathbb{D}^1_K \,\middle|\, v_\alpha(x) < 1\right\}$$

we remarked above that the latter is *not* an open subset of $\mathbb{D}^1_K$. In fact, the topological space $\mathbb{D}^1_K$ is connected, exactly as one would hope.

Definition B.16 A morphism of affinoid algebras $(R, R^+) \to (S, S^+)$ is said to be a quotient morphism if $R \to S$ is continuous, open and surjective, and S^+ is the integral closure of the image of R^+. A morphism $(R, R^+) \to (S, S^+)$ is said to be topologically of finite type if there exists a commutative diagram

$$\begin{array}{ccc} & & (R, R^+)\langle x_1, \ldots, x_n\rangle \\ & \nearrow & \downarrow \\ (R, R^+) & \longrightarrow & (S, S^+) \end{array}$$

with the right hand vertical arrow a quotient map.

A morphism $X \to Y$ of adic spaces over K is then (locally) of finite type if locally on Y (and on X) it is of the form $\mathrm{Spa}\left(S, S^+\right) \to \mathrm{Spa}\left(R, R^+\right)$ with $(R, R^+) \to (S, S^+)$ topologically of finite type.

Remark B.17 Note that if we fix a base affinoid algebra (R, R^+) then for any topologically finite type map $(R, R^+) \to (S, S^+)$ the ring S^+ is determined entirely by R^+ and S, namely it is the integral closure of the image of $R^+\langle x_1, \ldots, x_n\rangle$ under any presentation $R\langle x_1, \ldots, x_n\rangle \to S$. Hence, provided that we are understood to be working in the category of adic spaces locally of finite type over $\mathrm{Spa}\left(R, R^+\right)$, there is no ambiguity in writing $\mathrm{Spa}\,(S)$ instead of $\mathrm{Spa}\left(S, S^+\right)$.

Unfortunately, it is not true that fibre products of adic spaces over K are always representable, however, they are representable in a reasonable amount of generality, and it is straightforward to describe them locally.

Proposition B.18 ([4], Proposition 1.2.2) *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of adic spaces over K, and assume that either f or g is locally of finite type. Then the fibre product $X \times_Z Y$ exists in the category of adic spaces over K.*

Let $Z = \mathrm{Spa}\left(R, R^+\right)$ be an affinoid adic space, let $X = \mathrm{Spa}\left(S, S^+\right)$ be topologically of finite type over X and let $Y = \mathrm{Spa}\left(T, T^+\right)$ be an arbitrary adic space over X. Choose rings of definition R_0, S_0, T_0 for R, S, T. Then we may complete $S \otimes_R T$ with respect to the topology induced by the subring $S_0 \otimes_{R_0} T_0$, to obtain a Tate ring $U := S\widehat{\otimes}_R T$, and by letting U^+ denote the integral closure of the image of the completion of $S^+ \otimes_{R^+} T^+$ inside U we obtain an affinoid algebra (U, U^+). Then the fibre product $X \times_Z Y$ is then just $\mathrm{Spa}\left(U, U^+\right)$.

Since we may define open and closed immersions of adic spaces (in exactly the same way as one does for schemes), we may therefore speak of a locally finite type morphism $X \to Y$ being separated, meaning that the diagonal map $X \to X \times_Y X$ is a closed immersion.

Theorem B.19 ([4], Sect. 1.1.11) *The association $X \mapsto X^{\mathrm{ad}}$ from Example B.12 induces a fully faithful functor*

$$\{\textit{q.s. rigid analytic varieties } /K\} \to \{\textit{adic spaces } /K\}$$

from quasi-separated rigid analytic varieties over K to adic spaces over K. The essential image of this functor is exactly the quasi-separated adic spaces which are locally of finite type over K. Moreover the topos associated to the G-topology on X coincides with the topos associated to the topological space X^{ad}.

Finally, we will need to know how to associate adic spaces over K to formal schemes over $\mathscr{V}$, or more specifically Noetherian, π-adic formal schemes over $\mathscr{V}$ (but not necessarily of finite type). By localising it suffices to do this for affine formal schemes $\mathfrak{X} = \mathrm{Spf}(A)$. But in this case $(A \otimes_{\mathscr{V}} K, A^+)$, where A^+ is the integral closure of the image of A in $A \otimes_{\mathscr{V}} K$, is an affinoid algebra with a Noetherian ring of definition (namely the image of A), and hence $\mathfrak{X}_K := \mathrm{Spa}\left(A \otimes_{\mathscr{V}} K, A^+\right)$ is an adic space over K.

Note that if $\mathfrak{X}$ is topologically of finite type over $\mathscr{V}$ then there are two potential ways to come up with an adic space. Firstly, one simply takes the generic fibre $\mathfrak{X}_K$, as an adic space. But one can also consider the generic fibre as a rigid analytic space in the sense of Tate, and then take the associated adic space. Needless to say, these two constructions give the same answer.

B.2 Rigid Spaces of Fujiwara–Kato

The Fujiwara–Kato approach to analytic geometry, as developed in [2], essentially turns a theorem of Raynaud, describing the category of rigid analytic spaces over some complete non-archimedean field K in terms of the category of formal schemes over its ring of integers $\mathscr{V}$, on its head, and uses the category of formal schemes to describe the category of analytic spaces. To recall Raynaud's result, we assume K is discretely valued, choose a uniformiser $\pi \in \mathscr{V}$, and let $\mathbf{FSch}_{\mathscr{V}}$ denote the category of quasi-compact, quasi-separated formal schemes topologically of finite type over $\mathrm{Spf}(\mathscr{V})$.

Definition B.20 We say a blow-up $\mathfrak{Y} \to \mathfrak{X}$ of formal schemes along a coherent ideal $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ is admissible if $\mathscr{I}$ locally contains a power of the ideal (π).

Theorem B.21 ([5]) *The generic fibre functor*

$$\mathfrak{X} \mapsto \mathfrak{X}_K$$

induces an equivalence between the category $\mathbf{An}_K$ *of quasi-compact, quasi-separated rigid analytic varieties over K (in the sense of Tate) and the localisation of* $\mathbf{FSch}_{\mathscr{V}}$ *at the class of admissible blow-ups.*

To obtain a more general theory of rigid spaces, then, Fujiwara and Kato start with a more general category of formal schemes, and then perform the above localisation formally to obtain a more general category of quasi-compact, quasi-separated rigid spaces. They then use these as the basic building blocks to glue together to obtain rigid spaces which are not necessarily quasi-compact or quasi-separated. Since we will only be interested in the case where K is discretely valued (and so $\mathscr{V}$ is Noetherian), we will only work with Noetherian formal schemes, although Fujiwara and Kato are able to deal with a more general class of formal schemes, namely those which are 'universally rig-Noetherian', the definition of which would be too much of a distraction to go into.

Definition B.22 A formal scheme is understood to be a Noetherian, adic, quasi-separated formal scheme, that is a quasi-compact, quasi-separated locally ringed space $\mathfrak{X}$ which is covered by the formal spectra $\mathrm{Spf}(A)$ of Noetherian topological rings A, whose topology is that induced by some (necessarily finitely generated) ideal $I \subset A$.

An adic morphism of formal schemes $\mathfrak{Y} \to \mathfrak{X}$ is one which is locally of the form $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ with $A \to B$ a continuous homomorphism, such that for some ideal of definition I of A, $I \cdot B$ is an ideal of definition for B. We let $\mathbf{AcFs}^*$ denote the category of formal schemes with adic morphisms.

Then the key definition is again that of an admissible blow-up.

Definition B.23 A blow-up $\mathfrak{Y} \to \mathfrak{X}$ along an ideal $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ is said to be admissible if locally on $\mathfrak{X}$, $\mathscr{I}$ contains an ideal of definition.

Definition B.24 ([2], Sect. II) Define the category $\mathbf{CRf}$ of coherent (i.e. quasi-compact, quasi-separated) rigid spaces to be the localisation of $\mathbf{AcFs}^*$ with respect to the class of admissible blow-ups. We will denote the canonical functor

$$\mathbf{AcFs}^* \to \mathbf{CRf}$$

by $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{rig}}$. □

Example B.25 Let $\mathfrak{X} = \widehat{\mathbb{A}}^1_{\mathscr{V}}$ be the formal affine line over $\mathscr{V}$. An example of an admissible blow-up of $\mathfrak{X}$ is given by blowing up finitely many k-rational points of the special fibre $\mathbb{A}^1_k$, producing a formal model whose special fibre has one component isomorphic to $\mathbb{A}^1_k$ and several isomorphic to $\mathbb{P}^1_k$, each disjoint from the others and meeting $\mathbb{A}^1_k$ in exactly one k-rational point. Iterating this produces other formal models whose special fibres are trees.

They then proceed to construct the category of general (i.e. not necessarily quasi-compact or quasi-separated) rigid spaces by a process of 'formal gluing'. This is a somewhat delicate, and we won't go into the details here, the interested reader should consult Sect. II.2 of [2].

The study of rigid spaces then becomes, at least locally, the study of formal schemes modulo admissible blowups, and unlike Huber's theory where many properties have to be defined from scratch (albeit in an entirely predictable manner), in the Fujiwara–Kato theory, they may be defined simply by using the corresponding property for formal schemes.

For example, a morphism of rigid spaces $\mathscr{Y} \to \mathscr{X}$ is finite, of finite type, locally of finite type, proper, separated, a closed immersion, or an open immersion if, locally on $\mathscr{X}$ (and on $\mathscr{Y}$ in the case of a locally of finite type or separated morphism, or an open immersion), it has a formal model $\mathfrak{Y} \to \mathfrak{X}$ which has the same property. Similarly, fibre products of rigid spaces are constructed on the level of formal schemes, in other words one shows that the functor $\mathfrak{X} \mapsto \mathfrak{X}_K$ commutes with finite limits.

The link between the two approaches to rigid geometry comes from the Zariski–Riemann spaces attached to their rigid spaces by Fujiwara–Kato. In their theory, a coherent rigid space $\mathscr{X}$ is in some sense a collection of its formal models, and to produce a ringed space associated to this data, we take the inverse limit of these formal schemes (considered as locally ringed spaces).

Definition B.26 Let $\mathscr{X}$ be a coherent rigid space. Then we define the doubly ringed space

$$\mathbf{ZR}(X) = (\langle \mathscr{X} \rangle, \mathscr{O}_{\mathscr{X}}, \mathscr{O}^+_{\mathscr{X}})$$

as follows. Let $C_{\mathscr{X}}$ denote the category of all formal models of $\mathscr{X}$, then we set

$$(\langle \mathscr{X} \rangle, \mathscr{O}^+_{\mathscr{X}}) = \varprojlim_{C_{\mathscr{X}}} (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}).$$

If $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ is an ideal of definition for some model of $\mathscr{X}$, then we set

$$\mathscr{O}_{\mathscr{X}} = \mathscr{O}^+_{\mathscr{X}}[\mathscr{I}^{-1}].$$

This does not depend on the choice of $\mathfrak{X}$ or $\mathscr{I}$.

The major comparison theorem between Fujiwara–Kato's rigid spaces and Huber's adic spaces is then the following (there is a much more general comparison theorem, but the following simpler version is all we will need).

Theorem B.27 ([2], Sect. II.A) *The map*

$$\mathscr{X} \mapsto \mathbf{ZR}(\mathscr{X})$$

gives rise to a functor from the category $\mathbf{Rf}_K$ *of 'locally universally Noetherian' rigid spaces over* $\mathrm{Spf}(\mathscr{V})^{\mathrm{rig}}$ *to the category* $\mathbf{Ad}_K$ *of adic spaces over* K*. Moreover, for any fixed Noetherian,* π*-adic formal scheme* $\mathfrak{X}$ *over* $\mathscr{V}$ *(not necessarily of finite type) this induces an equivalence of categories between rigid spaces locally of finite type over* $\mathfrak{X}^{\mathrm{rig}}$*, and adic spaces locally of finite type over* $\mathfrak{X}_K$*.*

Remark B.28 The condition of being locally universally Noetherian is somewhat technical, however, since we will always be working with objects locally of finite type over $\mathrm{Spf}(\mathscr{V}[\![t]\!])^{\mathrm{rig}}$ it will always be satisfied.

Example B.29 This functor takes the rigid space $\mathrm{Spf}(\mathscr{V}[\![t]\!])^{\mathrm{rig}}$ to the bounded open unit disc $\mathbb{D}_K^b := \mathrm{Spa}\,(S_K, \mathscr{V}[\![t]\!])$ defined above.

Example B.30

1. Let $\beta \in \mathbb{A}_k^1$ be a rational point, let $\mathfrak{X}$ be the model of $\mathbb{D}_K^1$ obtained by blowing up $\widehat{\mathbb{A}}_{\mathscr{V}}^1$ at β, and let C denote the exceptional divisor. For any admissible blowup $\mathfrak{Y} \to \mathfrak{X}$, the generic point of the strict transform of C maps to the generic point of C, and we therefore get a point in the inverse limit $\mathbb{D}_K^1$. The corresponding valuation on $K\langle T\rangle$ is the height one valuation $K\langle x\rangle \to \mathbb{R}_{\geq 0}$ given by $f \mapsto \sup_{|\alpha - \tilde{\beta}| \leq 1/p} |f(\alpha)|$ where $\tilde{\beta}$ is some lift of β to $\mathscr{V}$. This is a Type II point in the general classification of points on $\mathbb{D}_K^1$, and it does not depend on the choice of lift $\tilde{\beta}$.
2. Let $\mathfrak{X}_0 = \mathfrak{X}$ be as above and write $\beta_0 = \beta$, except now parametrise the exceptional divisor so that the intersection point with the strict transform of $\mathbb{A}_k^1$ is the point at ∞, and blowup some point on this divisor corresponding to a rational point $\beta_1 \in \mathbb{A}_k^1$ to obtain $\mathfrak{X}_1$. Continue indefinitely to obtain a series of points $\beta_0, \beta_1, \beta_2, \ldots \in \mathbb{A}_k^1$ and a series of models $\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \ldots$ of $\mathbb{D}_K^1$. Now choose points $\tilde{\beta}_i \in \mathscr{V}$ starting with $\tilde{\beta}_0$ lifting β_0, such that $\tilde{\beta}_i \equiv \tilde{\beta}_{i-1} \bmod \pi^i$ and $\frac{1}{\pi^i}(\tilde{\beta}_i - \tilde{\beta}_{i-1}) \equiv \beta_i \bmod \pi$. Then the $\tilde{\beta}_i$ converge to some point $\alpha \in \mathscr{V}$, and the Type I point $f \mapsto |f(\alpha)|$ is a valuation on $K\langle x\rangle$ which maps to each point β_i on $\mathfrak{X}_i$ under the natural specialisation maps. This appears to depend on the choice of uniformiser, however, changing the uniformiser really corresponds to changing the parametrisation of each exceptional divisor in the tower of models $\{\mathfrak{X}_i\}$.

As a general rule, we will therefore identify a rigid space $\mathscr{X}$ with the corresponding adic space $\mathbf{ZR}(\mathscr{X})$ over K (at least when working with things locally of finite type over a fixed base). This equivalence preserves fibre products, and also respects the notion of a morphism being of finite type, an open or closed immersion, or separated. The question of whether or not it respects morphisms being proper or finite is dealt with in Sect. 2.3.

Of course, the various comparison results and functors between formal schemes, rigid analytic varieties, rigid spaces and adic spaces over K are compatible, in the sense that the diagram

$$\begin{array}{ccc} \mathbf{FSch}_{\mathcal{V}} & \xrightarrow{(-)^{\mathrm{rig}}} & \mathbf{CRf}_K \\ {\scriptstyle (-)_K}\downarrow & & \downarrow{\scriptstyle \mathbf{ZR}(-)} \\ \mathbf{An}_K & \xrightarrow{(-)^{\mathrm{ad}}} & \mathbf{Ad}_K \end{array}$$

is 2-commutative.

B.3 Valuations, Norms and Tubes

Fix some base formal scheme $\mathfrak{S}$ over $\mathcal{V}$, Noetherian and π-adic, though not necessarily of finite type over $\mathcal{V}$. We will let $\mathcal{S}$ denote the generic fibre of $\mathfrak{S}$ as an object of $\mathbf{Rf}_K$. We will work in the category $\mathbf{Rf}^{\mathrm{lft}}_{\mathcal{S}}$ of rigid spaces $\mathcal{X}$ locally of finite type over $\mathcal{S}$, and for any such space $\mathcal{X}$ we will abuse notation and also write $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}, \mathcal{O}^+_{\mathcal{X}})$ for the adic space $\mathbf{ZR}(\mathcal{X})$ over K associated to $\mathcal{X}$. We will also implicitly invoke the equivalence between $\mathbf{Rf}^{\mathrm{lft}}_{\mathcal{S}}$ and the category $\mathbf{Ad}^{\mathrm{lft}}_{\mathcal{S}}$ of adic spaces locally of finite type over $\mathcal{S}$.

For any point $x \in \mathcal{X}$ there is a canonical valuation

$$v_x : \mathcal{O}_{\mathcal{X},x} \to \Gamma_x \cup \{0\}$$

with values in some totally ordered group Γ_x.

Proposition B.31 ([3], Lemma 1.5 and Proposition 1.6) *Let $U \subset \mathcal{X}$ be an open subset, and $f \in \mathcal{O}_{\mathcal{X}}(U)$. Then $f \in \mathcal{O}^+_{\mathcal{X}}(U)$ if and only if $v_x(f) \le 1$ for all $x \in U$. Similarly, if $x \in \mathcal{X}$ and $f \in \mathcal{O}_{\mathcal{X},x}$ then $f \in \mathcal{O}^+_{\mathcal{X},x}$ if and only if $v_x(f) \le 1$.*

We may characterise the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{\mathcal{X},x}$ as the ideal of elements f such that $v_x(f) = 0$, the valuation $v_x : \mathcal{O}_{\mathcal{X},x} \to \Gamma_x \cup \{0\}$ therefore descends to a valuation on the quotient field $K_x := \mathcal{O}_{\mathcal{X},x}/\mathfrak{m}_x$ whose valuation ring V_x is just the image of $\mathcal{O}^+_{\mathcal{X},x}$ inside K_x. In the usual way, we may therefore view sections $f \in \mathcal{O}_{\mathcal{X}}$ as functions on $\mathcal{X}$, taking values in varying valued fields K_x, and we will abuse notation in the standard way by referring to the valuation $v(f(x)) := v_x(f)$ of a function at a point (we avoid using the notation $|f(x)|$ to emphasise the fact that we are working with valuations of rank possibly larger than 1). If $U \subset \mathcal{X}$ is an open subspace and $f \in \mathcal{O}_{\mathcal{X}}(U)$, then $f \in \mathcal{O}^\times_{\mathcal{X}}$ is a unit if and only if $f(x) \neq 0$ for all $x \in U$.

Example B.32 Let $\mathcal{X} = \mathrm{Spa}\left(R, R^+\right)$ be an affinoid rigid space over $\mathcal{S}$, and $f_1, \ldots, f_n, g \in R = \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ such that $(f_1, \ldots, f_n) = R$. Then the rational subset

$$U\left(\frac{f_1, \ldots, f_n}{g}\right)$$

is exactly the set of points $x \in \mathcal{X}$ such that $v(f_i(x)) \le v(g(x))$ for all i.

Another reason for using the notation $v(f(x))$ instead of $|f(x)|$ is to contrast with a genuine norm that we can define on point/function pairs, which is closely related to the 'tubular' subsets so important in rigid cohomology. This uses the concept of the 'maximal generisation' of a valuation.

Let $v : R \to \Gamma\ \{0\}$ be a valuation on some K-algebra R, and let $I \subset R$ denote its support. Let $v : \mathrm{Frac}(R/I) \to \Gamma \cup \{0\}$ denote the induced valuation, and V its valuation ring, with valuation ideal $P_v \subset V$. Then $\mathfrak{p} = \sqrt{(p)} \subset P_v$ is a height one prime ideal of V, and hence corresponds to a rank one valuation $v_{\mathfrak{p}} : R \to \mathbb{R}_{>0} \cup \{0\}$. This rank one valuation is called the maximal generisation of v. This extends to a function

$$[\cdot] : \mathscr{X} \to \mathscr{X}$$

from any rigid space to itself, which lands in the subset $[\mathscr{X}]$ of rank 1 points, i.e. points whose corresponding valuation has rank 1. If we normalise all these rank 1 valuations by decreeing that $v(p) = p^{-1}$, then we obtain continuous functions

$$\|f(\cdot)\| : \mathscr{X} \to \mathbb{R}_{\geq 0}$$

by setting $\|f(x)\| = v_{[x]}(f)$. These behave much more similarly to the functions $|f(\cdot)|$ on Tate's rigid analytic varieties (or the functions $\|f(\cdot)\|$ on Berkovich spaces), as the following example illustrates.

Example B.33 Let $\mathscr{X} = \mathbb{D}^1_K$. Then $\{x \in \mathbb{D}^1_K \,|\, \|x\| < 1\}$ is the open unit disc $\mathbb{D}^{1,\circ}_K$ over K.

If we equip $[\mathscr{X}]$ with the quotient topology via the surjective map $\mathscr{X} \to [\mathscr{X}]$, then we essentially obtain the Berkovich space corresponding to $\mathscr{X}$.

Definition B.34 We say that a rigid space $\mathscr{X}$ over K is *taut* if it is quasi-separated and the closure of every quasi-compact open is quasi-compact.

Recall from [1] that for a Banach K-algebra R the Berkovich spectrum $\mathscr{M}(R)$ is defined to be the set of continuous rank 1 valuations on R. By gluing the 'affinoid' spaces $\mathscr{M}(R)$ for affinoid K-algebras R one obtains the general notion of a Berkovich space.

Theorem B.35 ([4], Proposition 8.3.1 and Lemma 8.1.8) *The functor*

$$\mathscr{M}(R) \mapsto \mathrm{Spa}\left(R, R^+\right)$$

extends to an equivalence of categories $X \mapsto X^{\mathrm{ad}}$ from strictly K-analytic Berkovich spaces to taut rigid spaces, locally of finite type over K. There is a canonical homeomorphism of topological spaces $X \cong [X^{\mathrm{ad}}]$.

If $\mathfrak{X}$ is a formal scheme (of finite type) over $\mathfrak{B}$, then by the definition of the Zariski–Riemann space associated to its generic fibre $\mathscr{X}$ we get a continuous specialisation map

$$\mathrm{sp} : \mathscr{X} \to \mathfrak{X}.$$

If $Z \subset \mathfrak{X}$ is an closed subset then we define $]Z[_{\mathfrak{X}} := \mathrm{sp}^{-1}(Z)^{\circ}$ to be the interior of the closed subset $\mathrm{sp}^{-1}(Z)$.

Lemma B.36 ([2], Proposition 4.2.11) *Let $f_1, \ldots, f_r \in \mathscr{O}_{\mathfrak{X}}$ be such that $Z = V(f_1, \ldots, f_r)$. Then*

$$]Z[_{\mathfrak{X}} = \{x \in \mathscr{X} \mid \| f_i(x) \| < 1 \ \forall i\}.$$

Example B.37 Let $\mathfrak{X} = \mathrm{Spf}(\mathscr{V}\langle x \rangle)$, so that $\mathscr{X} = \mathbb{D}^1_K$ is the closed unit disc, and let $Z = \mathrm{Spf}(V) \hookrightarrow \mathfrak{X}$ be the point $x = 0$. Then $]Z[_{\mathfrak{X}} = \left\{\alpha \in \mathbb{D}^1_K \,\middle|\, \|\alpha\| < 1\right\}$ is equal to the open unit disc $\mathbb{D}^{1,\circ}_K$. This coincides with what we expect to happen from the Tate theory.

If $U \subset \mathfrak{X}$ is however an open subset then we define $]U[_{\mathfrak{X}} := \overline{\mathrm{sp}^{-1}(U)}$ to be the closure of the open subset $\mathrm{sp}^{-1}(U)$. Then the analogue of Lemma B.36 (which in fact follows from it) is the following.

Lemma B.38 *Let $g_1, \ldots, g_r \in \mathscr{O}_{\mathfrak{X}}$ be such that $Z = D(g_1) \cup \ldots \cup D(g_r)$. Then*

$$]Z[_{\mathfrak{X}} = \{\alpha \in \mathscr{X} \mid \exists i \text{ s.t. } \|g_i(\alpha)\| \geq 1\}.$$

Note that unlike the tubes associated to closed subsets of $\mathfrak{X}$, those associated to open subsets of $\mathfrak{X}$ do *not* in general have the structure of rigid spaces over K. One may use the tubes for closed and open subsets of $\mathfrak{X}$ to define the tube of any *constructible* subset of $\mathfrak{X}$, for example if $Y = U \cap Z$ is a locally closed subset, then we define $]Y[_{\mathfrak{X}} =]Z[_{\mathfrak{X}} \cap]U[_{\mathfrak{X}}$. The specialisation map $\mathscr{X} \to \mathfrak{X}$ factors through the quotient map $\mathscr{X} \to [\mathscr{X}]$, and the induced map

$$\mathrm{sp} : [\mathscr{X}] \to \mathfrak{X}$$

is *anticontinuous*. For any constructible subset $Y \subset \mathfrak{X}$ the tube $]Y[_{\mathfrak{X}} = [\cdot]^{-1}\mathrm{sp}^{-1}(Y)$ is the inverse image of the 'naïve' tube $\mathrm{sp}^{-1}(Y) \subset [\mathscr{X}]$ via $\mathscr{X} \to [\mathscr{X}]$.

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Appendix C
Cohomological Descent

In this appendix we give a brief introduction to the theory of cohomological descent. The original reference is [4], but for a more leisurely introduction we recommend the notes [1]. The basic idea of cohomological descent is two-fold: firstly to generalise the theory of descent for sheaves to the derived category, and therefore to cohomology (hence the name) and secondly to 'compute' the cohomology of a space using 'covers' that bear little or no relation to any natural topology on this space.

C.1 Simplicial Objects and Coskeleta

The first fundamental notion in the theory is that of a simplicial object in a category, which is a far reaching common generalisation of the data appearing when one computes the Čech cohomology associated to an (unordered) open cover, or when one computes the singular cohomology of a topological space.

Definition C.1 We define the simplex category Δ, whose objects are finite ordered sets $[n] := \{0, \ldots, n\}$ (one for each $n \geq 0$) and whose morphisms are order-preserving maps. The augmented simplex category Δ^+ is defined by adding an initial object $[-1] := \emptyset$ to Δ.

Then one can check that Δ is the category generated by the morphisms

$$\begin{aligned} \partial_n^i : [n] &\to [n+1] \\ \{0, \ldots, n\} &\mapsto \{0, \ldots, i-1, i+1, \ldots, n+1\} \\ \sigma_n^i : [n] &\to [n-1] \\ \{0, \ldots, n\} &\mapsto \{0, \ldots, i, i, \ldots, n-1\} \end{aligned}$$

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subject to the relations

$$\partial^i_{n+1}\partial^j_n = \partial^j_{n+1}\partial^{i-1}_n \quad \text{for } 0 \le j < i \le n+1$$
$$\sigma^i_n\sigma^j_{n+1} = \sigma^j_n\sigma^{i+1}_{n+1} \quad \text{for } 0 \le j \le i \le n$$
$$\sigma^i_{n-1}\partial^j_n = \begin{cases} \partial^j_{n-1}\sigma^{i-1}_{n-2} & \text{for } 0 \le j < i \le n-1 \\ \text{id} & \text{for } 0 \le i \le j \le i+1 \le n \\ \partial^{j-1}_{n-1}\sigma^i_{n-2} & \text{for } 0 \le i+1 < j \le n. \end{cases}$$

Definition C.2 Let $\mathscr{C}$ be a category. Then we define the category of simplicial objects in $\mathscr{C}$ to be the functor category $\mathscr{C}^{\Delta^{\text{op}}} := \text{Func}(\Delta^{\text{op}}, \mathscr{C})$. In other words, a simplicial object in $\mathscr{C}$ is a collection of objects X_n together with maps $d^i_n : X_{n+1} \to X_n$ and $s^i_n : X_{n-1} \to X_n$ satisfying the 'opposites' of the simplicial relations above, i.e.

$$d^j_n d^i_{n+1} = d^{i-1}_n d^j_{n+1} \quad \text{for } 0 \le j < i \le n+1$$
$$s^j_{n+1}s^i_n = s^{i+1}_{n+1}s^j_n \quad \text{for } 0 \le j \le i \le n$$
$$d^j_n s^i_{n-1} = \begin{cases} s^{i-1}_{n-2}d^j_{n-1} & \text{for } 0 \le j < i \le n-1 \\ \text{id} & \text{for } 0 \le i \le j \le i+1 \le n \\ s^i_{n-2}d^{j-1}_{n-1} & \text{for } 0 \le i+1 < j \le n. \end{cases}$$

We may similarly define the category $\mathscr{C}^{(\Delta^+)^{\text{op}}}$ of augmented simplicial objects in $\mathscr{C}$, as well as the category $\mathscr{C}^{\Delta}$ of cosimplicial objects, i.e. *co*variant functors $\Delta \to \mathscr{C}$ or augmented cosimplicial objects $\mathscr{C}^{\Delta^+}$.

Note that augmented simplicial objects $X : (\Delta^+)^{\text{op}} \to \mathscr{C}$ are the same thing as simplicial objects $X : \Delta^{\text{op}} \to \mathscr{C}_{/X_{-1}}$ in the slice category. We will use this to extend constructions and results from simplicial objects to augmented simplicial objects.

Example C.3 1. Let X be a topological space and $\mathfrak{U} = \{U_i\}$ an open cover of X. Write $U = \coprod_i U_i$ and set $U_n := U \times_X \ldots \times_X U$ (with $n+1$ copies of U). Then we can organise the U_n into a simplicial topological space $U_\bullet$ where the face maps d^i_n are appropriate projections and the degeneracy maps s^i_n inclusions. Moreover, we have a natural map from each U_n to X and hence we may view $U_\bullet \to X$ as an augmented simplicial object. Note that each U_n is a disjoint union of unordered n by n intersections of the U_i, and this simplicial object arises when computing the Čech cohomology of the cover $\mathfrak{U}$.

2. Let X be a topological space, and let X_n denote the set of n-simplices of X, that is the set of continuous maps $\Delta^n \to X$. Using the natural face and degeneracy maps $\Delta^n \to \Delta^{n-1}$ and $\Delta^n \to \Delta^{n+1}$ we can turn the X_n into a simplicial set $X_\bullet$. This arises when computing the singular cohomology of X.

Note that in the first example, the whole augmented simplicial object $U_\bullet \to X$ is determined by the first stage $U_0 \to X$. One of the original applications of simplicial objects was to be able to consider more exotic coverings that ordinary open coverings, where at each stage we are allowed to replace each open cover U_n of by a refinement. This leads to the notion of a hypercover, and to explain this we must first introduce the coskeleton functor.

Let $\Delta_{\leq n}$ denote the truncated simplex category, that is the full subcategory of Δ on objects $[m]$ for $m \leq n$. There is an obvious notion of an n-truncated simplicial object in a category $\mathscr{C}$, and we denote the category of such objects by $\mathscr{C}^{\Delta_{\leq n}^{\mathrm{op}}}$. We have a natural 'restriction' functor

$$\mathrm{sk}_n : \mathscr{C}^{\Delta^{\mathrm{op}}} \to \mathscr{C}^{\Delta_{\leq n}^{\mathrm{op}}}.$$

The coskeleton functor is constructed as a right adjoint to sk_n.

Theorem C.4 ([3], V.II (1.15)) *Assume that $\mathscr{C}$ has all finite limits. Then* sk_n *admits a right adjoint*

$$\mathrm{cosk}_n : \mathscr{C}^{\Delta_{\leq n}^{\mathrm{op}}} \to \mathscr{C}^{\Delta^{\mathrm{op}}}.$$

We won't go into the details of how one constructs cosk_n, but we will describe one very important special case.

Example C.5 Consider the 0-skeleton functor

$$\mathrm{sk}_0 : \mathscr{C}^{\Delta^{\mathrm{op}}} \to \mathscr{C}$$

which simply takes $X_\bullet$ to X_0. For every n we set

$$(\mathrm{cosk}_0 X_0)_n := X_0 \times \cdots \times X_0$$

(with $n+1$ copies of X_0) and define face and degeneracy maps

$$(\mathrm{cosk}_0 X_0)_{n+1} \to (\mathrm{cosk}_0 X_0)_n$$
$$(\mathrm{cosk}_0 X_0)_n \to (\mathrm{cosk}_0 X_0)_{n+1}$$

to be the various natural projections and diagonal-style inclusions respectively. This forms a simplicial object in $\mathscr{C}$, and it is not too difficult to check the defining property

$$\mathrm{Hom}_{\mathscr{C}^{\Delta^{\mathrm{op}}}}(Y_\bullet, \mathrm{cosk}_0(X_0)) = \mathrm{Hom}_{\mathscr{C}}(Y_0, X_0)$$

for any $Y_\bullet \in \mathscr{C}^{\Delta^{\mathrm{op}}}$, showing that this concrete cosk_0 *is* the same as the one whose existence is claimed by the above theorem. This is the sort of simplicial object that comes up when computing cohomology via Čech theory, and is hence often referred to as the Čech diagram associated to X_0.

Of course, by replacing $\mathscr{C}$ by $\mathscr{C}_{/X}$ we obtain coskeleta for augmented simplicial objects, note that here products are fibre products over X. In this situation we also have a right adjoint

$$\begin{aligned} \mathrm{cosk}_{-1} : \mathscr{C} &\to \mathscr{C}^{(\Delta^+)^{\mathrm{op}}} \\ X &\mapsto \text{constant simplicial object on } X \end{aligned}$$

to the '-1-skeleton' functor

$$\begin{aligned} \mathrm{sk}_{-1} : \mathscr{C}^{(\Delta^+)^{\mathrm{op}}} &\to \mathscr{C} \\ (X_\bullet \to X) &\mapsto X. \end{aligned}$$

C.2 Sheaves and Cohomology on Simplicial Spaces

The general idea behind the theory of cohomological descent is that it enable us to calculate the cohomology of some space X in terms of the cohomology of 'better behaved spaces' by choosing an appropriate 'resolution' $Y_\bullet \to X$ of X by a simplicial space $Y_\bullet$. In order to describe how this works, we will need to develop the theory of sheaves and cohomology on such 'simplicial spaces', and this is the topic of this section. So let $\mathscr{C}$ be a category of 'spaces', and for every $X \in \mathscr{C}$ let $\mathrm{Shv}(X)$ denote the category of 'sheaves' on X. We don't want to be too precise about exactly what a space, or a sheaf should mean, but the following examples should be kept in mind my the reader.

1. $\mathscr{C}$ is the category of topological spaces, and $\mathrm{Shv}(X)$ is the category of all abelian sheaves on X.
2. $\mathscr{C}$ is the category of schemes on which some integer n is invertible, and $\mathrm{Shv}(X)$ is the category of n-torsion étale sheaves on X.
3. $\mathscr{C}$ is a topos $\mathscr{T}$, and for $\mathscr{F} \in \mathscr{T}$, the category of sheaves $\mathrm{Shv}(\mathscr{F})$ is the category abelian group objects in the overcategory $\mathscr{T}_{/\mathscr{F}}$.
4. $\mathscr{C}$ is the category of smooth schemes over a field of characteristic zero, and $\mathrm{Shv}(X)$ is the category of sheaves on the infinitesimal site of X.

Let $Y_\bullet$ be a simplicial object in $\mathscr{C}$.

Definition C.6 A sheaf on $Y_\bullet$ is a sheaf $\mathscr{F}^n$ on Y_n for each n, together with morphisms

$$f_\phi : \phi^* \mathscr{F}^n \to \mathscr{F}^m$$

for every map $\phi : [n] \to [m]$ in Δ such that:

- $f_{\mathrm{id}} = \mathrm{id}$;
- $f_\phi \circ \phi^* f_\psi = f_{\phi \circ \psi}$ for any composable morphisms ϕ, ψ in Δ.

We leave it to the reader to explicate this in terms of face and degeneracy maps.

Remark C.7 Note that the definition is formally similar to that of a *cosimplicial* sheaf on a single space X, i.e. a covariant functor from Δ to the category $\mathrm{Shv}(X)$ of sheaves on X. The reason that it is not precisely a cosimplicial sheaf is that there is no single space on which all the $\mathscr{F}^n$ live.

Just as for single spaces, one can pushforward and pullback these sheaves via morphisms $Y_\bullet \to Y'_\bullet$ of simplicial spaces, the case that will most interest us (and the one we will go through in detail) is when $Y'_\bullet$ is a *constant* simplicial object, or in other words we are given an augmented simplicial space $Y_\bullet \to X$ in $\mathscr{C}$.

Pullback is straightforward to describe: given a sheaf $\mathscr{F}$ on X we may pull $\mathscr{F}$ back to obtain a sheaf $\mathscr{F}^n := p_n^*\mathscr{F}$ on each Y_n, which fit together to form a sheaf $p_\bullet^*\mathscr{F}$ on the simplicial space $Y_\bullet$. Pushforward is marginally more subtle: given a sheaf $\mathscr{F}^\bullet$ on $Y_\bullet$, then the sheaves $p_{n*}\mathscr{F}^n$ fit together to form a cosimplicial sheaf $p_{\bullet *}^\Delta\mathscr{F}^\bullet$ on X (the reasons for the notation $p_{\bullet *}^\Delta$ is to distinguish this cosimplicial sheaf from a *single* sheaf on X that will play the role of the pushforward). To obtain a single sheaf out of this, we will first introduce the chain complex associated to cosimplicial objects in an abelian category.

Definition C.8 Let $A^\bullet$ be a cosimplicial object in an abelian category. We define $C(A^\bullet)$ to be the chain complex

$$0 \to A^0 \to A^1 \to A^2 \to \dots$$

with differentials

$$d : A^n \to A^{n+1}$$
$$d = \sum_i (-1)^i d_n^i.$$

We now define

$$p_{\bullet *}\mathscr{F}^\bullet := H^0(C(p_{\bullet *}^\Delta\mathscr{F}^\bullet)),$$

and one can check that this does give a pair of adjoint functors

$$p_\bullet^* : \mathrm{Shv}(X) \leftrightarrows \mathrm{Shv}(Y_\bullet) : p_{\bullet *}$$

between sheaves on X and those on $Y_\bullet$. Given the definition of $p_{\bullet *}\mathscr{F}^\bullet$, one might guess that the complex

$$C(p_{\bullet *}^\Delta\mathscr{F}^\bullet)$$

should represented the total derived functor $\mathbf{R}p_{\bullet *}$ applied to $\mathscr{F}^\bullet$. This is not true, because it does not take into account the higher cohomologies $\mathbf{R}^i p_{n*}\mathscr{F}^n$, however, this is essentially the only reason why this naïve guess does not work.

So now let $I^{\bullet,*}$ be a bounded below complex of injective sheaves on $Y_\bullet$, in particular this implies that each $I^{n,m}$ is an injective sheaf on X_n. Then the pushforward

$p^{\Delta}_{\bullet *}I^{\bullet,*}$ is a cosimplicial complex of sheaves on X, and hence $C(p^{\Delta}_{\bullet *}I^{\bullet,*})$ is a double complex on X. We define

$$\mathbf{R}p_{\bullet *}I^{\bullet,*} := \mathrm{Tot}(C(p^{\Delta}_{\bullet *}I^{\bullet,*}))$$

to be the associated simple complex. This induces a functor

$$\mathbf{R}p_{\bullet *} : D^{+}(\mathrm{Shv}(Y_{\bullet})) \to D^{+}(\mathrm{Shv}(X))$$

which is the total derived functor of $p_{\bullet *}$. One may also check that we get adjoint funtors

$$p^{*}_{\bullet} : D^{+}(\mathrm{Shv}(X)) \leftrightarrows D^{+}(\mathrm{Shv}(Y_{\bullet})) : \mathbf{R}p_{\bullet *}.$$

Now simply using the spectral sequence associated to the double complex $C(p^{\Delta}_{\bullet *}I^{\bullet,*})$ gives the following result.

Proposition C.9 *Let $p_{\bullet} : Y_{\bullet} \to X$ be an augmented simplicial space, and $\mathscr{F}^{\bullet}$ a sheaf (or complex of sheaves) on $Y_{\bullet}$. Then there is a spectral sequence*

$$E_1^{n,i} = \mathbf{R}^{i}p_{n*}\mathscr{F}^{n} \Rightarrow \mathbf{R}^{n+i}p_{\bullet *}\mathscr{F}^{\bullet}.$$

In particular when $X = \{*\}$ is a point, there is a spectral sequence

$$E_1^{n,i} = H^{i}(Y_n, \mathscr{F}^{n}) \Rightarrow H^{n+i}(Y_{\bullet}, \mathscr{F}^{\bullet}).$$

□

Example C.10 As mentioned above, the only failure of $\mathbf{R}p_{\bullet *}$ to be given by $p^{\Delta}_{\bullet *}$ is that the sheaves $\mathscr{F}^{n}$ might have non-trivial higher pushforwards. An example where these are all trivial (and hence we can calculate $\mathbf{R}p_{\bullet *}$ explicitly) is the familiar one of computing the Čech cohomology of a scheme using open affines.

So let $U = \coprod_{i \in I} U_i \to X$ be an open affine cover of a Noetherian, separated scheme X, and $\mathscr{F}$ a quasi-coherent sheaf on X. Let $p_{\bullet} : U_{\bullet} \to X$ be the corresponding Čech diagram over X, where

$$U_n = U \times_X \ldots \times_X U = \coprod_{(i_0,\ldots,i_n) \in I^{n+1}} U_{i_0} \cap \ldots \cap U_{i_n}.$$

Then $\mathscr{G}^{\bullet} := p^{*}_{\bullet}\mathscr{F}$ is a sheaf on $U_{\bullet}$, with each G^{n} quasi-coherent on the affine scheme U_n, and hence all the higher cohomologies vanish. Hence the total cohomology

$$\mathbf{R}\Gamma(U_{\bullet}, \mathscr{G}^{\bullet})$$

of $\mathscr{G}^\bullet$ is represented by the usual (unordered) Čech complex

$$0 \to \prod_{i_0 \in I} \Gamma(U_{i_0}, \mathscr{F}) \to \prod_{(i_0,i_1)\in I^2} \Gamma(U_{i_0} \cap U_{i_1}, \mathscr{F}) \to \dots .$$

C.3 Hypercovers and Cohomological Descent

In order to be able to compute the cohomology of a space X in terms of some simplicial resolution $p_\bullet : Y_\bullet \to X$, and in particular to use the spectral sequence from Proposition C.9 we to be able to relate the cohomology $H^i(X, \mathscr{F})$ of a sheaf on X to that of $H^i(Y_\bullet, p_{\bullet *}\mathscr{F})$. This is formalised in the notion of cohomological descent.

Definition C.11 Let $p_\bullet : Y_\bullet \to X$ be an augmented simplicial space, and $\mathscr{S} \subset \mathrm{Shv}(X)$ a subcategory of sheaves on X. Then we say that $p_\bullet$ is of cohomological descent for $\mathscr{S}$ if for all $\mathscr{F} \in \mathscr{S}$, the adjunction morphism

$$\mathscr{F} \to \mathbf{R}p_{\bullet *}p_\bullet^*\mathscr{F}$$

is an isomorphism. We say that $p_\bullet$ is universally of cohomological descent (with respect to $\mathscr{S}$) if it remains so after any base change $X' \to X$. If $\mathscr{S} = \mathrm{Shv}(X)$ then we will drop the reference to $\mathscr{S}$, and simply talk of being (universally) of cohomological descent. Finally, if $p : Y \to X$ is a morphism in $\mathscr{C}$, then we will say that p is (universally) of cohomological descent (with respect to $\mathscr{S}$) if the corresponding Čech diagram $p_\bullet : Y_\bullet \to X$ is.

Corollary C.12 *If $p_\bullet$ is of cohomological decent for $\mathscr{S}$, then for all $\mathscr{F} \in \mathscr{S}$ there is a spectral sequence*

$$E_1^{n,i} = H^i(Y_n, p_n^*\mathscr{F}) \Rightarrow H^{n+i}(X, \mathscr{F}).$$

□

To see the analogy with usual descent theory, we use the following result.

Lemma C.13 *An augmented simplicial space $p_\bullet : Y_\bullet \to X$ is of cohomological descent if and only if the map*

$$p_\bullet^* : D^+(\mathrm{Shv}(X)) \to D^+(\mathrm{Shv}(Y_\bullet))$$

is fully faithful.

Proof Since $\mathbf{R}p_{\bullet *}$ and $p_\bullet^*$ are adjoint functors, we note that for all $\mathscr{K}, \mathscr{K}' \in D^+(\mathrm{Shv}(X))$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{D^+(\mathrm{Shv}(X))}(\mathscr{K}', \mathscr{K}) & \longrightarrow & \mathrm{Hom}_{D^+(\mathrm{Shv}(X))}(\mathscr{K}', \mathbf{R}p_{\bullet *}p_\bullet^*\mathscr{K}) \\ & \searrow & \downarrow \\ & & \mathrm{Hom}_{D^+(\mathrm{Shv}(Y_\bullet))}(p_\bullet^*\mathscr{K}', p_\bullet^*\mathscr{K}) \end{array}$$

with right hand arrow an isomorphism. The 'only if' direction is now clear, the 'if' direction follows from Yoneda's lemma. □

A key result is the following.

Proposition C.14 ([4], Théorème 3.3.3) *Let $Y_\bullet \to X$ be an augmented simplicial object, and $\mathscr{S} \subset \mathrm{Shv}(X)$. Suppose that for each $n \geq -1$ the morphism*

$$Y_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n Y_n)_{n+1}$$

is universally of cohomological descent with respect to $\mathscr{S}$. Then $Y_\bullet \to X$ is universally of cohomological descent with respect to $\mathscr{S}$.

So far things have been fairly formal, the non-formal (i.e. geometric) part of the theory consists of constructing interesting examples for which cohomological descent holds.

Definition C.15 Let **P** be a class of morphisms in $\mathscr{C}$, stable under base change, preserved under composition and containing all isomorphisms. An augmented simplicial object in $Y_\bullet \to X$ in $\mathscr{C}$ is said to be a **P**-hypercover if for all $n \geq -1$ the map

$$X_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_\mathrm{n} X_\bullet)_{n+1}$$

induced by the adjunction $\mathrm{sk}_n \dashv \mathrm{cosk}_n$ is in **P**.

Example C.16 Let $U \to X$ be a morphism in **P**. Then the associated Čech diagram $U_\bullet \to X$ is a **P**-hypercover.

The basic idea behind hypercovers is to start with a given 'cover' $Y_0 \to X$ of X, and gradually 'refine' it. Slightly more concretely, then, at the second stage, rather than simply taking $Y_1 = Y_0 \times_X Y_1$ to be the 'intersection' of this cover with itself, one is allowed to take another, finer cover $Y_1 \to Y_0 \times_X Y_1$. This then happens all the way up, at each stage we may replace the 'simplest possible' extension $\mathrm{cosk}_n Y_{\leq n}$ by a refinement $Y_{n+1} \to \mathrm{cosk}_n Y_{\leq n}$. To illustrate this, we will briefly explain some of the ideas behind the proof of the following theorem.

Theorem C.17 *Let k be a perfect field, $\mathscr{C}$ the category of separated schemes of finite type over k. Let* **P** *be the class of proper, surjective morphisms. Then for any $X \in \mathscr{C}$ there exists a* **P***-hypercover $Y_\bullet \to X$ such that each Y_n is smooth over k.*

Proof (Sketch) By de Jong's theorem on alterations, there exists a proper surjective map $Y_0 \to X$ such that Y_0 is smooth. The fibre product $Y_0 \times_X Y_0$ might not be smooth, but we can find a proper surjective map $\tilde{Y}_1 \to Y_0 \times_X Y_0$ such that $\tilde{Y}_1$ is smooth. We have natural face maps

$$d_0^i : \tilde{Y}_1 \rightrightarrows Y_0$$

given by composing with the two projections $Y_0 \times_X Y_0 \to Y_0$ but we don't necessarily have a degeneracy map $Y_0 \to \tilde{Y}_1$. To fix this, we let $Y_1 = \tilde{Y}_1 \coprod Y_0$, we can extend the face maps by requiring them to be the identity map on the copy of Y_0 inside Y_1, and we now have a degeneracy map $Y_0 \to Y_1$ given by the natural inclusion.

We continue similarly: at each stage we form $(\mathrm{cosk}_n Y_{\leq n})_{n+1}$ - this might not be smooth, but we can find a proper surjective map $\tilde{Y}_{n+1} \to (\mathrm{cosk}_n Y_{\leq n})_{n+1}$ with $\tilde{Y}_{n+1}$ smooth. Adding in copies of Y_n to ensure that we have the required face and degeneracy maps gives us Y_{n+1}. □

Thanks to Proposition C.14 we immediately have a trivial example of hypercovers of cohomological descent. Specifically, if we take **P** to be the collection of all covering morphisms $U \to X$ for the given topology, then all **P**-hypercovers are universally of cohomological descent.

Example C.18 Let $\mathscr{C}$ be the category of schemes on which n is invertible, and $\mathrm{Shv}(X)$ the category of n-torsion étale sheaves on X. Let **P** be the collection of surjective étale maps. Then all **P**-hypercovers are universally of cohomological descent.

The real power of the theory, though, comes when consider classes of morphisms **P** which have nothing to do with the topology on X.

Theorem C.19 ([4], Corollaire 4.1.6, Proposition 4.3.2) *Let $\mathscr{C}$ be the category of topological spaces (resp. schemes over $\mathbb{Z}[1/n]$), and* $\mathrm{Shv}(X)$ *the category of abelian sheaves on X (resp. n-torsion étale sheaves on X). Let* $\mathbf{P} \subset \mathscr{C}$ *be the collection of proper, surjective maps. Then all* **P***-hypercovers are universally of cohomological descent.*

By applying this, for example, to the category of schemes over a perfect field k, we obtain the following, allowing us to deduce results concerning the cohomology of arbitrary varieties, from those on 'better behaved' varieties.

Corollary C.20 *Let k be a perfect field, and n and integer coprime to the characteristic. Then for any variety X/k there exist smooth varieties $p_n : Y_n \to X$ such that for any n-torsion étale sheaf $\mathscr{F}$ on X there exists a spectral sequence*

$$E_1^{n,i} : H^i_{\text{ét}}(Y_n, p_n^*\mathscr{F}) \Rightarrow H^{n+i}_{\text{ét}}(X, \mathscr{F}).$$

□

Of course, there are far more sophisticated applications of this sort of idea. For example, Deligne in [2] uses these techniques to show that the singular cohomology $H^i(X(\mathbb{C}), \mathbb{Q})$ of any algebraic variety $X/\mathbb{C}$ has a canonical mixed Hodge structure.

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