General Appendices

Appendix I
The Dirac Formalism

For the convenience of the reader we gather here some important equations dealing with the Dirac formalism for the description of spin 1/2 particles. For derivations and detailed presentation, we refer the reader to Chapter 16 of Manoukian [6].

The Dirac equation is given by

$$\left(\frac{\gamma^{\mu}\partial_{\mu}}{\mathrm{i}} + m\right)\psi \ = \ 0,\ \{\gamma^{\mu},\gamma^{\nu}\} = -2\eta^{\mu\nu},\ [\eta^{\mu\nu}] = \mathrm{diag}[-1,1,1,1] = [\eta_{\mu\nu}]. \tag{I.1}$$

In the presence of an external electromagnetic field, the Dirac equation reads

$$\Big[\gamma^{\mu}\left(\frac{\partial_{\mu}}{\mathrm{i}} - \mathrm{e}A_{\mu}(x)\right) + m\Big]\psi(x) \ = \ 0, \tag{I.2}$$

from which one obtains the equation $(\overline{\psi} = \psi^{\dagger}\gamma^{0})$

$$\Big[\gamma^{\mu}\left(\frac{\partial_{\mu}}{\mathrm{i}} + \mathrm{e}A_{\mu}(x)\right) + m\Big]\psi^{\mathscr{C}}(x) = 0, \quad \psi^{\mathscr{C}}(x) = \mathscr{C}\overline{\psi}^{\top}(x), \quad \mathscr{C} = \mathrm{i}\gamma^{2}\gamma^{0}, \tag{I.3}$$

thus introducing, in turn, the charge conjugation matrix $\mathscr{C}$, and the charge conjugate spinor $\psi^{\mathscr{C}}$.

Under a homogeneous Lorentz transformation which may include a 3D rotation (see (2.2.1), (2.2.3), (2.2.6))

$$x'^{\mu} \ = \ \Lambda^{\mu}{}_{\nu}\, x^{\nu}, \qquad \partial_{\nu} \ = \ \Lambda^{\mu}{}_{\nu}\,\partial'_{\mu}, \qquad \partial'_{\mu} = \Lambda_{\mu}{}^{\nu}\partial_{\nu}. \tag{I.4}$$

the Dirac equation reads

$$\Big[\frac{\gamma^{\nu}\partial'_{\nu}}{\mathrm{i}} + m\Big]K\psi(x) \ = \ 0, \qquad K\,\psi(x) \ = \ \psi\,'(x'), \tag{I.5}$$

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where the matrix K satisfies the relations

$$K^{\dagger}\gamma^{0}K = \gamma^{0}, \qquad \Lambda^{\mu}{}_{\nu}\gamma^{\nu} = K^{-1}\gamma^{\mu}K. \tag{I.6}$$

For infinitesimal transformations, we recall from (2.2.18), that $\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$, $\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}$, and we may set $K \simeq I + (\mathrm{i}/2)\,\delta\omega^{\mu\nu}S_{\mu\nu}$, where $S_{\mu\nu}$ is to be determined. By substituting this expression in the last equation in (I.6) gives

$$[S^{\lambda\mu},\gamma^{\nu}] = \mathrm{i}\left(\eta^{\lambda\nu}\gamma^{\mu} - \eta^{\mu\nu}\gamma^{\lambda}\right), \quad \text{with solution} \quad S^{\lambda\mu} = \frac{\mathrm{i}}{4}[\gamma^{\lambda},\gamma^{\mu}]. \tag{I.7}$$

Some of the properties of the gamma matrices, based on their anti-commutations relations in (I.1) are given in Box I.1.

Box I.1: Some properties of the gamma matrices

$$\gamma^{\mu}\gamma^{\nu} = -\eta^{\mu\nu}I + \frac{1}{2}[\gamma^{\mu},\gamma^{\nu}], \quad \mathrm{Tr}[\gamma^{\mu}] = 0,$$

$$\left(\gamma^{0}\right)^2 = I, \quad \left(\gamma^{i}\right)^2 = -I, \quad i = 1,2,3.$$

$$\eta_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = -4I,$$

$$\eta_{\mu\nu}\gamma^{\mu}\left(\gamma^{\sigma}\right)\gamma^{\nu} = 2\gamma^{\sigma},$$

$$\eta_{\mu\nu}\gamma^{\mu}\left(\gamma^{\sigma}\gamma^{\lambda}\right)\gamma^{\nu} = 4\eta^{\sigma\lambda},$$

$$\eta_{\mu\nu}\gamma^{\mu}\left(\gamma^{\sigma}\gamma^{\lambda}\gamma^{\rho}\right)\gamma^{\nu} = 2\gamma^{\rho}\gamma^{\lambda}\gamma^{\sigma},$$

$$\left[\gamma^{\mu},\left[\gamma^{\sigma},\gamma^{\rho}\right]\right] = 4\left(\gamma^{\sigma}\eta^{\mu\rho} - \gamma^{\rho}\eta^{\mu\sigma}\right),$$

$$\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}] = -4\eta^{\mu\nu},$$

$$\mathrm{Tr}\left[\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right] = 4\left(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}\right),$$

$$\mathrm{Tr}\left[\gamma^{5}\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right] = -4\,\mathrm{i}\,\varepsilon^{\alpha\beta\mu\nu},$$

$\varepsilon^{\alpha\beta\mu\nu}$ totally anti-symmetric with $\varepsilon^{0123} = +1$.

Tr[odd number of γ's] $= 0$.

$$\gamma^{5} = \mathrm{i}\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}, \quad \mathrm{Tr}[\gamma^{5}] = 0, \quad \mathrm{Tr}[\gamma^{5}\gamma^{\mu}] = 0,$$

$$\left(\gamma^{5}\right)^2 = I, \quad \{\gamma^{5},\gamma^{\mu}\} = 0,$$

$$\left(\gamma^{\mu}a_{\mu}\right)^2 = -I\left[\mathbf{a}^2 - (a^0)^2\right], \quad \left(\boldsymbol{\gamma}\cdot\mathbf{a}\right)^2 = -I\,\mathbf{a}^2,$$

$$\mathbf{a} = \left(a_1,a_2,a_3\right), \quad a_0 = -a^0, \quad a_i = a^i, \quad i = 1,2,3.$$

The Dirac representation of the γ^μ matrices, in particular, is defined by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad \gamma^5 \equiv \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{I.8}$$

and $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{I.9}$$

satisfying, the relations $\{\sigma^i, \sigma^j\} = 2\,\delta^{ij}$.

In the momentum description, on the mass shell $p^0 = \sqrt{\mathbf{p}^2 + m^2}$, where m is the mass of a particle, we have two sets of solutions: $u(\mathbf{p}, \sigma)$, $v(\mathbf{p}, \sigma)$, with $\sigma = \pm 1$ specifying spin states, satisfying the equations

$$(\gamma p + m) u(\mathbf{p}, \sigma) = 0, \qquad \overline{u}(\mathbf{p}, \sigma)(\gamma p + m) = 0, \tag{I.10}$$

$$(-\gamma p + m) v(\mathbf{p}, \sigma) = 0, \qquad \overline{v}(\mathbf{p}, \sigma)(-\gamma p + m) = 0, \tag{I.11}$$

and $u(\mathbf{p}, \sigma)$ may be taken as

$$u(\mathbf{p}, \sigma) = \sqrt{\frac{p^0 + m}{2m}} \begin{pmatrix} \xi_\sigma \\ \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0 + m}\xi_\sigma \end{pmatrix}, \qquad \sigma = \pm 1, \tag{I.12}$$

involving two normalized two component spinors satisfying $\xi^\dagger_\sigma\, \xi_{\sigma'} = \delta_{\sigma\sigma'}$, which, for an arbitrary unit vector $\mathbf{N} = (\cos\phi\ \sin\theta, \sin\phi\ \sin\theta, \cos\theta)$, may be taken as

$$\xi_{+\mathbf{N}} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} \cos\frac{\theta}{2} \\ \\ \mathrm{e}^{\mathrm{i}\phi/2} \sin\frac{\theta}{2} \end{pmatrix}, \quad \xi_{-\mathbf{N}} = \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\phi/2} \sin\frac{\theta}{2} \\ \\ \mathrm{e}^{\mathrm{i}\phi/2} \cos\frac{\theta}{2} \end{pmatrix}, \tag{I.13}$$

$$\boldsymbol{\sigma} \cdot \mathbf{N}\, \xi_{\pm\mathbf{N}} = \pm\, \xi_{\pm\mathbf{N}}, \qquad \mathrm{i}\,\sigma^2\, \xi^*_{\pm\mathbf{N}} = \mp\, \xi_{\mp\mathbf{N}}. \tag{I.14}$$

We note that the spin matrix is defined by

$$S^i = \frac{1}{2}\varepsilon^{ijk} S^{jk} = \frac{\mathrm{i}}{8}\varepsilon^{ijk}[\gamma^j, \gamma^k], \qquad \mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}. \tag{I.15}$$

In the Dirac representation, the charge conjugation matrix $\mathscr{C} = \mathrm{i}\,\gamma^{2}\gamma^{0}$ is

$$\mathscr{C}|_{\mathrm{D}} = \mathrm{i}\gamma^{2}\gamma^{0}\Big|_{\mathrm{D}} = \begin{pmatrix} 0 & -\mathrm{i}\sigma^{2} \\ -\mathrm{i}\sigma^{2} & 0 \end{pmatrix}, \tag{I.16}$$

with the charge conjugate of $u(\mathbf{p},\sigma)$ given by $\mathscr{C}\,\overline{u}^{\top}(\mathbf{p},\pm 1) = \pm\,\gamma^{5}\,u(\mathbf{p},\mp 1)$. Thus we may define the second set of spinors in the momentum description by

$$v(\mathbf{p},\pm 1) = \gamma^{5}u(\mathbf{p},\mp 1), \qquad \overline{v}(\mathbf{p},\pm 1) = -\,\overline{u}(\mathbf{p},\mp 1)\gamma^{5}, \tag{I.17}$$

up to phase factors, with $v(\mathbf{p},\sigma)$ as charge conjugate of $u(\mathbf{p},\sigma)$.

Under space reflection: $x' = (x^{0}, -\mathbf{x})$ the Dirac equation reads

$$\left[\frac{\gamma^{\mu}\partial'_{\mu}}{\mathrm{i}} + m\right]\gamma^{0}\psi(x) = 0, \qquad \psi'(x') = \gamma^{0}\,\psi(x), \tag{I.18}$$

up to a phase factor for the latter. Hence under space reflection, $u(\mathbf{p},\sigma) \rightarrow \gamma^{0}u(-\mathbf{p},\sigma) = +\,u(-\mathbf{p},\sigma)$, $v(\mathbf{p},\sigma) \rightarrow \gamma^{0}v(-\mathbf{p},\sigma) = -\,v(-\mathbf{p},\sigma)$, having opposite (intrinsic) parities.

We have the following normalization conditions

$$\overline{u}(\mathbf{p},\sigma)u(\mathbf{p},\sigma') = \delta_{\sigma\sigma'}, \quad \overline{v}(\mathbf{p},\sigma)\,v(\mathbf{p},\sigma') = -\,\delta_{\sigma\sigma'}, \quad \overline{u}(\mathbf{p},\sigma)\,v(\mathbf{p},\sigma') = 0, \tag{I.19}$$

$$u^{\dagger}(\mathbf{p},\sigma)\,u(\mathbf{p},\sigma') = v^{\dagger}(\mathbf{p},\sigma)\,v(\mathbf{p},\sigma') = \frac{p^{0}}{m}\,\delta_{\sigma\sigma'}, \qquad u^{\dagger}(-\mathbf{p},\sigma)\,v(\mathbf{p},\sigma') = 0, \tag{I.20}$$

and the completeness relations

$$\sum_{\sigma} u(\mathbf{p},\sigma)\,\overline{u}(\mathbf{p},\sigma) = \frac{(-\gamma p + m)}{2m} \equiv \mathbb{P}_{+}(p), \tag{I.21}$$

$$-\sum_{\sigma} v(\mathbf{p},\sigma)\,\overline{v}(\mathbf{p},\sigma) = \frac{(\,\gamma p + m)}{2m} \equiv \mathbb{P}_{-}(p). \tag{I.22}$$

Note that for

$$\mathbf{N} = \mathbf{p}/|\mathbf{p}|, \tag{I.23}$$

in (I.13),

$$u(\mathbf{p}, \pm) = \sqrt{\frac{p^0 + m}{2m}} \begin{pmatrix} \xi_\pm \\ \pm \frac{|\mathbf{p}|}{p^0 + m} \xi_\pm \end{pmatrix}. \tag{I.24}$$

For a massless Dirac particle, it is convenient to work in the *chiral* representation (see (2.3.3)), and with the unit vector $\mathbf{N}$ chosen along the momentum of the particle, as given in (I.23), we have spinors[1]

$$u(\mathbf{p}, +1) = \begin{pmatrix} \xi_+ \\ 0 \end{pmatrix} = \frac{I + \gamma^5}{2} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}, \quad u(\mathbf{p}, -1) = \begin{pmatrix} 0 \\ \xi_- \end{pmatrix} = \frac{I - \gamma^5}{2} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}, \tag{I.25}$$

$u^\dagger(\mathbf{p}, \pm 1)\, u(\mathbf{p}, \pm 1) = 1$. We may conveniently write

$$u(\pm\mathbf{p}, \pm 1) = \begin{pmatrix} \xi_\pm \\ 0 \end{pmatrix}, \quad u(\pm\mathbf{p}, \mp 1) = \begin{pmatrix} 0 \\ \xi_\mp \end{pmatrix}, \tag{I.26}$$

where $\pm$ with $\pm$ means spin projection in the same direction as of the momentum, while $\pm$ with $\mp$ means in the opposite directions. With an appropriately chosen phases, the latter are related by a parity transformation: $-\gamma^0 u(-\mathbf{p}, +1) \to u(+\mathbf{p}, +1)$, $-\gamma^0 u(+\mathbf{p}, -1) \to u(-\mathbf{p}, -1)$, with spin and momentum in opposite directions to spin and momentum in the same direction, and vice versa. Accordingly, if parity is not conserved, nature picks up only one of the helicities.

The interest in the equations in (I.25) is that $(I + \gamma^5)/2$ projects out a state corresponding to a right-handed particle, with spin along its momentum, while $(I - \gamma^5)/2$ projects out a left-handed one, with spin in opposite direction to its momentum. The corresponding particles are referred to, respectively, as right-handed and left-handed. The situation may be demonstrated as shown in Fig. I.1

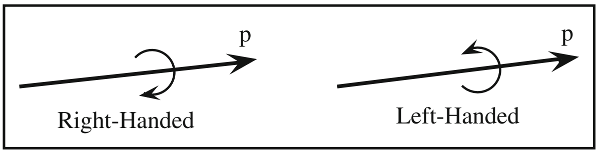

Fig. I.1 Diagrams defining the handedness of a particle, where $\mathbf{p}$ is its 3-momentum

[1] For the relevant details, see Manoukian [6], p. 912.

The eigenvectors $u(\mathbf{p}, \pm 1)$ then satisfy the simultaneous eigenvalue equations:

$$\gamma^0 \boldsymbol{\gamma} \cdot \mathbf{p}\, u(\mathbf{p}, \pm 1) = |\mathbf{p}| \; u(\mathbf{p}, \pm 1), \tag{I.27}$$

$$\gamma^5 u(\mathbf{p}, \pm 1) = \pm \; u(\mathbf{p}, \pm 1), \tag{I.28}$$

$$\mathbf{S} \cdot \mathbf{N}\, u(\mathbf{p}, \pm 1) = \pm (1/2)\, u(\mathbf{p}, \pm 1), \tag{I.29}$$

where $\{\gamma^5, \ \mathbf{S}\cdot\mathbf{N}, \ \gamma^0 \boldsymbol{\gamma}\cdot\mathbf{p}\}$ is a commuting set of operators,[2] and specify the state of a particle. The eigenvalues of γ^5 are referred to as the chiralities (handedness) of a particle, and $\mathbf{S}\cdot\mathbf{N} \equiv (\boldsymbol{\Sigma}/2)\cdot\mathbf{N}$ defines the helicity operator, with $u(\mathbf{p}, \pm)$ corresponding to a massless particle with two-helicity states. As seen above, if parity is not conserved a massless particle is to be considered to have only one helicity.

[2]Note that γ^5 does not commute with $\gamma^0(\boldsymbol{\gamma}\cdot\mathbf{p} + m)$ for $m \neq 0$.

Appendix II
Doing Integrals in Field Theory

For a real, consider the following two integrals

$$\int_{-\infty}^{\infty} \mathrm{d}y\ \cos(a\,y^2) \;=\; \sqrt{\frac{\pi}{2|a|}}, \qquad \int_{-\infty}^{\infty} \mathrm{d}y\ \sin(a\,y^2) \;=\; \sqrt{\frac{\pi}{2|a|}}\ \mathrm{sgn}\, a. \tag{II.1}$$

Using the fact that $\sin\pi/4 = \cos\pi/4 = 1/\sqrt{2}$, they may be combined into an exponential form as

$$\int_{-\infty}^{\infty} \mathrm{d}y\ \mathrm{e}^{\mathrm{i}\,a\,y^2} \;=\; \sqrt{\frac{\pi}{|a|}}\ \exp\Big[\,\mathrm{i}\frac{\pi}{4}\ \mathrm{sgn}\, a\Big]. \tag{II.2}$$

In view of applications in n-dimensional spacetime, we apply this integral to a typical expression of the denominator of a propagator, which may be first rewritten as

$$\frac{1}{(k^2+M^2-\mathrm{i}\,\epsilon)} \;=\; \mathrm{i}\int_0^{\infty} \mathrm{d}s\ \exp[-\mathrm{i}s(k^2+M^2-\mathrm{i}\,\epsilon)], \quad k^2 \;=\; \sum_{i=1}^{n-1} k_i^2 - (k^0)^2. \tag{II.3}$$

In n-dimensional spacetime, our metric is defined by $[\eta_{\mu\nu}] = \mathrm{diag}[-1, 1, \dots, 1]$.

From (II.2) we obtain the following integrals , with $s > 0$,

$$\int_{-\infty}^{\infty} \mathrm{d}k^0 \exp[\,\mathrm{i}\,s\,(k^0)^2] \;=\sqrt{\frac{\pi}{s}}\ \mathrm{e}^{\,\mathrm{i}\,\frac{\pi}{4}}, \tag{II.4}$$

$$\int_{-\infty}^{\infty}\mathrm{d}k^1 \int_{-\infty}^{\infty}\mathrm{d}k^2 \dots \int_{-\infty}^{\infty}\mathrm{d}k^{n-1} \exp[-\mathrm{i}\,s\,\big((k^1)^2+\cdots+(k^{n-1})^2\big)] \;=\; \bigg(\sqrt{\frac{\pi}{s}}\ \mathrm{e}^{-\mathrm{i}\,\frac{\pi}{4}}\bigg)^{n-1}, \tag{II.5}$$

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and hence

$$\int_{-\infty}^{\infty}\mathrm{d}k^0\int_{-\infty}^{\infty}\mathrm{d}k^1\ldots\int_{\infty}^{\infty}\mathrm{d}k^{n-1}\,\mathrm{e}^{-\mathrm{i}sk^2}=\left(\frac{1}{\mathrm{i}}\right)^{\frac{n-2}{2}}\left(\frac{\pi}{s}\right)^{\frac{n}{2}},\qquad k^2=\sum_{i=1}^{n-1}(k^i)^2-(k^0)^2,\quad s>0. \tag{II.6}$$

In the remaining part of this appendix, we work only with $n = 4$, i.e., in 4-dimensional spacetime. We also suppress the $-\mathrm{i}\,\epsilon$ factor in (II.3).

If we integrate the expression in (II.3) over k, we encounter a singularity in the s-integral at $s = 0$. We may, however, differentiate (II.3) twice with respect to M^2 to obtain

$$\frac{1}{(k^2+M^2)^3} = -\frac{\mathrm{i}}{2}\int_0^{\infty}\mathrm{d}s\,s^2\exp\big[-\mathrm{i}s(k^2+M^2)\big], \tag{II.7}$$

leading from (II.6) to the useful integral

$$\int\frac{(\mathrm{d}k)}{(k^2+M^2)^3} = \frac{\mathrm{i}\,\pi^2}{2}\frac{1}{M^2},\qquad (\mathrm{d}k)=\mathrm{d}k^0\mathrm{d}k^1\mathrm{d}k^2\mathrm{d}k^3, \tag{II.8}$$

where we have used the fact that $\int_0^{\infty}\mathrm{d}s\,\exp-\mathrm{i}s(M^2-\mathrm{i}\,\epsilon)=-\mathrm{i}/(M^2)$.

A further differentiation of (II.8) with respect to M^2, for example, leads to

$$\int\frac{(\mathrm{d}k)}{(k^2+M^2)^4} = \frac{\mathrm{i}\,\pi^2}{6}\frac{1}{(M^2)^2},\qquad \int\frac{k^2\,(\mathrm{d}k)}{(k^2+M^2)^4} = \frac{\mathrm{i}\,\pi^2}{3}\frac{1}{M^2}. \tag{II.9}$$

We note that quite generally one may write,

$$\int(\mathrm{d}k)\,\frac{k^{\mu}k^{\nu}}{(k^2+M^2)^4} = A\,\eta^{\mu\nu}, \tag{II.10}$$

$$\int(\mathrm{d}k)\,\frac{k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}}{(k^2+M^2)^5} = B\,(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}+\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}+\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}). \tag{II.11}$$

To determine the coefficient A, for example, we may make a contraction over $\mu,\ \nu$, to obtain from the second equation in (II.9), $A = \mathrm{i}\pi^2/12M^2$, using the fact that $\eta^{\mu}{}_{\mu} = 4$. Similarly, contractions over $\mu_1,\ \mu_2$, and $\mu_3,\ \mu_4$, give $24B$ to the right-hand side of (II.10), leading to $B = \mathrm{i}\,\pi^2/96M^2$. These values are worth recording here:

$$A = \frac{\mathrm{i}\,\pi^2}{12M^2},\qquad B = \frac{\mathrm{i}\,\pi^2}{96M^2}. \tag{II.12}$$

Thus we are bound to encounter the following readily evaluated integrals:

$$\int (\mathrm{d}k)\, \frac{(k^2)^{m-2}}{(k^2+M^2)^n} = \frac{\mathrm{i}\,\pi^2}{(M^2)^{n-m}} \frac{\Gamma(m)\,\Gamma(n-m)}{\Gamma(n)}, \tag{II.13}$$

which obviously exist for $n > m > 0$, where $\Gamma(z)$ is the gamma function.

Now we consider more complicated integrals involving an additional (external) momentum. For convenience, we enumerate the type of integrals considered.

1.

$$I(p^2) = \int (\mathrm{d}k) \left(\frac{1}{[(k-p)^2 + M^2(p^2)]^2} - \frac{1}{[k^2 + M^2(p^2)]^2} \right) = 0, \tag{II.14}$$

where, as indicated, $M^2(p^2)$ may be a function of p^2. To show that this integral is indeed zero, we take its derivative with respect to p_σ. It is easily checked that the derivative of the integrand is zero giving $\partial I(p^2)/\partial p_\sigma = 0$. With the boundary condition that $I(0) = 0$, gives $I(p^2) = 0$. Here we note that if we define the *degree of divergence of an integral as the of power of k in the numerator minus the power of k in the denominator plus four*, then the degree of divergence of the above integral restricted to the first term is zero. This then allows one to make a shift of the integration variable $k \rightarrow k + p$ in the first part of the integral just mentioned leading to the net result zero for the integral $I(p^2)$. Of course such a shift of an integration variable is obviously valid if the degree of divergence of an integral is negative.

The degree of divergence of the following integrals are negative, thus by making a shift of the integration variable lead to the stated results by using, in the process, (II.8):

$$\int \frac{(\mathrm{d}k)}{[\,(k-p)^2 + M^2(p^2)\,]^3} = \int \frac{(\mathrm{d}k)}{[\,k^2 + M^2(p^2)\,]^3} = \frac{\mathrm{i}\,\pi^2}{2M^2}, \tag{II.15}$$

$$\int (\mathrm{d}k) \frac{k^\mu}{[\,(k-p)^2 + M^2(p^2)\,]^3} = p^\mu \int \frac{(\mathrm{d}k)}{[\,k^2 + M^2(p^2)\,]^3} = \frac{\mathrm{i}\,\pi^2 p^\mu}{2M^2}, \tag{II.16}$$

where in the latter equation, we have used the property that an integral which is odd in k^μ is zero. Another integral, where the shift of the integration variable in the first term is obviously permissible is

$$\int (\mathrm{d}k) \left(\frac{k^\mu k^\nu}{[\,(k-p)^2 + M^2(p^2)\,]^3} - \frac{k^\mu k^\nu}{[\,k^2 + M^2(p^2)\,]^3} - \frac{p^\mu p^\nu}{[\,k^2 + M^2(p^2)\,]^3} \right) = 0. \tag{II.17}$$

2. To obtain the following integral

$$\int (\mathrm{d}k) \left(\frac{k^\mu k^\nu k^\sigma}{[\,(k-p)^2 + M^2(p^2)\,]^3} - \frac{1}{4} \frac{\eta^{\mu\nu} p^\sigma + \eta^{\mu\sigma} p^\nu + \eta^{\nu\sigma} p^\mu}{[\,k^2 + M^2(p^2)\,]^2} \right)$$
$$= -\frac{5\,\mathrm{i}\,\pi^2}{24} (\eta^{\mu\nu} p^\sigma + \eta^{\mu\sigma} p^\nu + \eta^{\nu\sigma} p^\mu) + \frac{\mathrm{i}\,\pi^2}{2M^2(p^2)} p^\mu p^\nu p^\sigma, \tag{II.18}$$

we denote the result of the integration by

$$C(p^2)\,(\eta^{\mu\nu} p^\sigma + \eta^{\mu\sigma} p^\nu + \eta^{\nu\sigma} p^\mu) + D(p^2) p^\mu p^\nu p^\sigma, \tag{II.19}$$

and determine $C(p^2)$ and $D(p^2)$. To this end, note that the integral is zero for $p = 0$ since the resulting integrand is an odd function of k. Secondly, note that the degree of divergence of the integral restricted to the first term involving $k^\mu k^\nu k^\sigma$ is positive and hence we cannot simply make a shift of the integration variable k to $k + p$ in evaluating this term. What we do instead is take the derivative of the integral with respect to p_λ as shown below.

The derivative of the first term in the integrand, for example, is

$$(-3) \left(2p^\lambda + \frac{\partial M^2(p^2)}{\partial p_\lambda} \right) \frac{k^\mu k^\nu k^\sigma}{[\,(k-p)^2 + M^2(p^2)\,]^4} + 6\, \frac{k^\mu k^\nu k^\sigma k^\lambda}{[\,(k-p)^2 + M^2(p^2)\,]^4}.$$

The first term leads to an integral of degree of divergence −1, and the second to a degree of divergence 0. Hence we can make a shift of the integration variable p to $p + k$ here. Continuing in this manner and using any of the integrations obtained before (II.18), we obtain for the derivative of the integral on the left-hand side of (II.18) explicitly

$$-\frac{5\,\mathrm{i}\,\pi^2}{24} (\eta^{\mu\nu}\eta^{\sigma\lambda} + \eta^{\mu\sigma}\eta^{\nu\lambda} + \eta^{\nu\sigma}\eta^{\mu\lambda}) - \frac{\mathrm{i}\,\pi^2}{2(M^2(p^2))^2} \frac{\partial M^2(p^2)}{\partial p_\lambda} p^\mu p^\nu p^\sigma$$
$$+ \frac{\mathrm{i}\,\pi^2}{2M^2(p^2)} (\eta^{\mu\lambda} p^\nu p^\sigma + \eta^{\nu\lambda} p^\mu p^\sigma + \eta^{\sigma\lambda} p^\mu p^\nu). \tag{II.20}$$

This is to be compared with the derivative of the expression in (II.19) with respect to k_λ,

$$\frac{\partial C(p^2)}{\partial p_\lambda} (\eta^{\mu\nu} p^\sigma + \eta^{\mu\sigma} p^\nu + \eta^{\nu\sigma} p^\mu) + C(p^2)\,(\eta^{\mu\nu}\eta^{\sigma\lambda} + \eta^{\mu\sigma}\eta^{\nu\lambda} + \eta^{\nu\sigma}\eta^{\mu\lambda})$$
$$+ \frac{\partial D(p^2)}{\partial p_\lambda} p^\mu p^\nu p^\sigma + D(p^2)\,(\eta^{\mu\lambda} p^\nu p^\sigma + \eta^{\nu\lambda} p^\mu p^\sigma + \eta^{\sigma\lambda} p^\mu p^\nu), \tag{II.21}$$

from which we obtain

$$C(p^2) = -\frac{5\,\mathrm{i}\,\pi^2}{24}, \qquad D(p^2) = \frac{\mathrm{i}\,\pi^2}{2M^2(p^2)}, \tag{II.22}$$

as given on the right-hand side of (II.18).

The following integrals is obtained in a similar manner

$$\int (\mathrm{d}p)\left(\frac{p^\mu p^\nu}{[\,(p-k)^2 + M^2(k^2)\,]^2} - \frac{k^\mu k^\nu}{[p^2 + M^2(k^2)\,]^2} - \frac{1}{4}\frac{p^2\,\eta^{\mu\nu}}{[p^2 + M^2(k^2)\,]^2}\right)$$
$$= -\frac{5\,\mathrm{i}\,\pi^2}{6} k^\mu k^\nu - \frac{\mathrm{i}\,\pi^2}{6} k^2\,\eta^{\mu\nu}. \tag{II.23}$$

Here we have used p as the *integration* variable for a direct application of this in Appendix A of Chap. 3. The following integral is also quite useful

$$\int (\mathrm{d}k)\left(\frac{k^\mu}{[\,(k-p)^2 + M^2(p^2)\,]^2} - \frac{p^\mu}{[k^2 + M^2(p^2)\,]^2}\right) = -\frac{\mathrm{i}\,\pi^2}{2} p^\mu. \tag{II.24}$$

The reader is strongly encouraged to go through the above details and realize the simplicity in evaluating these integrals by the method just developed.

Integrands appearing in (II.8)… (II.24) arise when one combines the product of two (or more) factors such as

$$\frac{1}{(k-p)^2 + M^2}\,\frac{1}{k^2 + m^2}, \tag{II.25}$$

into one factor in the following manner. To this end, for two c-numbers A, B, consider the integral

$$\int_0^1 \mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{[Ax + B(1-x)\,]} = \frac{1}{A} - \frac{1}{B}. \tag{II.26}$$

By carrying out the differentiation with respect to x and by dividing the resulting integral by $(B-A)$ gives the useful formula

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\,\frac{1}{[Ax + B(1-x)\,]^2}. \tag{II.27}$$

This is referred to as a Feynman parameter representation of the product on the left-hand side of (II.27).

As an interesting example for applying this formula and comparing it with the method developed above for evaluating some integrals, we will evaluate the following

$$I(p^2) = \int (\mathrm{d}k) \left(\frac{1}{(k-p)^2 + m^2} - \frac{1}{k^2 + m^2} \right), \quad I(p^2)|_{p=0} = 0. \tag{II.28}$$

Let us apply the method developed earlier first to evaluate this integral by differentiating it with respect to p_μ. This gives

$$\frac{\partial}{\partial p_\mu} I(p^2) = -2 \int (\mathrm{d}k) \left(\frac{p^\mu}{[(k-p)^2 + m^2]^2} - \frac{k^\mu}{[(k-p)^2 + m^2]^2} \right) = -\mathrm{i}\pi^2 p^\mu, \tag{II.29}$$

$$I(p^2) = -\frac{\mathrm{i}\pi^2}{2} p^2, \tag{II.30}$$

where we have used the fact that the degree of divergence of the integral in (II.29) *restricted* to the *first* term is zero to make a shift of the integration variable in the first, and then used (II.24), to finally obtain $I(p^2)$. Now follow the *explicit* evaluation of (II.28).

Let us use (II.27). This leads from the latter to the evaluation of the integral

$$\begin{aligned} I(p^2) &= -\int_0^1 \mathrm{d}x \int (\mathrm{d}k) \frac{p^2 - 2\,kp}{[(k-px)^2 + p^2 x(1-x) + m^2]^2} \\ &= -\frac{\mathrm{i}\pi^2}{2} p^2 + p^2 \int (\mathrm{d}k) \int_0^1 \frac{(2x-1)\,\mathrm{d}x}{[k^2 + k^2 x(1-x) + m^2]^2}, \end{aligned} \tag{II.31}$$

where in obtaining the latter equality, we have, in the process, used (II.24). By noting that $p^2(1-2x)\,\mathrm{d}x = \mathrm{d}\,(p^2 x(1-x))$, we may carry out the integration over x in (II.31) by parts, and then carry out the resulting integral over k using (II.8). The integral on the right-hand side of (II.31), as the coefficient of p^2, then becomes

$$\begin{aligned} &-\mathrm{i}\pi^2 \int_0^1 \mathrm{d}x \, \frac{x(1-x)p^2(1-2x)}{[m^2 + p^2 x(1-x)]} \\ &= -\mathrm{i}\pi^2 \int_0^1 \mathrm{d}x \; x(1-x) \frac{\mathrm{d}}{\mathrm{d}x} \ln\left[1 + \frac{p^2}{m^2} x(1-x)\right] \\ &= +\mathrm{i}\pi^2 \int_0^1 \mathrm{d}x \; (1-2x) \ln\left[1 + \frac{p^2}{m^2} x(1-x)\right]. \end{aligned} \tag{II.32}$$

The last integral is zero since the change of variable $x \to 1-x$ gives minus the same integral. That is, the integral on the right-hand side of (II.31) is zero and we obtain the result given in (II.30).

Another example that is worth considering is

$$\int (\mathrm{d}k) \left(\frac{k^\mu}{(k-p)^2 + m^2} - \frac{1}{2} \frac{p^\mu k^2}{(k^2+m^2)^2} \right) = -\frac{\mathrm{i}\,\pi^2}{3} p^{\,2} p^\mu, \tag{II.33}$$

and we leave it as an exercise for the reader to verify it by both methods given above.

A useful formula that follows from (II.27) is obtained by differentiating the latter with respect to A. This gives

$$\frac{1}{A^2 B} = 2 \int_0^1 x \, \mathrm{d}x \, \frac{1}{[Ax + B(1-x)\,]^3}. \tag{II.34}$$

More generally,

$$\frac{1}{AB\,C} = 2 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}z \, \frac{1}{[A(1-x) + Bz + C(x-z)\,]^3}. \tag{II.35}$$

The latter may be generalized for the product of N, not necessarily identical, factors, and more conveniently written, as follows

$$\frac{1}{A_1 A_2 \cdots A_N} = (N-1)! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{N-2}} \mathrm{d}x_{N-1} \times \frac{1}{[A_1 x_{N-1} + A_2(x_{N-2} - x_{N-1}) + \cdots + A_N(1-x_1)\,]^N}, \tag{II.36}$$

which, for $N = 3$, coincides with (II.35) upon setting $A_1 = B$, $A_2 = C$, $A_3 = A$, $x_1 = x$, $x_2 = z$.

Appendix III
Analytic Continuation in Spacetime Dimension and Dimensional Regularization

Upon differentiation of the expression in (II.3)

$$\frac{1}{(k^2+M^2-\mathrm{i}\,\epsilon)} = \mathrm{i}\int_0^\infty \mathrm{d}s\,\exp[-\mathrm{i}\,s(k^2+M^2-\mathrm{i}\,\epsilon)], \quad k^2 = \sum_{i=1}^{n-1} k_i^2-(k^0)^2. \tag{III.1}$$

$(\nu-1)$ number of times, we obtain

$$\frac{1}{(k^2+M^2-\mathrm{i}\,\epsilon)^\nu} = \frac{(\mathrm{i})^\nu}{\Gamma(\nu)}\int_0^\infty \mathrm{d}s\,s^{\nu-1}\exp[-\mathrm{i}\,s(k^2+M^2-\mathrm{i}\,\epsilon)]. \tag{III.2}$$

Hence from (II.6)

$$\int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2+M^2-\mathrm{i}\,\epsilon)^\nu} = \frac{(\mathrm{i})^{\nu-\frac{n}{2}+1}}{(4\pi)^{\frac{n}{2}}\,\Gamma(\nu)}\int_0^\infty \mathrm{d}s\,s^{\nu-\frac{n}{2}-1}\,\mathrm{e}^{-s\,(\mathrm{i}\,M^2+\epsilon)}. \tag{III.3}$$

Upon using the integral

$$\int_0^\infty \mathrm{d}s\,s^{\nu-\frac{n}{2}-1}\,\mathrm{e}^{-s\,(\mathrm{i}\,M^2+\epsilon)} = \frac{1}{(\mathrm{i}\,M^2+\epsilon)^{\nu-\frac{n}{2}}}\,\Gamma\Big(\nu-\frac{n}{2}\Big), \tag{III.4}$$

we obtain

$$\int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2+M^2-\mathrm{i}\,\epsilon)^\nu} = \frac{\mathrm{i}}{(4\pi)^{\frac{n}{2}}}\,\frac{1}{(M^2-\mathrm{i}\,\epsilon)^{\nu-\frac{n}{2}}}\,\frac{\Gamma(\nu-\frac{n}{2})}{\Gamma(\nu)}, \quad \nu > \frac{n}{2}. \tag{III.5}$$

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From the above integral it easily follows that

$$\int \frac{d^n k}{(2\pi)^n} \frac{k^2}{(k^2 + M^2 - \mathrm{i}\,\epsilon)^\nu} = \frac{\mathrm{i}}{(4\pi)^{\frac{n}{2}}} \frac{n}{2} \frac{1}{(M^2 - \mathrm{i}\,\epsilon)^{\nu - \frac{n}{2} - 1}} \frac{\Gamma(\nu - \frac{n}{2} - 1)}{\Gamma(\nu)}, \tag{III.6}$$

valid for $\nu > (n/2) + 1$.

One then defines integrals by an analytic extension of spacetime dimension to D as follows

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + M^2 - \mathrm{i}\,\epsilon)^\nu} = \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{(M^2 - \mathrm{i}\,\epsilon)^{\nu - \frac{D}{2}}} \frac{\Gamma(\nu - \frac{D}{2})}{\Gamma(\nu)}, \tag{III.7}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 + M^2 - \mathrm{i}\,\epsilon)^\nu} = \frac{\mathrm{i}\,D}{(4\pi)^{\frac{D}{2}}} \frac{(M^2 - \mathrm{i}\,\epsilon)}{(M^2 - \mathrm{i}\,\epsilon)^{\nu - \frac{D}{2}}} \frac{\Gamma(\nu - \frac{D}{2})}{(2\nu - D - 2)\Gamma(\nu)}. \tag{III.8}$$

Dimensional regularization consists of choosing

$$D = 4 - \varepsilon, \qquad \varepsilon > 0, \tag{III.9}$$

with ε not to be confused with ϵ earlier. The so-called ultraviolet divergences in quantum field theory, i.e., at high energies, are equivalently expressed in terms of singularities in ε, with the latter *kept* > 0.

We may use Feynman parameters representation, as given in (II.27), to write

$$\frac{1}{(k-p)^2\, k^2} = \int_0^1 \mathrm{d}x \, \frac{1}{[\,(k - p\,x)^2 + p^2\, x(1-x)\,]^2}, \tag{III.10}$$

which from (III.7) leads to

$$\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k-p)^2\, k^2} = \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \int_0^1 \mathrm{d}x \, \frac{1}{\big[\,p^2 x(1-x)\,\big]^{\varepsilon/2}}\, \Gamma(\varepsilon/2). \tag{III.11}$$

Upon using the integral

$$\int_0^1 \mathrm{d}x \, x^{\alpha-1}\,(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \alpha > 0,\ \beta > 0, \tag{III.12}$$

we obtain

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k-p)^2\, k^2} = \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \left(\frac{1}{p^2}\right)^{\varepsilon/2} \frac{\Gamma(\frac{\varepsilon}{2})\,\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}. \tag{III.13}$$

By shifting the variable k: $k \to k + px$, using the expression on the right-hand side of (III.10), and (III.12), in the process, we also obtain

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu}{(k-p)^2 k^2} = \frac{p^\mu}{2} \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \Big(\frac{1}{p^2}\Big)^{\varepsilon/2} \frac{\Gamma(\frac{\varepsilon}{2})\, \Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}. \tag{III.14}$$

Here we have also used the property that the integral of an odd function of k is zero. Another useful integral is obtained again by shifting the variable k: $k \to k+px$, by using in the process (II.8) and noting that in D dimensions $\eta^\mu{}_\mu = D$ to obtain

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k-p)^2 k^2} = \frac{1}{4(3-\varepsilon)} \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \Big(\frac{1}{p^2}\Big)^{\varepsilon/2} \frac{\Gamma(\frac{\varepsilon}{2})\, \Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)} \times [-\eta^{\mu\nu} p^2 + (4-\varepsilon) p^\mu p^\nu]. \tag{III.15}$$

In particular, using the fact that $\eta^\mu{}_\mu = (4-\varepsilon)$, dimensional regularization gives from the above equation the interesting result

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k-p)^2} = 0. \tag{III.16}$$

The following properties of the Γ- function should be noted

$$\Gamma(z+1) = z\, \Gamma(z), \qquad \Gamma\Big(1-\frac{\varepsilon}{2}\Big) = 1 + \gamma_{\mathrm{E}} \frac{\varepsilon}{2} + \mathscr{O}(\varepsilon^2),$$
$$\Gamma\Big(\frac{\varepsilon}{2}\Big) = \frac{2}{\varepsilon} - \gamma_{\mathrm{E}} + \mathscr{O}(\varepsilon), \tag{III.17}$$

where $\gamma_{\mathrm{E}} = 0.5772157\ldots$ is Euler's constant.

Finally, we note that due to dimensional reasons, when the dimension of space-time is analytically continued to D, an integral in four dimensions is then defined by introducing an arbitrary mass parameter μ_D, and by making, in the process, the replacement $(\mathrm{d}k) \to (\mu_D)^\varepsilon d^D k$. Therefore, a four dimensional regularized integral may be defined as

$$\int \frac{(\mathrm{d}k)}{(2\pi)^4} F(k,p)\Big|_{\mathrm{Reg}} = (\mu_D)^\varepsilon \int \frac{d^D k}{(2\pi)^D} F(k,p). \tag{III.18}$$

We note that the integrals (III.13), (III.14), (III.15), using the definition in (III.18), all involve the following factor, when expanded in powers of ε,

$$\frac{1}{(4\pi)^{\frac{D}{2}}} \Big(\frac{\mu_D^2}{p^2}\Big)^{\varepsilon/2} \Gamma\Big(\frac{\varepsilon}{2}\Big) = \frac{1}{16\pi^2} \Big[\frac{2}{\varepsilon} - \gamma_{\mathrm{E}} - \ell n(p^2/\mu_D^2) + \ell n(4\pi) + \mathscr{O}(\varepsilon)\Big], \tag{III.19}$$

up to overall multiplicative finite constant terms coming from the other gamma functions occurring in their factors, for $\varepsilon \to 0$. The parameter ε has now replaced the ultraviolet cut-off.

It is also worth knowing that the following two integrals obtained by the same method as above, and by power counting, are ultraviolet finite (UV finite):

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(p_1 - k)^2 (p_2 - k)^2 k^2} = \text{UV finite}, \tag{III.20}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(p_1 - k)^2 (p_2 - k)^2 k^2} = \text{UV finite}, \tag{III.21}$$

while with $Q = p_2 - p_1 \neq 0$,

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(p_1 - k)^2 (p_2 - k)^2 k^2} = \frac{\mathrm{i}\, \eta^{\mu\nu}}{(4\pi)^{\frac{D}{2}}} \left(\frac{1}{Q^2}\right)^{\varepsilon/2} \frac{\Gamma(\frac{\varepsilon}{2}) \Gamma^2(1 - \frac{\varepsilon}{2})}{\Gamma(3)\Gamma(3 - \varepsilon)} + \text{UV finite}. \tag{III.22}$$

In $D = 4 - \varepsilon$ dimensions, the gamma matrices are defined to satisfy the following relations:

Box III.1: Some properties of the gamma matrices in $D = 4 - \varepsilon$ dimensions

$$\{\gamma^\mu, \gamma^\nu\} = -2\, \eta^{\mu\nu} I, \quad \eta^\mu{}_\mu = (4 - \varepsilon), \quad \gamma^\mu \gamma_\mu = -(4 - \varepsilon) I,$$

$$\gamma^\mu \gamma^\sigma \gamma_\mu = (2 - \varepsilon)\, \gamma^\sigma, \quad \mathrm{Tr}\, I = 4, \quad \mathrm{Tr}\,[\gamma^\mu \gamma^\nu] = -4\, \eta^{\mu\nu},$$

$$\mathrm{Tr}\left[\gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu\right] = 4\left(\eta^{\alpha\beta} \eta^{\mu\nu} - \eta^{\alpha\mu} \eta^{\beta\nu} + \eta^{\alpha\nu} \eta^{\beta\mu}\right),$$

$$\mathrm{Tr}\,[\,\text{odd number of } \gamma\text{ 's}\,] = 0.$$

Appendix IV
Schwinger's Point Splitting Method of Currents: Arbitrary Orders

The vacuum-to-vacuum transition amplitude corresponding to Dirac's equation $[\gamma^\mu(\partial_\mu/\mathrm{i} - \mathrm{e}A_\mu) + m]\psi = 0,$ in the presence of an external electromagnetic field $A_\mu,$ is given in (3.6.26) to be given by

$$\langle\, 0_+ \mid 0_- \rangle^{(\mathrm{e})} \;=\; \exp \mathrm{i}\, W, \tag{IV.1}$$

which is normalized to unity for $\mathrm{e} = 0.$ Here ($x_\pm \;=\; x \pm \epsilon/2$)

$$\mathrm{i}\, W = -\int_0^{\mathrm{e}} \mathrm{d}\mathrm{e}'\,(\mathrm{d}x)\,\Big(\mathrm{Tr}\,[\,S(x_-, x_+; \mathrm{e}'A)\gamma A\,]\,\mathrm{e}^{\mathrm{i}\,\mathrm{e}'\int_{x-}^{x+}\mathrm{d}\xi^\mu A_\mu(\xi)}\Big), \quad \epsilon \to 0, \tag{IV.2}$$

as given in (3.6.27), with a priori dependence on ϵ arising from Schwinger's point splitting of the current as given in (3.6.24) and spelled out to be

$$\frac{\langle\, 0_+ \mid j^\mu(x) \mid 0_- \rangle^{(\mathrm{e})}}{\langle\, 0_+ \mid 0_- \rangle^{(\mathrm{e})}} = \mathrm{i}\,\mathrm{Tr}\,\mathrm{Av}\left([\,S_+(x_-, x_+; \mathrm{e}A)\gamma^\mu\,]\exp\,\mathrm{i}\;\mathrm{e}\int_{x_-}^{x_+}\mathrm{d}\xi^\mu\, A_\mu(\xi)\right), \tag{IV.3}$$

where Av stands for an average over $\epsilon^0 > 0$ and $\epsilon^0 < 0,$ with ϵ in a space-like direction. Here Tr denotes a trace over the spinor indices of gamma matrices. We also note that $S^A(x_-, x_+) \equiv S_+(x_-, x_+; \mathrm{e}A).$

The purpose of this appendix, is to develop a perturbation expansion of (IV.2) to arbitrary orders in $\mathrm{e}A_\mu$ in the limit $\epsilon \to 0.$ We will see, in particular, that the perturbative expansion, due to the presence of Schwinger's line-integral, is modified from its naïve expression, only up to fourth order in the external electromagnetic field.

We first consider the expression

$$\mathrm{Tr}\,[\,S(x_-, x_+; \mathrm{e}A)\gamma^{\,\mu}\,]\,A_\mu(x) \;\equiv\; \mathrm{Tr}\,[\,\gamma^\mu S(x_-, x_+; \mathrm{e}A)\,]\,A_\mu(x), \tag{IV.4}$$

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multiplying the Schwinger line-integral in (IV.2) for a coupling e. From the integral equation for $S(x_-, x_+; \mathrm{e}A) \equiv S_+^A(x_-, x_+)$, in (3.2.12), we may carry out an expansion in powers of the external potential A_μ as

$$\begin{aligned}
&\mathrm{Tr}\,[\,S(x_-, x_+; \mathrm{e}A)\gamma^{\,\mu}\,]A_\mu(x) = A_{\mu_1}(x)\,\mathrm{Tr}\,[S_+(x_- - x_+)\gamma^{\,\mu_1}] \, + \\
&+ \sum_{N\geq 2} (\mathrm{e})^{N-1} \int (\mathrm{d}x_2)\cdots(\mathrm{d}x_N)\, A_{\mu_1}(x)A_{\mu_2}(x_2)\cdots A_{\mu_N}(x_N) \times \\
&\times\, \mathrm{Tr}\,[\,S_+(x_- - x_N)\gamma^{\,\mu_N} S_+(x_N - x_{N-1})\gamma^{\,\mu_{N-1}}\cdots\gamma^{\,\mu_2} S_+(x_2 - x_+)\gamma^{\mu_1}\,],
\end{aligned} \tag{IV.5}$$

where $S_+(x - x')$ is the fermion propagator in (3.1.9). Equivalently for the expression on the right-hand side of (IV.5), we have

$$\begin{aligned}
&\mathrm{Tr}\,[\,\gamma^{\,\mu} S(x_-, x_+; \mathrm{e}A)\,]A_\mu(x) = A_{\mu_1}(x)\,\mathrm{Tr}\,[\,\gamma^{\,\mu_1} S_+(x_- - x_+)\,] \, + \\
&+ \sum_{N\geq 2} (\mathrm{e})^{N-1} \int (\mathrm{d}x_2)\cdots(\mathrm{d}x_N)\, A_{\mu_1}(x)A_{\mu_2}(x_2)\ldots A_{\mu_N}(x_N) \times \\
&\times \mathrm{Tr}\,[\,\gamma^{\,\mu_1} S_+(x_- - x_2)\gamma^{\,\mu_2}\ldots\gamma^{\,\mu_{N-1}} S_+(x_{N-1} - x_N)\gamma^{\,\mu_N} S_+(x_N - x_+)\,].
\end{aligned} \tag{IV.6}$$

We may rewrite the $\mathrm{Tr}[(\,.\,)]$ term within the multiple integral in (IV.5) as $\mathrm{Tr}\,[\,(\,.\,)^\top\,] \equiv \mathrm{Tr}[\mathscr{C}^{-1}(\,.\,)^\top\mathscr{C}\,]$, where $\mathscr{C}$ is the charge conjugation matrix in (I.3), satisfying, in particular,

$$\mathscr{C}^{-1}(\gamma^{\,\mu})^\top\mathscr{C} = -\gamma^{\,\mu}, \qquad \mathscr{C}^{-1}(S_+(x - x'))^\top\mathscr{C} = S_+(x' - x). \tag{IV.7}$$

Accordingly, (IV.5) may be re-expressed as

$$\begin{aligned}
&\mathrm{Tr}\,[\,S(x_-, x_+; \mathrm{e}A)\gamma^{\,\mu}\,]A_\mu(x) = -A_{\mu_1}(x)\,\mathrm{Tr}\,[\,\gamma^{\,\mu_1} S_+(x_+ - x_-)\,] \, + \\
&+ \sum_{N\geq 2} (-1)^N (\mathrm{e})^{N-1} \int (\mathrm{d}x_2)\cdots(\mathrm{d}x_N)\, A_{\mu_1}(x)A_{\mu_2}(x_2)\cdots A_{\mu_N}(x_N) \times \\
&\times \mathrm{Tr}\,[\,\gamma^{\,\mu_1} S_+(x_+ - x_2)\,\gamma^{\,\mu_2}\cdots\gamma^{\,\mu_{N-1}} S_+(x_{N-1} - x_N)\gamma^{\,\mu_N} S_+(x_N - x_-)\,].
\end{aligned} \tag{IV.8}$$

Eqs. (IV.6), (IV.8) allow us to write (IV.5) as the average of the just mentioned two equivalent expressions. That is

$$\begin{aligned}
&\mathrm{Tr}\,[\,S(x_-, x_+; \mathrm{e}A)\gamma^{\,\mu}\,]A_\mu(x) = \frac{1}{2} A_{\mu_1}(x)\,\mathrm{Tr}\Big[\,\gamma^{\,\mu_1}[\,S_+(x_- - x_+) - S_+(x_+ - x_-)\,]\,\Big] \\
&+ \sum_{N\geq 2} (\mathrm{e})^{N-1} \int (\mathrm{d}x_2)\ldots(\mathrm{d}x_N)\, A_{\mu_1}(x)A_{\mu_2}(x_2)\cdots A_{\mu_N}(x_N) \times \\
&\times \frac{1}{2}\,[\,F^{\mu_1\mu_2\cdots\mu_N}(x_-, x_2, \ldots, x_N, x_+) \; + \; (-1)^N\, F_N^{\mu_1\mu_2\cdots\mu_N}(x_+, x_2, \cdots, x_N, x_-)\,],
\end{aligned} \tag{IV.9}$$

where

$$F^{\mu_1\mu_2\cdots\mu_N}(x,x_2,\ldots,x_N,x') = \mathrm{Tr}\,[\,\gamma^{\,\mu_1}S_+(x-x_2)\gamma^{\,\mu_2}S_+(x_2-x_3)\,\cdots\,\gamma^{\,\mu_N}S_+(x_N-x')\,]. \tag{IV.10}$$

Note that $S_+(x_- - x_+)$, $S_+(x_+ - x_-)$, are *independent* of x.

By carrying out Fourier transforms,

$$A_\mu(x) = \int \frac{(\mathrm{d}Q)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}Qx}A_\mu(Q), \qquad S_+(x-x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}p(x-x')}\,S_+(p), \tag{IV.11}$$

(IV.9) becomes

$$\begin{aligned}
&\mathrm{Tr}\,[\,S(x_-,x_+;\mathrm{e}A)\gamma^{\,\mu}\,]\,A_\mu(x)\\
&= \int \frac{(\mathrm{d}p)}{(2\pi)^4}\frac{(\mathrm{d}Q_1)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}Q_1x}\,\frac{1}{2}\,[\,\mathrm{e}^{-\mathrm{i}p\epsilon}\mathrm{e}^{\mathrm{i}Q_1\epsilon/2} - \mathrm{e}^{\mathrm{i}p\epsilon}\mathrm{e}^{-\mathrm{i}Q_1\epsilon/2}\,]\,A_{\mu_1}(Q_1)\mathrm{Tr}\,[\,\gamma^{\,\mu_1}S_+(p)\,] +\\
&\quad + \sum_{N\geq 2}\int \frac{(\mathrm{d}p)}{(2\pi)^4}\frac{(\mathrm{d}Q_1)}{(2\pi)^4}\cdots\frac{(\mathrm{d}Q_N)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}\,(Q_1+\cdots+Q_N)x}\,\times\\
&\times\,\frac{1}{2}\,\mathrm{e}^{N-1}\,[\,\mathrm{e}^{-\mathrm{i}p\epsilon}\,\mathrm{e}^{\mathrm{i}\sum' Q_i\epsilon/2} + (-1)^N\mathrm{e}^{\mathrm{i}p\epsilon}\,\mathrm{e}^{-\mathrm{i}\sum' Q_i\epsilon/2}\,]\,A_{\mu_1}(Q_1)\cdots A_{\mu_N}(Q_N)\times\\
&\times\,\mathrm{Tr}\Big[\gamma^{\,\mu_1}S_+\big(p-\frac{Q_1}{2}\big)\gamma^{\,\mu_2}S_+\big(p-\frac{Q_1}{2}-Q_2\big)\ldots\gamma^{\,\mu_N}S_+\big(p-\frac{Q_1}{2}-Q_2-\ldots-Q_N\big)\Big].
\end{aligned} \tag{IV.12}$$

Now we consider the contribution of the gauge compensating factor, i.e., of the Schwinger line-integral in (IV.2), for a given coupling e. Since we will eventually take the limits of ϵ going to zero, we may carry out an expansion as follows

$$\exp\,[\,\mathrm{i}\,\mathrm{e}\int_{x_-}^{x_+}\mathrm{d}\xi^\mu A_\mu(\xi)\,] \;=\; 1 + I_1 + I_2 + I_3 + \cdots. \tag{IV.13}$$

To carry out the ξ^μ - integration, we make a change of variable $\xi^\mu \to \lambda : \xi^\mu = x^\mu + (\epsilon^\mu/2)\lambda$, $-1 \leq \lambda \leq +1$, make a Fourier transform of $A_\mu(x)$, and expand in powers of ϵ to obtain for $\epsilon \simeq 0$

$$I_1 = 2\left(\frac{\mathrm{i}\,\mathrm{e}}{2}\right)\int\frac{(\mathrm{d}Q_1)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}\,Q_1x}\,A_{\mu_1}(Q_1)\Big[\,1 - \frac{(Q_1\epsilon)^2}{24} + \cdots\,\Big]\epsilon^{\mu_1}, \tag{IV.14}$$

$$I_2 = 4\left(\frac{\mathrm{i}\,\mathrm{e}}{2}\right)^2\frac{1}{2!}\int\frac{(\mathrm{d}Q_1)}{(2\pi)^4}\frac{(\mathrm{d}Q_2)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}(Q_1+Q_2)x}A_{\mu_1}(Q_1)\,A_{\mu_2}(Q_2)[\,1 + \cdots]\,\epsilon^{\mu_1}\epsilon^{\mu_2}, \tag{IV.15}$$

$$\begin{aligned}
I_3 = 8\left(\frac{\mathrm{i}\,\mathrm{e}}{2}\right)^3\frac{1}{3!}\int\frac{(\mathrm{d}Q_1)}{(2\pi)^4}\frac{(\mathrm{d}Q_2)}{(2\pi)^4}\frac{(\mathrm{d}Q_3)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}\,(Q_1+Q_2+Q_3)x}A_{\mu_1}(Q_1)\,A_{\mu_2}(Q_2)\times\\
\times A_{\mu_3}(Q_3)[\,1 + \cdots]\,\epsilon^{\mu_1}\epsilon^{\mu_2}\epsilon^{\mu_3}.
\end{aligned} \tag{IV.16}$$

We will show that all the terms represented by $\ldots$ in (IV.14), (IV.15), (IV.16), as well as $I_4,\ I_5,\ \ldots$ in (IV.13), will not contribute in the limiting procedure.

We multiply (IV.12) by the expression $[1 + I_1 + I_2 + I_3 + \ldots]$ in (IV.13), extract the $(N-1)$th order terms in e of the product, integrate over e as indicated in (IV.2), and integrate over x as well. Here we note the basic property of the factor $[\mathrm{e}^{-\mathrm{i}p\epsilon}\mathrm{e}^{\,\mathrm{i}\sum' Q_i\epsilon/2} + (-1)^N \mathrm{e}^{\mathrm{i}p\epsilon}\mathrm{e}^{\,-\mathrm{i}\sum' Q_i\epsilon/2}]/2$ in (IV.12), in reference to (IV.13), (IV.14), (IV.15), (IV.16):

$$\begin{aligned}&\epsilon^{\mu_1}\ldots\epsilon^{\mu_k}\,\frac{1}{2}\big[\,\mathrm{e}^{-\mathrm{i}p\epsilon}\,\mathrm{e}^{\,\mathrm{i}\sum' Q_i\,\epsilon/2} + (-1)^{N-k}\mathrm{e}^{\mathrm{i}p\,\epsilon}\mathrm{e}^{-\mathrm{i}\sum' Q_i\,\epsilon/2}\,\big]\\ &= \Big(\mathrm{i}\frac{\partial}{\partial p_{\mu_1}}\Big)\ldots\Big(\mathrm{i}\frac{\partial}{\partial p_{\mu_k}}\Big)\frac{1}{2}\big[\,\mathrm{e}^{-\mathrm{i}p\,\epsilon}\mathrm{e}^{\,\mathrm{i}\sum' Q_i\,\epsilon/2} + (-1)^N\mathrm{e}^{\mathrm{i}p\,\epsilon}\mathrm{e}^{-\mathrm{i}\sum' Q_i\,\epsilon/2}\,\big].\end{aligned} \tag{IV.17}$$

We may then integrate by parts over p to apply these derivatives with respect to the p_{μ_j}, replacing the ϵ^{μ_j} in (IV.14), (IV.15), (IV.16), as indicated above, to the $\mathrm{Tr}\,[\ .\ ]$ factor in (IV.12), and take the limit $\epsilon \to 0$. The process is straightforward, and we obtain for the Nth- order term in e for $\mathrm{i}\,W$ in (IV.2)

$$\begin{aligned}\mathrm{i}\,W\Big|_{(N)} &= -\,\frac{(\mathrm{e})^N}{N}\int(\mathrm{d}x)\frac{(\mathrm{d}Q_1)}{(2\pi)^4}\cdots\frac{(\mathrm{d}Q_N)}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}\,(Q_1+\cdots+Q_N)x}\\ &\quad\times A_{\mu_1}(Q_1)\cdots A_{\mu_N}(Q_N)\,L^{\mu_1\ldots\mu_N}(Q_1,\ldots,Q_N),\end{aligned} \tag{IV.18}$$

where

$$L^{\mu_1\ldots\mu_N}(Q_1,\ldots,Q_N) = \frac{1}{2}\big[\,1+(-1)^N\,\big]\int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\Pi^{\mu_1\ldots\mu_N}(p\,;Q_1,\ldots,Q_N), \tag{IV.19}$$

and we immediately note that only terms even in N, i.e., for $N = 2,4,\ldots,$ contribute. This is known as Furry's Theorem . It is a consequence of charge conjugation invariance of electrodynamics (if A_μ is also made to transform to $-A_\mu$ consistently with Maxwell's equations).

Before spelling out the explicit expressions for $\Pi^{\mu_1\ldots\mu_N}(p\,;Q_1,\ldots,Q_N)$, for the various N, we first recall Gauss' Theorem in 4D conveniently written in the way it presents itself in the above formulae for a function $f(p)$ involving a product of Dirac propagators. It reads

$$\int(\mathrm{d}p)\,\frac{\partial}{\partial p_\mu}f(p) \;=\; \int\mathrm{d}\sigma^\mu f(p), \tag{IV.20}$$

with the right-hand side defined as a surface integral, where $\mathrm{d}\sigma^\mu$ is an element of surface. With $p^0 \to \mathrm{i}p^0, \mathrm{d}\sigma^\mu = n^\mu\mathrm{d}\sigma,\ n^\mu$ a unit vector along p^μ, if $\mathrm{d}\sigma$ scaled by a parameter λ, it grows like λ^3. Accordingly, if $f(\lambda p)$ vanishes faster than λ^{-3}, then the integral on the right-hand side is zero.

According to the rule in (IV.17), the gauge compensating terms I_n in (IV.13), each of them, lead to n or more derivatives $\partial/\partial p_\mu$, corresponding to the ϵ^μ appearing in the I_n,[3] and *acting* on the $\mathrm{Tr}[\ .\ ]$ factor in (IV.12). That is, a gauge compensating term I_n leads explicitly to integrals of the form

$$\int (\mathrm{d}p)\, \frac{\partial}{\partial p_{\nu_1}} \left(\frac{\partial}{\partial p_{\nu_2}} \cdots \frac{\partial}{\partial p_{\nu_n}} \right)$$
$$\times\ \mathrm{Tr}\Big[\gamma^{\mu_1} S_+(p - \frac{Q_1}{2})\gamma^{\mu_2} S_+(p - \frac{Q_1}{2} - Q_2)\ldots\gamma^{\mu_k} S_+(p - \frac{Q_1}{2} - Q_2 - \cdots - Q_k)\Big], \tag{IV.21}$$

and may involve even more derivatives with respect to p. By Gauss' Theorem, such integrals would vanish for $n-1+k \geq 4$, since $S(\lambda p) = \mathscr{O}(\lambda^{-1})$. Accordingly, for $k = 1$, only, I_1, I_2, I_3 may contribute, and for $k = 2$, only I_1, I_2 may contribute, and finally for $k = 3$ only I_1 may contribute. Hence we may infer that only $\Pi^{\mu_1\mu_2}$, and $\Pi^{\mu_1\mu_2\mu_3\mu_4}$ would involve an addition to the expressions in (IV.2) coming from gauge compensating terms. Needless to say $\Pi^{\mu_1\mu_2\mu_3}$ is zero, corresponding to $N = 3$.

Accordingly for $N = 6, 8, \ldots,$

$$\Pi^{\mu_1\ldots\mu_N}(p\,; Q_1, \ldots, Q_N)$$
$$= \mathrm{Tr}\Big[\gamma^{\mu_1} S_+(p - \frac{Q_1}{2})\gamma^{\mu_2} S_+(p - \frac{Q_1}{2} - Q_2)\ldots\gamma^{\mu_N} S_+(p - \frac{Q_1}{2} - Q_2 - \cdots - Q_N)\Big]_{\mathrm{sym}}, \tag{IV.22}$$

where $[\,.\,]_{\mathrm{sym}}$ means to symmetrize over the $(N-1)$ pairs of "generalized indices" (μ_2, Q_2) , ... ,(μ_N, Q_N).

For $N = 4$, we have

$$\Pi^{\mu_1\mu_2\mu_3\mu_4}(p; Q_1, \ldots, Q_4) =$$
$$\mathrm{Tr}\Big[\gamma^{\mu_1} S_+(p - \frac{Q_1}{2})\gamma^{\mu_2} S_+(p - \frac{Q_1}{2} - Q_2)\ldots\gamma^{\mu_4} S_+(p - \frac{Q_1}{2} - Q_2 - Q_3 - Q_4)\Big]$$
$$+ \frac{\partial}{\partial p_{\mu_4}} \mathrm{Tr}\Big[\gamma^{\mu_1} S_+(p - \frac{Q_1}{2})\gamma^{\mu_2} S_+(p - \frac{Q_1}{2} - Q_2)\gamma^{\mu_3} S_+(p - \frac{Q_1}{2} - Q_2 - Q_3\Big]$$
$$+ \frac{1}{2\,!} \frac{\partial}{\partial p_{\mu_3}} \frac{\partial}{\partial p_{\mu_4}} \mathrm{Tr}\Big[\gamma^{\mu_1} S_+(p - \frac{Q_1}{2})\gamma^{\mu_2} S_+(p - \frac{Q_1}{2} - Q_2)\Big]$$
$$+ \frac{1}{3!} \frac{\partial}{\partial p_{\mu_2}} \frac{\partial}{\partial p_{\mu_3}} \frac{\partial}{\partial p_{\mu_4}} \mathrm{Tr}\Big[\gamma^{\mu_1} S_+(p)\Big]. \tag{IV.23}$$

[3] See, e.g., (IV.14), (IV.15), (IV.16).

A symmetrization is necessary. However much simplification of this expression may be done first. To this end, note that for any given q, we may write

$$\frac{1}{\gamma(p-Q)+m} = \frac{1}{\gamma p+m} + \frac{1}{\gamma(p-Q)+m}\gamma Q\frac{1}{\gamma p+m}. \tag{IV.24}$$

Clearly, the second term introduces one more power of the momentum p in the denominator, and hence invoking Gauss' Theorem, we may infer that we may set the Q momenta in the second and third terms on the right-hand side of (IV.23) equal to zero. On the other hand symmetrizing the second term on the right-hand side of (IV.23) over μ_2, μ_3, after setting the Q momenta equal to zero, it takes the simple form

$$\frac{1}{2}\frac{\partial}{\partial p_{\mu_4}}\mathrm{Tr}\big[\gamma^{\mu_1}S_+(p)\gamma^{\mu_2}S_+(p)\gamma^{\mu_3}S_+(p)\big]+\frac{1}{2}\frac{\partial}{\partial p_{\mu_4}}\mathrm{Tr}\big[\gamma^{\mu_1}S_+(p)\gamma^{\mu_3}S_+(p)\gamma^{\mu_2}S_+(p)\big]$$
$$\equiv -\frac{1}{2}\frac{\partial}{\partial p_{\mu_3}}\frac{\partial}{\partial p_{\mu_4}}\mathrm{Tr}\big[\gamma^{\mu_1}S_+(p)\gamma^{\mu_2}S_+(p)\big], \tag{IV.25}$$

which exactly cancels the third term on the right-hand side of (IV.23) (after setting the Q's equal to zero). Hence (IV.23) finally reduces to

$$\Pi^{\mu_1\mu_2\mu_3\mu_4}(p\,;Q_1,\dots,Q_4)$$
$$= \mathrm{Tr}\Big[\gamma^{\mu_1}S_+(p-\frac{Q_1}{2})\gamma^{\mu_2}S_+(p-\frac{Q_1}{2}-Q_2)\dots\gamma^{\mu_4}S_+(p-\frac{Q_1}{2}-Q_2-Q_3-Q_4)\Big]_{\mathrm{sym}}$$
$$+\frac{1}{3}\frac{\partial}{\partial p_{\mu_2}}\frac{\partial}{\partial p_{\mu_3}}\frac{\partial}{\partial p_{\mu_4}}\mathrm{Tr}\big[\gamma^{\mu_1}S_+(p)\big]. \tag{IV.26}$$

For $N=2$, we simply have $(Q_2=-Q_1)$

$$\Pi^{\mu_1\mu_2}(p\,;Q_1,-Q_1) = \mathrm{Tr}\Big[\gamma^{\mu_1}S_+(p-\frac{Q_1}{2})\gamma^{\mu_2}S_+(p+\frac{Q_1}{2})\Big]$$
$$+\,Q_1^{\nu_1}Q_1^{\nu_2}\frac{1}{24}\frac{\partial}{\partial p^{\nu_1}}\frac{\partial}{\partial p^{\nu_2}}\frac{\partial}{\partial p_{\mu_2}}\mathrm{Tr}\,[\gamma^{\mu_1}S_+(p)\,]+\frac{\partial}{\partial p_{\mu_2}}\mathrm{Tr}\,[\gamma^{\mu_1}S_+(p)\,]. \tag{IV.27}$$

Thus we have obtained all the terms in (IV.18), (IV.19), for the Nth order contribution to $\mathrm{i}\,W$ in (IV.2). After integrating (IV.18) over x, the following expression emerges for $\mathrm{i}\,W$

$$\mathrm{i}W = -\sum_{N=2,4,..}\frac{(\mathrm{e})^N}{N}\int\frac{(\mathrm{d}Q_1)}{(2\pi)^4}\dots\frac{(\mathrm{d}Q_N)}{(2\pi)^4}(2\pi)^4\delta^4(Q_1+\dots+Q_N)$$
$$\times A_{\mu_1}(Q_1)\dots A_{\mu_N}(Q_N)\int\frac{(\mathrm{d}p)}{(2\pi)^4}\,\Pi^{\mu_1\dots\mu_N}(p\,;Q_1,\dots,Q_N), \tag{IV.28}$$

where $\Pi^{\mu_1 \cdots \mu_N}(p\,; Q_1, \ldots, Q_N)$ is given in (IV.22) for $N = 6, 8, \ldots$, and in (IV.26) for $N = 4$, and in (IV.27) for $N = 2$. The expression in (IV.27) for $N = 2$ was possibly first derived in details by K. Johnson[4], and by Schwinger himself.

[4]See, e.g., K. Johnson (1965). Brandeis University Summer Institute in Theoretical Physics. In *Lectures on particles and fields*. Englewood Cliffs: Prentice Hall.

Appendix V
Renormalization and the Underlying Subtractions

In this appendix, we learn how to define the renormalized Feynman integral, associated with a graph, given its unrenormalized Feynman integrand.

One defines a graph by specifying a set of vertices $\mathscr{V} = (\upsilon_1, \ldots, \upsilon_r)$, a set of lines $\mathscr{L} = (\ell_1, \ldots, \ell_s)$, together with a rule describing which of these lines join which of the vertices. By removing all the external lines of a graph, one generates an amputated graph. Vertices $\{\upsilon_i\}$ in an amputated graph, to which external lines were attached are referred to as external vertices, while their remaining vertices are referred to as internal vertices. From now on we will consider only amputated graphs and amputated subdiagrams with the latter to be defined below. By specifying a subset $\mathscr{V}' \subset \mathscr{V}$ of the vertices and all the lines in $\mathscr{L}$ of a graph G that join these vertices, but not those lines joining vertices not in $\mathscr{V}'$, one defines a subgraph G' of G. By specifying a subset $\mathscr{V}' \subset \mathscr{V}$ of the vertices and not necessarily all the lines in $\mathscr{L}$ of the original graph that join these vertices, one defines a subdiagram g of G. Examples of an (amputated) graph, together with a subgraph and a subdiagram are given in Fig. V.1.

A vertex of an amputated subdiagram at which an external line was attached previous to the removal of the latter is referred to as an external vertex, while the other vertices are referred to as internal vertices. All the lines of an amputated subdiagram are, by definition, internal lines.

A subdiagram is called disconnected if it is connected out of two or more subdiagrams in which any two of them have no vertices and no lines in common. If upon cutting or removing an internal line of a subdiagram, the number of the connected parts of the latter is increased, it is called an improper line. A subdiagram is called a proper connected subdiagram, if it is connected and has no improper lines. A subdiagram is called proper but not connected, if it is constructed out of two or more subdiagrams none of which have improper lines. Two subdiagrams g_1, g_2 specified by the pairs $(\mathscr{V}_1, \mathscr{L}_1)$, $(\mathscr{V}_2, \mathscr{L}_2)$, for which $\mathscr{V}_1 \subset \mathscr{V}_2$, and $\mathscr{L}_1 \subset \mathscr{L}_2$, we write $g_1 \subset g_2$. In this work, the symbol $\subset$ may include equality, while $\subsetneqq$ is used to exclude an equality.

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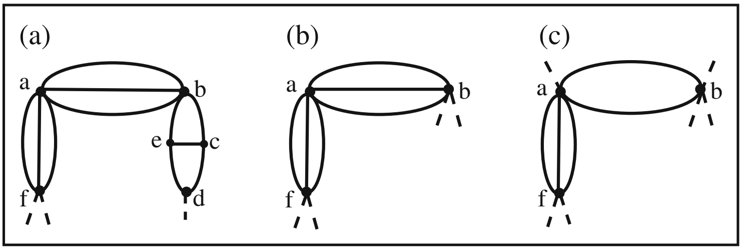


Fig. V.1 A graph is introduced in part (**a**). The *dashed lines* stand for external lines that have been removed defining an amputated graph. Vertices at f and d represent external vertices for the graph, while the remaining ones represent internal vertices. A subgraph is introduced in part (**b**) with four external lines that have been removed. In this case f and b represent external vertices of the subgraph, while a represents an internal vertex. A subdiagram is introduced in part (**c**) where one of the lines joining the vertices at a and b has been cut with the resulting two external lines removed. The vertices at a, b and f represent external vertices for the subdiagram in part (**c**)

Since the subtractions of renormalization involve a sequence of Taylor expansions in the momenta associated with the external lines of a proper graph and in the momenta of the external momenta of proper subdiagrams, we have to spell out how such momenta are defined for the proper graph and the proper subdiagrams. This is considered next.

Suppose that a proper and connected graph G has m external vertices with total external momenta $\{q_1, \ldots, q_m\} \equiv q^G$, $q_r = (q_r^0, q_r^1, q_r^2, q_r^3)$. The integration variables associated with the graph are denoted by $\{k_1^0, k_1^1, \ldots, k_n^3\} \equiv k^G$. A line labeled ℓ, joining a vertex υ_i to a vertex υ_j, carries a momentum denoted by $Q_{ij\,\ell}$ which may be written as

$$Q_{ij\,\ell} = k_{ij\,\ell} + q_{ij\,\ell}, \quad k_{ij\,\ell} = \sum_{s=1}^{n} a_{ij\,\ell}^{s} k_s, \quad q_{ij\,\ell} = \sum_{r=1}^{m} b_{ij\,\ell}^{r} q_r, \quad Q_{ij\,\ell} = -Q_{ji\,\ell}. \tag{V.1}$$

At each external vertex υ_j of G,

$$\sum_{i\ell}{}^{G} Q_{ij\,\ell} = q_j^G, \quad \text{and by momentum conservation} \quad \sum_j q_j^G = 0, \tag{V.2}$$

where the sum is over all i corresponding to all the vertices with lines ℓ joining them to the external vertex υ_j. For an internal vertex, the right-hand side of the above first equality is set equal to zero.

A line joining a vertex υ_i to a vertex υ_j in G, will be represented by

$$\Delta_{+ij\,\ell}(Q_{ij\,\ell}, \mu_{ij\,\ell}) = f_{ij\,\ell}(Q_{ij\,\ell}, \mu_{ij\,\ell})[Q_{ij\,\ell}^2 + \mu_{ij\,\ell}^2]^{-1}, \tag{V.3}$$

involving a familiar multiplicative factor $f_{ij\,\ell}(Q_{ij\,\ell}, \mu_{ij\,\ell})$ when one is dealing with particles of non-zero spin. Let j be fixed, and consider the set $\{\upsilon_{i(j)}\}_{1\leq i\leq r_j}$ of vertices attached by lines to the vertex υ_j in G, and consider the set $\mathscr{L}^G(\upsilon_j)$ of all lines joining the vertices $\upsilon_{i(j)}$ to the vertex υ_j. Moreover, let $\{Q_{ij\ell}\}_{1\leq\ell\leq s_{ij}}$ be the set of momenta carried by these lines. Then we assign to the veretx υ_j a polynomial $\mathscr{P}_j = \mathscr{P}_j(Q_{ij1},\ldots,Q_{r_j j s_{r_j j}})$. The unrenormalized Feynman integrand associated with the proper and connected graph G, up to an overall multiplicative constants (involving couplings, etc.), is of the form

$$I_G = \prod\nolimits^G_{ij\ell,i<j} \mathscr{P}_j\, \Delta_{+ij\ell}. \tag{V.4}$$

We introduce canonical variables,[4] by choosing

$$\sum_{i\ell}{}^G q_{ij\ell} = q^G_j, \quad \text{and} \quad q_{ij\ell} = u_i - u_j, \tag{V.5}$$

where u_i, u_j are four vectors, and the second equality above means that the external variables of the lines joining a vertex υ_i to the vertex υ_j are all chosen to be equal. In particular, we also note that (V.2), (V.5) imply that

$$\sum_{i\ell}{}^G k_{ij\ell} = 0, \qquad \sum_{ij\ell}{}^G k_{ij\ell} = 0, \tag{V.6}$$

with the latter defining a constraint. Equation (V.5) provides $(\#\mathscr{V}-1)$ independent solutions of the $(\#\mathscr{V}-1)$ independent differences (u_j-u_i), where $\#\mathscr{V}$ denotes the number of vertices. We may write the variables $q_{ij\ell}$ as a linear combination of the elements in $\{q^G_j\}\equiv q^G$: $q_{ij\ell} = q_{ij\ell}(q^G)$.

If $4n$ denotes the number of independent integration variables associated with G then $n = \#\mathscr{L}^G - \#\mathscr{V}^G + 1$. Eqs. (V.6) imply that only n of the $\#\mathscr{L}^G\, k_{ij\ell}$ are independent, and we may write $k_{ij\ell} = k_{ij\ell}(k)$. We provide examples of canonical decompositions of some graphs and some diagrams obtained from them.

Example V.1 Consider the graph in Fig. V.2a, with external vertices at 1 and 2. At the vertex υ_1

$$q = q_{211} + q_{212} + q_{213}, \quad \text{or} \quad u_2 - u_1 = \frac{1}{3}q, \quad \text{i.e.} \quad q_{21\ell} = \frac{1}{3}q, \quad \ell = 1,2,3. \tag{V.7}$$

At vertex υ_1, we also have: $k_{211}+k_{212}+k_{213}=0$. Let $k_{211}=k_1$, $k_{212}=k_2$, then $k_{213}=-k_1-k_2$. Therefore a canonical choice of variables are

$$Q_{211} = k_1 + \frac{1}{3}q, \quad Q_{212} = k_2 + \frac{1}{3}q, \quad Q_{213} = -k_1 - k_2 + \frac{1}{3}q. \quad \diamond \tag{V.8}$$

[4] Canonical variables were introduced by Zimmermann [9].

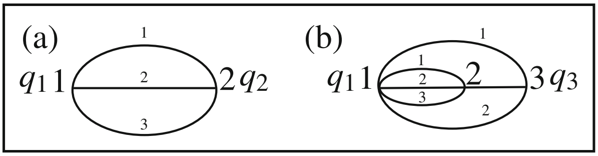


Fig. V.2 (**a**) A graph with external vertices at 1 and 2, and external momenta $q_1 \equiv q$, $q_2 \equiv -q$. (**b**) A graph with external vertices at 1 and 3 and an internal vertex at 2

Example V.2 Consider the graph in Fig. V.2b with $q_1 \equiv q$, $q_3 \equiv -q$. At the vertices, υ_1, υ_2, υ_3, we have, respectively, with υ_2 an internal vertex

$$q = q_{211} + q_{212} + q_{213} + q_{311} + q_{312} = 3(u_2 - u_1) + 2(u_3 - u_1), \tag{V.9}$$

$$0 = q_{121} + q_{122} + q_{123} + q_{32} = 3(u_1 - u_2) + (u_3 - u_2), \tag{V.10}$$

$$-q = q_{131} + q_{132} + q_{23} = 2(u_1 - u_3) + (u_2 - u_3). \tag{V.11}$$

The unique solutions of the differences are then: $u_1 - u_2 = -q/11$, $u_1 - u_3 = -4\,q/11$, $u_2 - u_3 = -3\,q/11$. That is, $q_{12\,\ell} = -q/11$, $\ell = 1, 2, 3$; $q_{13\,\ell} = -4\,q/11$, $\ell = 1, 2$; $q_{23} = -3\,q/11$. For 16 integration variables, we may write for the four vectors: $k_{121} = k_1$, $k_{122} = k_2$, $k_{123} = k_3$, $k_{131} = k_4$. For the remaining integration variables, we note that at υ_2, and υ_3, respectively, we have

$$k_{121} + k_{122} + k_{123} + k_{32} = 0, \qquad k_{131} + k_{132} + k_{23} = 0, \tag{V.12}$$

from which $k_{32} = -k_1 - k_2 - k_3$, and $k_{132} = -k_1 - k_2 - k_3 - k_4$. Accordingly a canonical set of variables are

$$Q_{121} = k_1 - \frac{1}{11}q, \quad Q_{122} = k_2 - \frac{1}{11}q, \quad Q_{123} = k_3 - \frac{1}{11}q \quad Q_{131} = k_4 - \frac{4}{11}\,q, \tag{V.13}$$

$$Q_{132} = -k_1 - k_2 - k_3 - k_4 - \frac{4}{11}\,q, \quad Q_{32} = -k_1 - k_2 - k_3 + \frac{3}{11}q. \quad \diamond \tag{V.14}$$

Example V.3 Consider the graph G in Fig. V.3 below, with $q_1 = -q$, $q_4 = q$. Let $k_{12} = 3\,(k_1 + 3\,k_2)/8$, $k_{34} = -3\,(3\,k_1 + k_2)/8$. Following the procedure in the above two examples gives rise to the following canonical decompositions:

$$Q_{12} = \frac{3}{8}\,k_1 + \frac{9}{8}\,k_2 + \frac{1}{2}\,q, \quad Q_{13} = -\frac{3}{8}\,k_1 - \frac{9}{8}\,k_2 + \frac{1}{2}\,q, \quad Q_{32} = \frac{3}{4}\,k_1 - \frac{3}{4}\,k_2, \tag{V.15}$$

$$Q_{34} = -\frac{9}{8}\,k_1 - \frac{3}{8}\,k_2 + \frac{1}{2}\,q, \qquad Q_{24} = \frac{9}{8}\,k_1 + \frac{3}{8}\,k_2 + \frac{1}{2}\,q. \quad \diamond \tag{V.16}$$

Canonical decomposition corresponding to a proper and connected subdiagram g of G, will be defined in a similar manner. We write the momentum associated with a line in g as

$$Q_{ij\ell} = k^G_{ij\,\ell} + q^G_{ij\,\ell} = k^{\,g}_{\,ij\,\ell} + q^{\,g}_{\,ij\,\ell}, \tag{V.17}$$

and at each external vertex υ_j of g,

$$\sum_{i\ell}{}^{g} q^{\,g}_{\,ij\,\ell}(k^G, q^G) = q^{\,g}_{j}(k^G, q^G), \quad \sum_{j}{}^{g} q^{\,g}_{j}(k^G, q^G) = 0 \quad \text{and} \quad q^{\,g}_{\,ij\,\ell} = w_i - w_j, \tag{V.18}$$

where w_i, w_j are four vectors, and the sum over $i\,\ell$ pertains to the subdiagram g. If υ_j is an internal vertex of g then we set $q^{\,g}_{j} = 0$. Once the $q^{\,g}_{\,ij\,\ell}$ are determined, then $k^{\,g}_{\,ij\,\ell} = Q_{ij\ell} - q^{\,g}_{\,ij\,\ell}$. We show that the $k^{\,g}_{\,ij\,\ell}$ are linear combination of the elements in k^G only and are independent of the elements in q^G. To see this, set the integration variables in k^G, equal to zero, then

$$\sum_{i\ell}{}^{g} q^{\,g}_{\,ij\,\ell}(0, q^G) = q^{\,g}_{j}(0, q^G), \qquad \sum_{j}^{g}\Big(\sum_{1\ell}^{g} q^{\,g}_{\,ij\,\ell}(0, q^G)\Big) = 0,$$
$$q^{\,g}_{\,ij\,\ell}(0, q^G) = (w_i - w_j)\big|_{k^G=0} = w_i{}' - w_j{}'. \tag{V.19}$$

But for $k^G = 0$, $Q_{ij\ell} = q^G_{ij\ell} = k^{\,g}_{\,ij\,\ell}(0, q^G) + q^{\,g}_{\,ij\,\ell}(0, q^G)$ and

$$\sum_{i\ell}{}^{g} q^G_{ij\,\ell} = \sum_{i\ell}{}^{g} Q_{ij\,\ell} = q^{\,g}_{j}(0, q^G), \quad \Big(\sum_{i\ell}{}^{g} q^G_{ij\ell}\Big) = 0, \quad q^G_{ij\,\ell} = u_i{}' - u_j{}', \tag{V.20}$$

for i,j,ℓ pertaining to the subdiagram g. Uniqueness of the solutions of the $(\#\mathscr{V}^g - 1)$ conditions of the $(\#\mathscr{V}^g - 1)$ independent differences $u_i{}' - u_j{}'$ or $w_i{}' - w_j{}'$, implies that $u_i{}' - u_j{}' = w_i{}' - w_j{}'$. That is $k^{\,g}_{\,ij\,\ell}(0, q^G) = 0$ and hence $k^{\,g}_{\,ij\,\ell} = k^{\,g}_{\,ij\,\ell}(k^G)$.

Similarly for two proper connected subdiagram $g' \subsetneqq g$, $k^{\,g'}_{\,ij\ell} + q^{\,g'}_{\,ij\ell} = k^{\,g}_{\,ij\,\ell} + q^{\,g}_{\,ij\,\ell}$, and $k^{\,g'}_{\,ij\ell} = k^{\,g'}_{\,ij\,\ell}(k^g)$, $q^{\,g'}_{\,ij\ell} = q^{\,g'}_{\,ij\,\ell}(k^g, q^g)$. In particular, the $q^{g'}_{j}$ are linear combinations of the $k^{g}_{ij\,\ell}$ in g/g'. Here we note that once the unrenormalized Feynman integrands I_g and $I_{g'}$ have been defined, $I_{g/g'}$ is defined through $I_g = I_{g/g'}\,I_{g'}$. Note that

$$\sum_{i\ell}{}^{g'} q^G_{ij\,\ell} = q^{g'}(k^g, k^g), \quad \text{at each vertex } \upsilon_j \text{ of } g', \tag{V.21}$$

with the sum over $i\ell$ pertaining to the subdiagram g'. Also

$$q_j^{\,g} = \sum_{i\ell}{}^{g} q_{ij\,\ell}^{G} = \sum_{i\ell}{}^{g'} Q_{ij\,\ell} + \sum_{i\ell}{}^{g/g'} Q_{ij\,\ell}, \tag{V.22}$$

and hence the above two equations imply that

$$q_j^{\,g'}(k^g, q^g) = q_j^{\,g} - \sum_{i\ell}{}^{g/g'} \big(k_{ij\,\ell}^{\,g} + q_{ij\,\ell}^{\,g}\big), \tag{V.23}$$

which establishes the stated result.

Example V.4 Consider the graph G and the two subdiagrams g_1 and g_2 in Fig. V.3. From (V.15), (V.16), (V.17), at the vertices 2 and of 3 of subdiagram g_1, respectively,

$$q_2^{g_1} = q_{42}^{g_1} + q_{32}^{g_1} = Q_{42} + Q_{32} \;=\; w_4 - w_2 + w_3 - w_2 = -\frac{3}{8}\,k_1 - \frac{9}{8}\,k_2 - \frac{1}{2}\,q, \tag{V.24}$$

$$q_3^{g_1} = q_{43}^{g_1} + q_{23}^{g_1} = Q_{43} + Q_{23} \;=\; w_4 - w_3 + w_2 - w_3 = +\frac{3}{8}\,k_1 + \frac{9}{8}\,k_2 - \frac{1}{2}\,q. \tag{V.25}$$

These lead to ($q_4^{g_1} = q$)

$$q_{34}^{g_1} = -\frac{1}{8}\,k_1 - \frac{3}{8}\,k_2 + \frac{1}{2}\,q, \quad q_{24}^{g_1} = \frac{1}{8}\,k_1 + \frac{3}{8}\,k_2 + \frac{1}{2}\,q, \quad q_{32}^{g_1} = -\frac{1}{4}\,k_1 - \frac{3}{4}\,k_2. \tag{V.26}$$

For the internal variables we then have

$$k_{34}^{g_1} = Q_{34} - q_{34}^{g_1} = -k_1, \quad k_{24}^{g_1} = Q_{24} - q_{24}^{g_1} = k_1, \quad k_{32}^{g_1} = Q_{32} - q_{32}^{g_1} = k_1. \tag{V.27}$$

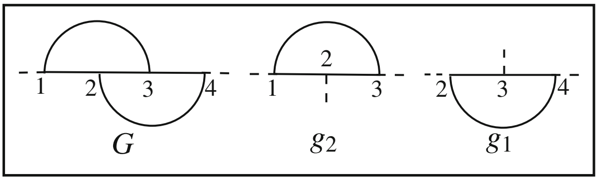


Fig. V.3 A graph G with external vertices at 1 and 4, showing only two of its proper subdiagrams g_1 and g_2 having each three external vertices. The *dashed lines* are just to remind us of the *external lines* that have been removed.

Similarly for the subdiagram g_2 we have $\big(q_1^{g_2} = -q,\, q_2^{g_2} = (9/8)\,k_1 + (3/8)\,k_2 + (q/2),\, q_3^{g_2} = -(9/8)\,k_1 - (3/8)\,k_2 + (q/2)\big)$

$$q_{32}^{g_2} = \frac{3}{4}k_1 + \frac{1}{4}k_2,\quad q_{13}^{g_2} = -\frac{3}{8}k_1 - \frac{1}{8}k_2 + \frac{1}{2}q,\quad q_{12}^{g_2} = \frac{3}{8}k_1 + \frac{1}{8}k_2 + \frac{1}{2}q, \tag{V.28}$$

$$k_{32}^{g_2} = -k_2, \qquad k_{13}^{g_2} = -k_2, \qquad k_{12}^{g_2} = k_2. \qquad \diamond \tag{V.29}$$

Once the external momentum variables $q_{ij\ell}^G$ of the graph G in question, as well as the external variables $q_{ij\ell}^g$ of all proper and connected subdiagrams of G have been defined, one may define Taylor operations T_G, T_g in their external momenta about the origin up to their (superficial) degree of divergences. The (superficial) degree of divergence $d(g)$ of a proper connected subdiagram g involving $4\,t$ integration variables is defined by

$$d(g) = \deg I_g + 4\,t. \tag{V.30}$$

where "deg" is here defined by scaling the integration variables in I_g by a parameter λ and by considering the power of λ for $\lambda \to \infty$. For a proper not necessarily connected subdiagram g involving proper and connected subdiagrams $g_1, g_2, \ldots, g_s$, the (superficial) degree of divergence of g is defined by $d(g) = \sum_{i=1}^{s} d(g_i)$.

We are particularly interested in the consecutive application of two or more Taylor operations. To this end, consider two proper subdiagrams $g' \subsetneqq g$, we define $T_g\, T_{g'} I_g$ by the following procedure. We first note that

$$T_{g'}\, I_{g'} = F(Q, k^{g'}, q^{g'}), \tag{V.31}$$

where the function $I_{g/g'} F(Q, k^{g'}, q^{g'})$ should be now expressed in terms the internal k^g, and external q^g variables of g. To do this, we use (V.17), and the functional relations $k^{g'} = k^{g'}(k^g)$, $q^{g'} = q^{g'}(k^g, q^g)$. The application of consecutive several Taylor operation $T_{g'_s} \ldots T_{g'_1} I_{g'_s}$ for proper subdiagrams $g'_1 \subsetneqq \ldots \subsetneqq g'_s$ is similarly handled.

Now we are ready to define the renormalized R Feynman integrand involving Taylor operations, about the origin, applied to the external variables of a proper and connected graph G, and to its proper, not necessarily connected subdiagrams. The external variables of such subdiagrams constitute the totality of the external variables of its connected parts.

The renormalized Feynman integrand R, involving subtractions, is defined as follows:

$$R = \Big[1 + \sum_{D} \prod_{g \in D} (-T_g)\Big] I_G, \tag{V.32}$$

where the sum is over *all* non-empty sets D such that

(i) If $g \in D$, then g is a proper, but not necessarily connected, subdiagram of G with $d(g) \geq 0$. If $d(G) \geq 0$, then one of the elements of D may be G itself.

(ii) If $g_1, g_2 \in D$, then either $g_1 \subsetneqq g_2$ or $g_2 \subsetneqq g_1$. If $g_1 \subsetneqq g_2$, then the ordering of the Taylor operations in (V.32) is as $\ldots(-T_{g_2})\ldots(-T_{g_1})\ldots$.

Note that the Taylor operations are applied directly to the integrand I_G in the momentum representation, and no questions of divergences arise in (V.32). The sets D are called renormalization sets.

We see that an important task is to find the renormalization sets D associated with a graph G. For example, for the graph G in Fig. V.3, the normalization sets, say for simplicity for scalar field self-coupling in 6 dimensional spacetime, are: $\{G\}$, $\{G, g_1\}$, $\{G, g_2\}$, $\{g_1\}$, $\{g_2\}$. Note that the subdiagram obtained from the graph G by cutting (i.e, removing) line 23 has a degree of divergence -2 in 6 dimensions. Note also that $\{g_1, g_2\}$ is not a D set, since neither $g_1 \subsetneqq g_2$ nor $g_2 \subsetneqq g_1$. The renormalized integrand then becomes

$$\Big[1 + (-T_G) + (-T_G)(-T_{g_1}) + (-T_G)(-T_{g_2}) + (-T_{g_1}) + (-T_{g_2})\Big] I_G$$
$$= (1 - T_G)\Big[1 + (-T_{g_1}) + (-T_{g_2})\Big] I_G, \tag{V.33}$$

with $d(G) = 2$, $d(g_1) = d(g_2) = 0$.

Remarks

(1) The key point in establishing the finiteness of the renormalized Feynman integral in Euclidean space for massive particles $\mu^2_{ij\,\ell} \neq 0$ is to show that when any subset of the integration variables become large, one can always group the Taylor operations in a particular way, and show that the renormalized integrand has just right behavior and vanishes rapidly asymptotically as a function of these variables and guaranties its integrability. A proof of this is given in Manoukian [3]). For a proof of the finiteness of the renormalized in Minkowski space, in the sense of distributions, see also the latter reference, and for a simpler and direct proof see Manoukian [4, 5], including for zero-mass particles.

(2) Once the finiteness of the renormalized theory is established, with subtractions at the origin, it may be then normalized at other points.

(3) The above subtraction procedure may be reduced to the one referred to as the Bogoliubov-Parasiuk-Hepp-Zimmermann scheme, and for the proof of the underlying theorem, referred to as the "Unifying Theorem of Renormalization",[5] see Manoukian [3].

(4) The renormalization procedure with subtractions is equivalent to adding terms to the original Lagrangian, referred to as counter-terms, and adjusting their numerical coefficients in such a way that parameters appearing in the theory

[5] See also Zeidler [8, p. 972] and Figueroa and Gracia-Bondia [1].

(masses, couplings,...) are taken from experiments. For the equivalence of the subtraction scheme and the counter-term formalism, see Manoukian [2, 3]. This establishes the criterion of renormalizabilty and is expressed as follows. One, *a prioi*, may generalize the original Lagrangian density by adding to it new terms. If the counter-terms needed to make the (modified) theory finite and consistent, have the same structures as of terms in the (modified) Lagrangian density and are *finite* in number, then the (modified) theory is called renormalizable. In the latter case only a finite number of parameters are taken from experiments. For example in QED, the counter-terms have exactly the same structures of terms in the Dirac-Maxwell Lagrangian density and the values of the two parameters the (renormalized) mass and (renormalized) charge of the electron are taken from experiments. If the number of counter-terms needed are infinite in number, then one may need to fix an infinite number of parameters and the theory loses its predictive power.

(5) Asymptotic behavior of the *renormalized* theory, such as at high-energy, or for small masses, or for large masses, and other variations, as well as of the proof of the decoupling theorem consistently used in QCD, see Manoukian [3].
(6) Regarding the author's work in the completion of the renormalization program stemming that of Salam's, Streater [7] writes: *"It is the end of a long chapter in the history of physics"*. He also states: *"Physicists found Salam's [method] easier than the BPH one"*.
(7) When you write down a local Lagrangian density, the parameters (couplings, masses,...) are introduced at infinite energies (specified by large ultraviolet cut-offs) and are unattainable. Renormalization theory eliminates these parameters in favor of physically measurable parameters.

Recommended Reading

1. Figueroa, H., & Gracia-Bondia, J. M. (2004). The uses of Connes and Kreimer's algebraic formulation of renormalization. *International Journal of Modern Physics, A19*, 2739–2754. hep–th/0301015v2.
2. Manoukian, E. B. (1979). Subtractions vs Counterterms. *Nuovo Cimento, 53A*, 345–358.
3. Manoukian, E. B. (1983a). *Renormalization*. New York/London/Paris: Academic Press.
4. Manoukian, E. B. (1983b). Elementary proof of $\epsilon \to +0$ limit of Renormalized Feynman amplitudes. *Journal of Physics: Mathematical and General, 16*, 4131–4133.
5. Manoukian, E. B. (1984b). Elementary proof of $\epsilon \to +0$ limit of Renormalized Feynman amplitudes. II: Theories involving zero mass particles. *Journal of Physics: Mathematical and General, 17*, 1931–1935.
6. Manoukian, E. B. (2006). *Quantum theory: A wide spectrum*. Dordrecht: Springer.
7. Streater, R. F. (1985). Review of Renormalization by E. B. Manoukian. *Bulletin of London Mathematical Society, 17*, pp. 509–510.
8. Zeidler, E. (2009). *Quantum field theory II: Quantum electrodynamics*. Berlin: Springer. pp. 972–975.
9. Zimmermann, W. (1969). Convergence of Bogoliubov's method of renormalization in momentum space. *Communications in Mathematical Physics, 15*, 208–234.

Solutions to the Problems

Chapter 2

2.1. Note that $\cos 2\theta = 1/\sqrt{|\mathbf{a}|^2 + 1}$, $\sin 2\theta = |\mathbf{a}|/\sqrt{\mathbf{a}^2 + 1}$, where we have used $\{\gamma^0, \boldsymbol{\gamma}\} = 0$, $(\boldsymbol{\gamma} \cdot \mathbf{a})^2 = -|\mathbf{a}|^2$. Also note that $(|\mathbf{a}| \equiv a)$

$$G\gamma^0 G^{-1} = \gamma^0 \Big[\cos 2\theta - \frac{\gamma \cdot \mathbf{a}}{|\mathbf{a}|} \sin 2\theta \Big]$$

$$G\gamma^0 \boldsymbol{\gamma} \cdot \mathbf{a}\, G^{-1} = \gamma^0 \big[\boldsymbol{\gamma} \cdot \mathbf{a} \cos 2\theta + a \sin 2\theta \big].$$

Hence $G\gamma^0(\boldsymbol{\gamma} \cdot \mathbf{a} + 1)\, G^{-1} = \gamma^0 \Big[\boldsymbol{\gamma} \cdot \mathbf{a} \big(\cos 2\theta - (\sin 2\theta/|\mathbf{a}|) \big) + (\cos 2\theta + a \sin 2\theta) \Big]$.

The statement of the problem then follows from the expressions of $\cos 2\theta$ and $\sin 2\theta$ given above.

2.2. Let $\mathbf{a} = \mathbf{p}/m$, $\cos\theta = \sqrt{(p^0 + m)/2p^0}$, $\sin\theta = |\mathbf{p}|/\sqrt{2p^0(p^0 + m)}$ in Problem 2.1. Then $GHG^{-1} = m\gamma^0 \sqrt{\mathbf{p}^2/m^2 + 1} = \gamma^0 \sqrt{\mathbf{p}^2 + m^2}$.

2.3. The integral in question is given by: $\int \mathrm{d}\rho_j\, (\rho_j - \beta_j)(\alpha_0 + c_1 \rho_j)$ which reduces to $\int \mathrm{d}\rho_j\, \rho_j (\alpha_0 + c_1 \beta_j) = \alpha_0 + c_1 \beta_j = f(\beta_j)$, where c_1 is a c number.

2.4. The integral is given by $\int \mathrm{d}\rho\, (\alpha_0 + c_1 \rho) = \int \mathrm{d}\rho\, c_1 \rho = c_1$, since c_1 is a c number. On the other hand, $(\partial/\partial\rho)(\alpha_0 + c_1\rho) = c_1(\partial/\partial\rho)\rho = c_1$.

2.5. (i) $\int \mathrm{d}\rho_{\mathrm{R}} \rho_{\mathrm{R}} = 1 = \Big(\int \mathrm{d}\rho_{\mathrm{R}} \rho_{\mathrm{R}} \Big)^*$. The latter, in turn, is equal to $\int \rho_{\mathrm{R}}\, (\mathrm{d}\rho_{\mathrm{R}})^* = -\int (\mathrm{d}\rho_{\mathrm{R}})^*\, \rho_{\mathrm{R}}$. That is, $(\mathrm{d}\rho_{\mathrm{R}})^* = -\,\mathrm{d}\rho_{\mathrm{R}}$, and similarly $(\mathrm{d}\rho_{\mathrm{I}})^* = -\,\mathrm{d}\rho_{\mathrm{I}}$. From the definition $\mathrm{d}\rho = (\mathrm{d}\rho_{\mathrm{R}} + \mathrm{i}\,\mathrm{d}\rho_{\mathrm{I}})$ we obtain $(\mathrm{d}\rho)^* = (-\,\mathrm{d}\rho_{\mathrm{R}}^* - (\mathrm{i})(-1)\mathrm{d}\rho_{\mathrm{I}}^*)$, from which the condition $(\mathrm{d}\rho)^* = -\,\mathrm{d}\rho^*$ follows. (ii) Using the latter result, we have $\Big(\int \mathrm{d}\rho\, \rho \Big)^* = \int \rho^* (-\,\mathrm{d}\rho^*) = \int \mathrm{d}\rho^* \rho^*$. Recall that complex conjugation reverses the order in a product.

E.B. Manoukian, *Quantum Field Theory I*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-30939-2

2.6. The integral in question may be rewritten as:

$$\int \mathrm{d}\overline{\eta}_1 \overline{\eta}_1 [(\mathrm{i})^2 (\rho_1 - \alpha_1)(\overline{\rho}_1 - \overline{\alpha}_1)]\, \eta_1 \mathrm{d}\eta_1 \ldots \mathrm{d}\overline{\eta}_n \overline{\eta}_n [(\mathrm{i})^2 (\rho_n - \alpha_n)(\overline{\rho}_n - \overline{\alpha}_n)]\, \eta_n \mathrm{d}\eta_n.$$

The result then follows from the integrals $\int \mathrm{d}\overline{\eta}\,\overline{\eta} = 1$, $\int \eta\, \mathrm{d}\eta = -1$. Note that every factor like $\overline{\eta}_j(\rho_j - \alpha_j)$, $(\overline{\rho}_j - \overline{\alpha}_j)\eta_j$, $(\rho_j - \alpha_j)(\overline{\rho}_j - \overline{\alpha}_j)$ commutes with everything for any j.

2.7. $[\delta/\delta\overline{\eta}_a(x)] \exp[\mathrm{i}\,\overline{\eta} A\, \eta] = \mathrm{i} \exp[\mathrm{i}\,\overline{\eta} A\, \eta] \int (\mathrm{d}x\,')\, A_{ab}(x, x\,')\eta_b(x\,')$. A further application of $[\delta/\delta\eta_c(y)]$ to the latter gives: $\exp[\mathrm{i}\,\overline{\eta}\, A\, \eta]\times$

$$\times \left[\Big((\mathrm{i})(-\mathrm{i})\int (\mathrm{d}x\,'')\overline{\eta}_d(x\,'') A_{dc}(x\,'', y)\Big)\Big(\int (\mathrm{d}y) A_{ad}(x, y)\eta_d(y)\Big) + \mathrm{i}\, A_{ac}(x, y)\right],$$

which coincides with (2.6.15), where we note that $\exp[\mathrm{i}\,\overline{\eta} A \eta]$ commutes with everything.

2.8. Using the identities in (2.6.11), (2.6.12), we may write to the leading order

$$Z[\overline{\eta}, \eta] = \left[1 + \mathrm{i}\,\mathrm{e} \int (\mathrm{d}x)(\mathrm{i}) \frac{\delta}{\delta\eta(x)} \gamma^{\mu} A_{\mu}(x)(-\mathrm{i}) \frac{\delta}{\delta\overline{\eta}(x)}\right]$$
$$\times \int \mathscr{D}\overline{\rho}\, \mathscr{D}\rho \exp\left(\mathrm{i}\left[-\overline{\rho}\Big(\gamma^{\mu}\frac{\partial_{\mu}}{\mathrm{i}} + m\Big)\rho + \overline{\rho}\eta + \overline{\eta}\rho\right]\right).$$

But the functional integral is from (2.6.27) is equal to $C \exp[\mathrm{i}\,\overline{\eta} S_{+} \eta]$, where the multiplicative factor C is independent of $(\overline{\eta}, \eta)$ and, of course, independent of e as well. Hence from (2.6.15), we have

$$\begin{aligned} Z[\overline{\eta}, \eta] &= C\left[1 + \mathrm{i}\,\mathrm{e} \int (\mathrm{d}x)(\mathrm{i}) \frac{\delta}{\delta\eta(x)} \gamma^{\mu} A_{\mu}(x)(-\mathrm{i}) \frac{\delta}{\delta\overline{\eta}(x)}\right] \exp[\mathrm{i}\,\overline{\eta}\, S_{+} \eta] \\ &= C\Big[1 - \mathrm{e}\int (\mathrm{d}x) \mathrm{Tr}[\gamma^{\mu} A_{\mu}(x) S_{+}(x, x)] + \mathrm{i}\,\mathrm{e}\int (\mathrm{d}x)(\mathrm{d}x\,')(\mathrm{d}y) \times \\ &\qquad \times \overline{\eta}(x)\, S_{+}(x, x\,')\gamma^{\mu} A_{\mu}(x\,') S_{+}(x\,', y)\, \eta(y)\Big] \exp[\mathrm{i}\,\overline{\eta}\, S_{+} \eta]. \end{aligned}$$

Therefore to the leading order, $\exp[\mathrm{i}\,\overline{\eta}\, S_{+}^{A} \eta] = Z[\overline{\eta}, \eta]/Z[0, 0]$ is equal to

$$\left[1 + \mathrm{i}\,\mathrm{e}\int (\mathrm{d}x)(\mathrm{d}x\,')(\mathrm{d}y)\overline{\eta}(x)\, S_{+}(x, x\,')\gamma^{\mu} A_{\mu}(x\,') S_{+}(x\,', y)\, \eta(y)\right] \exp[\mathrm{i}\,\overline{\eta}\, S_{+} \eta].$$

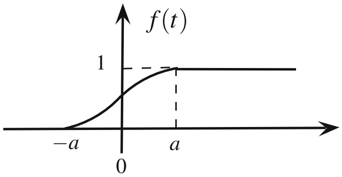
In particular carrying the functional derivatives: $(-\mathrm{i})[\delta/\delta\eta(y)][\delta/\delta\overline{\eta}(x)]$ of $\exp[\mathrm{i}\,\overline{\eta}\,S_+^A\eta]$ and then setting $\overline{\eta}=0$, $\eta=0$, we obtain from the above equation, to the leading order in A_μ,

$$S_+^A(x,y) = S_+(x,y) + \mathrm{e}\int(\mathrm{d}x')\,S_+(x,x')\gamma^\mu A_\mu(x')S_+(x',y),$$

where $S_+^{-1}(x,x') = (\gamma^\mu\partial_\mu/\mathrm{i}+m)\,\delta^{(4)}(x-x')$. S_+ is worked out in Sect. 3.1 with an appropriate boundary condition.

Chapter 3

3.1. At $t=-a$, the second expression is $(1/2)\exp(-\infty)=0$, and $f(t)$ is continuous at this point and vanishes. The continuity at $t=0$ is obvious too. For $t\to a$ from below, write $t=a-\epsilon$, $\epsilon\to+0$. Then the third expression is
$\big[1-(1/2)\exp\big(-2(a/\epsilon)-1\big)\big]\to 1$ for $\epsilon\to+0$, and $f(t)$ is continuous at $t=a$ as well and is equal to 1.



The derivative is given by

$$f'(t) = \begin{cases} 0, & t<-a, \\ \frac{a}{(t+a)^2}\exp\big[-2\big(\frac{a}{t+a}-1\big)\big], & -a\le t<0, \\ \frac{a}{(t-a)^2}\exp\big[-2\big(\frac{a}{-t+a}-1\big)\big], & 0\le t<a, \\ 0, & a\le t. \end{cases}$$

The continuity of $f'(t)$ is immediate by noting, in the process, that at $t=\pm a$ we have $(1/\epsilon)^2\exp(-1/\epsilon)\to 0$ for $\epsilon\to+0$, and is obviously continuous at $t=0$. It is easy to see that $f'(t)$ provides a continuous representation of a Dirac delta for $a\to 0$, which peaks at $t=0$ and vanishes for $t<-a$, $t\ge a$.

3.2. The left-hand side of the equation in question is given by

$$\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{-\gamma p+m}{p^2+m^2-\mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}p(x'-x)}\Big(-\frac{\gamma\overleftarrow{\partial}}{\mathrm{i}}+m\Big) = \delta^{(4)}(x'-x),$$

where we have used the identity $(-\gamma p+m)(\gamma p+m)=(p^2+m^2)$.

3.3. For a weak external electromagnetic potential we have seen in (3.2.14) that

$$S_+^A(x,x') \simeq S_+(x-x') + \mathrm{e}\int(\mathrm{d}x'')\, S_+(x-x'')\gamma^{\mu}A_\mu(x'')S_+^A(x'',x').$$

Using the expression for $S_+(x-x')$ in (3.1.10) for $x^0 > x'^0$, and the Fourier transform of the field $A_\mu(x)$ in (3.2.18), we obtain for the e-dependent part of $\langle\psi(x)\rangle_A$ in (3.2.13)

$$\mathrm{e}^2(\mathrm{i})^2\int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2p'^0}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2p^0}(-\gamma p'+m)\gamma^{\mu}(-\gamma p+m)A_\mu(p'-p)\mathrm{e}^{\mathrm{i}p'x}\mathrm{e}^{-\mathrm{i}px'},$$

for $x'^0 \ll x''^0 \ll x^0$. Using the projections in (I.21) for $(-\gamma p'+m)$, $(-\gamma p+m)$, as well as the contribution to $\langle\psi(x)\rangle_A$ coming from the first term on the right-hand side of the equation for $S_+^A(x,x')$ above, as given in (3.2.17), the result in (3.2.15)/(3.2.16) follows.

3.4. We explicitly have

$$\frac{\langle 0_+\mid\psi(x)\mid 0_-\rangle}{\langle 0_+\mid 0_-\rangle} = \mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2p^0}(\gamma p+m)\,\eta(-p)\,\mathrm{e}^{-\mathrm{i}px}.$$

Upon using the projection for $(\gamma p+m)$ in (I.22), the result follows.

3.5. The e-dependent part of $S_+^A(x',x)$, with $x^0 \gg x''^0 \gg x'^0$ is explicitly given by

$$\mathrm{e}(\mathrm{i})^2\int\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2p'^0}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2p^0}(\gamma p+m)\,\gamma^{\mu}\,(\gamma p'+m)\,\delta A_\mu(p'-p)\,\mathrm{e}^{-\mathrm{i}px'}\,\mathrm{e}^{\mathrm{i}p'x}$$

$$=\int\sum_{\sigma\sigma'}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2p'^0}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2p^0}\,[-\mathrm{i}\,v(\mathbf{p},\sigma)\,\mathrm{e}^{-\mathrm{i}px'}]\,[-\mathrm{i}\,\mathrm{e}\,\overline{v}(\mathbf{p},\sigma)\,\gamma^{\mu}\,v(\mathbf{p}',\sigma')]$$

$$\times\; v(\mathbf{p}',\sigma')\,\mathrm{e}^{\mathrm{i}p'x}\,\delta A_\mu(p'-p),$$

where we have used the projection in (I.22). Moreover by writing the e-independent part of $S_+^A(x',x)$, i.e., $S_+(x',x)$, as

$$S_+(x',x) = \int\sum_{\sigma,\sigma'}\frac{m\,\mathrm{d}^3\mathbf{p}'}{p'^0(2\pi)^3}\frac{m\,\mathrm{d}^3\mathbf{p}}{p^0(2\pi)^3}\,(2\pi)^3\,\frac{p^0}{m}\,\delta_{\sigma'\sigma}\,\delta^{(3)}(\mathbf{p}'-\mathbf{p})$$

$$\times\;[-\mathrm{i}\,v(\mathbf{p},\sigma)\,\mathrm{e}^{-\mathrm{i}px'}]\,\overline{v}(\mathbf{p}',\sigma')\,\mathrm{e}^{\mathrm{i}p'x},$$

the result stated for $\langle 0_+\mid\overline{\psi}(x)\mid 0_-\rangle_{\delta A}/\langle 0_+\mid 0_-\rangle_{\delta A}$ in (3.3.14) follows, by multiplying $S_+^A(x',x)$ by $\overline{\eta}(x')$, and integrating over x' as indicated in (3.3.13).

3.6. This is simply obtained by taking the absolute value squared of (3.3.41), and restrict the **p**-integration as indicated in the region specified by Δ, picking up the spin σ, and dividing by the normalization factor N.

3.7. By using the projection given in (I.21), we have

$$\frac{1}{2}\sum_{\sigma\sigma'}\big|\,\overline{u}(\mathbf{p}',\sigma')\,\gamma^{0}\,u(\mathbf{p},\sigma)\,\big|^{2} = \frac{1}{8m^{2}}\,\mathrm{Tr}\big[\,(-\gamma p' + m)\gamma^{0}(-\gamma p + m)\gamma^{0}\,\big]$$
$$= \frac{1}{8m^{2}}\{p'_{\mu}p_{\nu}\,\mathrm{Tr}\,[\,\gamma^{\mu}\gamma^{0}\gamma^{\nu}\gamma^{0}\,] + m^{2}\,\mathrm{Tr}\,[\,\gamma^{0}\gamma^{0}\,]\},$$

where we have used the fact that the trace of an odd number of gamma matrices is zero. Finally using the identities

$$\mathrm{Tr}\,[\,\gamma^{\mu}\gamma^{0}\gamma^{\nu}\gamma^{0}\,] = 4\big(\eta^{\mu 0}\eta^{\nu 0} - \eta^{\mu\nu}\eta^{00} + \eta^{\mu 0}\eta^{\nu 0}\big), \quad \mathrm{Tr}\,[\,\gamma^{0}\gamma^{0}\,] = 4,$$

and $pp' = |\mathbf{p}\,|^{2}\cos\theta - (p^{0})^{2}$, $|\mathbf{p}\,|^{2} = (p^{0})^{2} - m^{2}$, $(1+\cos\theta) = 2\cos^{2}\theta/2$, $(1-\cos\theta) = 2\sin^{2}(\theta/2)$, the result follows.

3.8. $\mathbb{P}_{+}(p)$, $\mathbb{P}_{-}(p)$, defined in (I.21), (I.22), satisfy, in particular, the following equations:

$$(\mathbb{P}_{\pm}(p)\,\gamma^{0})^{\dagger} = \mathbb{P}_{\pm}(p)\,\gamma^{0}, \qquad (2m)^{2}\mathbb{P}_{+}(p)\,\gamma^{0}\,\mathbb{P}_{+}(p) = 2p^{0}(-\gamma p + m),$$
$$(2m)^{2}\,\mathbb{P}_{-}(p)\,\gamma^{0}\,\mathbb{P}_{-}(p) = -2p^{0}(\gamma p + m), \quad \mathbb{P}_{\pm}(p^{0},\mathbf{p}))\,\gamma^{0}\,\mathbb{P}_{\mp}(p^{0},-\mathbf{p}) = 0.$$

Using the anti-commutation relation $\{\psi_{a}(y),\psi_{b}^{\dagger}(y\,')\} = \delta_{ab}\delta^{3}(\mathbf{y}-\mathbf{y}\,')$, for $y^{0} = y'^{\,0}\,(=0)$, in (3.5.11), we note that due to the last identity in the above set of equations that the cross term in $\{\psi_{a}(x),\overline{\psi}_{b}(x\,')\}$, for arbitrary x, $x\,'$, vanishes and we obtain from the remaining identities above

$$\{\psi_{a}(x),\overline{\psi}_{b}(x\,')\} = \Big(-\frac{\gamma\partial}{\mathrm{i}} + m\Big)_{ab}\int\frac{\mathrm{d}^{3}\mathbf{p}}{(2\pi)^{3}2p^{0}}\Big[\mathrm{e}^{\mathrm{i}p\,(x-x')} - \mathrm{e}^{-\mathrm{i}p\,(x-x')}\Big].$$

Upon changing the variable $\mathbf{p} \to -\mathbf{p}$ in the second term in the above integrand, the statement in the problem follows.

3.9. Note that $(-\gamma\partial/\mathrm{i} + m)_{ab}\,\exp[\,\mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}\,')]\,[\sin(p^{0} - p'^{\,0})]$ becomes simply

$-\gamma^{0}_{ab}\,p^{0}/\mathrm{i}$ at equal times, and the $-p^{0}/\mathrm{i}$ factor cancels the product of $-\mathrm{i}$ and $1/p^{0}$ in the integrand in $\Delta(x-x\,')$, giving $\gamma^{0}_{ab}\,\delta^{3}(\mathbf{x}-\mathbf{x}\,')$. The result then follows by multiplying by γ^{0}.

3.10. The ξ integral may be expressed in terms of the sine function, or equivalently

$$I = \frac{\varepsilon^\mu}{2}\int \frac{(\mathrm{d}Q)}{(2\pi)^4} A_\mu(Q)\ \mathrm{e}^{\mathrm{i}Qx}\int_{-1}^{+1} \mathrm{d}\lambda\ \mathrm{e}^{\mathrm{i}\,(\epsilon Q/2)\lambda}$$
$$= \epsilon^\mu \int \frac{(\mathrm{d}Q)}{(2\pi)^4} A_\mu(Q)\Big[1 - \frac{1}{3!}\Big(\frac{\epsilon\, Q}{2}\Big)^2 + \frac{1}{5!}\Big(\frac{\epsilon\, Q}{2}\Big)^4 - \cdots\Big]\mathrm{e}^{\mathrm{i}Qx},$$

and is an odd function in ϵ.

3.11. From (I.15) $\frac{\mathrm{i}}{4}[\gamma^i,\gamma^j]\varepsilon_{ijk} = \Sigma_k$. Moreover $[\gamma^0,\gamma^i] = 2\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0\end{pmatrix}$. Hence the expressions $F_{0i} = -F^{0i} = -E^i$, $F_{ij} = \varepsilon_{ijk}B^k$, lead from

$$\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} = 2\frac{\mathrm{i}}{4}[\gamma^0,\gamma^i]F_{0i} + \frac{\mathrm{i}}{4}[\gamma^i,\gamma^j]F_{ij},$$

to the stated result, consistent with (3.7.13) for $\mathbf{E}$, $\mathbf{B}$ along the third axis.

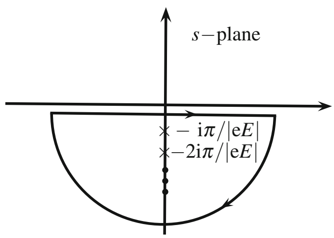
3.12. The formal substitution $s \to \mathrm{i}\,s$ amounts in replacing (3.8.5) by

$$\frac{2\,\mathrm{Im}W^{(\mathrm{e})}}{VT} = -\frac{1}{4\pi^2}\,\mathrm{Re}\,\mathrm{i}\int_0^\infty \frac{\mathrm{d}s}{s^3}\,\mathrm{e}^{-\mathrm{i}sm^2}\Big[\big(s|\mathrm{e}E|\coth s|\mathrm{e}E|\big) - \frac{(s\mathrm{e}E)^2}{3} - 1\Big].$$

The expression within the square brackets is real. Accordingly, we may make the replacement $\mathrm{Re}[\mathrm{i}\,\mathrm{e}^{-\mathrm{i}sm^2}] = \sin(sm^2)$, leading to an integrand which is an even function of s. Thus the above integral may be rewritten as

$$\frac{2\,\mathrm{Im}W^{(\mathrm{e})}}{VT} = -\frac{1}{8\pi^2}\int_{-\infty}^\infty \frac{\mathrm{d}s}{s^3}\,\sin(sm^2)\Big[\big(s|\mathrm{e}E|\coth s|\mathrm{e}E|\big) - \frac{(s\mathrm{e}E)^2}{3} - 1\Big],$$
$$= \frac{1}{8\pi^2}\mathrm{Im}\int_{-\infty}^\infty \frac{\mathrm{d}s}{s^3}\,\mathrm{e}^{-\mathrm{i}sm^2}\Big[\big(s|\mathrm{e}E|\coth s|\mathrm{e}E|\big) - \frac{(s\mathrm{e}E)^2}{3} - 1\Big].$$

It is precisely because of the $(s\mathrm{e}E)^2/3$, term within the square brackets that the point $s = 0$ is *not* a pole of the integrand. By closing the integral from below as shown in the c.w. direction in the complex s-plane, we enclose all the poles $-\mathrm{i}n\pi//|\mathrm{e}E|, n = 1,2,\ldots$, and note that $-\mathrm{i}\big(\mathrm{i}\,(\mathrm{Im}s)\big)\,m^2 = m^2(\mathrm{Im}s)$, thus $\mathrm{e}^{m^2(\mathrm{Im}s)} \to 0$ for $(\mathrm{Im}s) \to -\infty$.



For $s \simeq -\mathrm{i}n\pi/|\mathrm{e}E|$, with $n = 1, 2, \ldots$, the integrand approaches: $\mathrm{e}^{-\mathrm{i}sm^2}/\left[s^2\left(s + \frac{\mathrm{i}n\pi}{|\mathrm{e}E|}\right)\right]$, and the residue theorem gives

$$\frac{2\,\mathrm{Im}W^{(\mathrm{e})}}{VT} = \frac{1}{8\pi^2}\mathrm{Im}\,(-2\pi\mathrm{i})\sum_{n=1}^{\infty}\frac{|\mathrm{e}E|^2}{-n^2\pi^2}\mathrm{e}^{-n\,\pi\,m^2/|\mathrm{e}E|} = \frac{\alpha\,E^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-n\,\pi\,m^2/|\mathrm{e}E|}}{n^2},$$

coinciding with the one in (3.8.8).

3.13. The identity follows by considering the chain of equalities

$$\begin{aligned}
&\frac{1}{(\gamma(p+\delta p)+m)} - \frac{1}{(\gamma p + m)} \\
&= \frac{1}{(\gamma(p+\delta p)+m)}\Big[\big(\gamma p + m\big) - \big(\gamma(p+\delta p)+m\big)\Big]\frac{1}{(\gamma p + m)} \\
&= \frac{1}{(\gamma(p+\delta p)+m)}(-\gamma\delta p)\frac{1}{(\gamma p + m)},
\end{aligned}$$

and the identity follows upon taking the limit $\delta p_\nu \to 0$.

3.14. Due to the matrix nature of $U(x)$, $\partial_\mu U(x)$ and $U(x)$ do not commute. From the gauge transformation in (3.10.3) $A_\nu \to A'_\nu = U A_\nu U^{-1} + (\mathrm{i}/g)U\partial_\nu U^{-1}$, hence using the fact that $\partial_\mu(U^{-1}U) = 0$ i.e., $U^{-1}(\partial_\mu U) = -(\partial_\mu U^{-1})U$, then following expression emerges for $\partial_\mu\big(UA_\nu U^{-1} + (\mathrm{i}/g)\,U\partial_\nu U^{-1}\big) =$

$$U(\partial_\mu A_\nu)U^{-1} - \Big[U\partial_\mu U^{-1}, UA_\nu U^{-1}\Big] - \frac{\mathrm{i}}{g}\,(U\partial_\mu U^{-1})\,(U\partial_\nu U^{-1}) + \frac{\mathrm{i}}{g}\,U\partial_\mu\partial_\nu U^{-1}.$$

Accordingly,

$$\begin{aligned}
\partial_\mu A'_\nu - \partial_\nu A'_\mu &= U(\partial_\mu A_\nu - \partial_\nu A_\mu)U^{-1} - \Big[U\partial_\mu U^{-1}, UA_\nu U^{-1}\Big] \\
&\quad + \Big[U\partial_\nu U^{-1}, UA_\mu U^{-1}\Big] - \frac{\mathrm{i}}{g}\Big[U\partial_\mu U^{-1}, U\partial_\nu U^{-1}\Big],
\end{aligned}$$

$$\begin{aligned}
-\mathrm{i}\,g\,[A'_\mu, A'_\nu] &= -\mathrm{i}\,g\,U[A_\mu, A_\nu]U^{-1} - \Big[U\partial_\nu U^{-1}, UA_\mu U^{-1}\Big] \\
&\quad + \Big[U\partial_\mu U^{-1}, UA_\nu U^{-1}\Big] + \frac{\mathrm{i}}{g}\Big[U\partial_\mu U^{-1}, U\partial_\nu U^{-1}\Big].
\end{aligned}$$

The last two equations give

$$\partial_\mu A'_\nu - \partial_\nu A'_\mu - \mathrm{i}g\,[A'_\mu, A'_\nu] = U\Big(\partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\,[A_\mu, A_\nu]\Big)U^{-1},$$

which provides the gauge transformation of $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\,[A_\mu, A_\nu\,]$.

3.15. $\big(\gamma^\mu(\partial/\mathrm{i}\partial z^\mu)+m\big)S_+(z-y) = \delta^{(4)}(z-y)$. Upon multiplying the latter, from the right, by $S_+^{-1}(y-x)$, and integrating with respect to y, we obtain : $S_+^{-1}(z-x) = \big(\gamma^\mu(\partial/\mathrm{i}\partial z^\mu) + m\big)\delta^{(4)}(z-x) = \delta^{(4)}(z-x)\big(-\gamma^\mu \overleftarrow{\partial}/\mathrm{i}\,\overleftarrow{\partial} x^\mu + m\big)$. Hence

$$\exp[\,\mathrm{i}F(z)\,]\big[S_+^{-1}(z-x) + \delta^{(4)}(z-x)\,\gamma^\mu\frac{\partial}{\partial x^\mu}F(x)\big]$$

$$= \exp[\,\mathrm{i}F(x)\,]\,\delta^{(4)}(z-x)\Big(\gamma^\mu\Big[-\frac{\overleftarrow{\partial}_\mu}{\mathrm{i}} + \overrightarrow{\partial}_\mu F(x)\Big]+m\Big)$$

$$= \delta^{(4)}(z-x)\Big[-\frac{\gamma^\mu\overleftarrow{\partial}_\mu}{\mathrm{i}} + m\Big]\exp[\,\mathrm{i}F(x)\,] = S_+^{-1}(z-x)\,\exp[\,\mathrm{i}F(x)\,],$$

which is the statement in the problem.

Chapter 4

4.1. Using the facts that $\langle\,\mathbf{0}\sigma\,|\,\mathbf{P} = \mathbf{0}$, and $\langle\,\mathbf{0}\sigma\,|\,P^0 = m\,\langle\,\mathbf{0}\sigma\,|$, we may infer from (4.2.23)–(4.2.31) that

$$\langle\,\mathbf{0}\sigma\,|\,\exp[-\mathrm{i}\alpha J_{03}\,]\exp[-\mathrm{i}\theta\mathbf{n}\cdot\mathbf{J}\,]P^i = R^{ij}\langle\,\mathbf{0}\sigma\,|\,Y^j\exp[-\mathrm{i}\alpha J_{03}\,]\exp[-\mathrm{i}\theta\mathbf{n}\cdot\mathbf{J}\,]$$

$$= R^{ij}\,m\,\sinh\alpha\,\delta_{j3}\langle\,\mathbf{0}\sigma\,|\,\exp[-\mathrm{i}\alpha J_{03}\,]\exp[-\mathrm{i}\theta\mathbf{n}\cdot\mathbf{J}\,].$$

With $\mathbf{n}$ given in (4.2.24), we may refer to (2.2.11) to infer that $R^{33} = \cos\theta$, $R^{23} = \sin\phi\sin\theta$, $R^{13} = \cos\phi\sin\theta$, and use the fact that $\sinh\alpha = |\mathbf{p}\,|/m$, to conclude that the state in (4.2.23) becomes multiplied by $\mathbf{p}$ by the action of $\mathbf{P}$, where $\mathbf{p}$ is given in (4.2.33). On the other hand, $Y^0 = P^0\cosh\alpha + P^3\sinh\alpha$, as given in (4.2.30), and $m\cosh\alpha = \sqrt{|\mathbf{p}\,|^2 + m^2}$, which verifies (4.2.32) for the application of P^0 as well.

4.2. From (4.2.14) we note that W^0 commutes with $\mathbf{J}$. On the other hand, consider the expression: $K(\alpha) = \exp[-\mathrm{i}\alpha\,J_{03}\,]W^0\exp[\,\mathrm{i}\alpha\,J_{03}\,]$, with B.C. $K(0) = W^0$. We explicitly have $K'(0) = W^3$, $K''(\alpha) = K(\alpha)$, with the latter two equalities following from (4.2.14), from which we obtain $K(\alpha) = W^3\sinh\alpha + W^0\cosh\alpha$. This leads to the statement of the problem by finally noting that $\langle\,0\sigma|W^0 = 0$, $\langle\,0\sigma|W^3 = m\sigma\langle\,0,\sigma|$.

4.3. If $W^\mu = 0$, then we may write $W^\mu = 0\times P^\mu$, and there is nothing to prove. Accordingly suppose that $W^\mu \neq 0$, that is, $|W^0| = |\mathbf{W}| > 0$. From the orthogonality of the two vectors: $P^0\,W^0 = |\mathbf{P}|\,|\mathbf{W}|\,\cos\theta$ or $|W^0||P^0| = |\mathbf{P}|\,|\mathbf{W}|\,|\cos\theta|$ from which $|\cos\theta| = 1$. That is, $\mathbf{W} = \lambda\,\mathbf{P}$. On the other hand, the relation $W^0P^0 = \mathbf{W}\cdot\mathbf{P} = \lambda|\mathbf{P}|^2$ also implies that we also have $W^0 = \lambda P^0$, since $P^0 = |\mathbf{P}| > 0$, which completes the proof.

4.4. $W^0 = (1/2)\,\varepsilon^{ijk}P_iJ_{jk}$. But $(1/2)\,\varepsilon^{ijk}J_{jk} = J^i$, hence $W^0 = \mathbf{P}.\mathbf{J}$. On the other hand, $W^i = (1/2)\varepsilon^{i\nu\sigma\lambda}P_\nu J_{\sigma\lambda}$. The latter may be rewritten as $P^0J^i + \varepsilon^{0ijk}P_jJ_{0k}$. But $J_{0k} = -N_k$, from which $\mathbf{W} = P^0\mathbf{J} - \mathbf{P}\times\mathbf{N}$.

4.5. Upon writing $A_\mu(x) = t_cA_{c\mu}(x)$, and using the commutation rule $[t_a, t_b] = \mathrm{i}f_{abc}t_c$, as given in (3.10.4), the result follows.

4.6. We note that $\pi^{\alpha\mu} = \partial\mathscr{L}/\partial(\partial_\mu A_\alpha(x)) = F^{\alpha\mu}$. Hence from (4.4.6)

$$\Omega^{\mu\nu\lambda} = \frac{\mathrm{i}}{2}\Big(F^{\alpha\mu}(S^{\nu\lambda})_\alpha{}^\beta A_\beta + F^{\alpha\lambda}(S^{\mu\nu})_\alpha{}^\beta A_\beta + F^{\alpha\nu}(S^{\mu\lambda})_\alpha{}^\beta A_\beta\Big) = F^{\mu\lambda}A^\nu.$$

Maxwell's equations: $\partial_\lambda F^{\lambda\mu} = 0$, give $\partial_\lambda\Omega^{\mu\nu\lambda} = F^{\mu\lambda}\,\partial_\lambda A^\nu$. Accordingly, from (4.4.16)

$$\begin{aligned} T^{\mu\nu} &= (-(1/4)\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\alpha\mu}\partial^\nu A_\alpha) - F^{\mu\lambda}\partial_\lambda A^\nu \\ &= -(1/4)\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\alpha}F^\nu{}_\alpha, \end{aligned}$$

which is obviously symmetric and gauge invariant, i.e., under the transformation $A_\alpha \to A_\alpha + \partial_\alpha\lambda$, for the latter. Using the elementary identity $\partial^\nu F_{\alpha\beta} = \partial_\alpha F^\nu{}_\beta - \partial_\beta F^\nu{}_\alpha$, and the asymmetry of $F^{\alpha\beta}$ the conservation law follows $\partial_\mu T^{\mu\nu} = -\frac{1}{2}F^{\alpha\beta}\partial^\nu F_{\alpha\beta} + F^{\mu\alpha}\partial_\mu F^\nu{}_\alpha = 0$. $T^{\mu\nu}$ is traceless.

4.7. $\big(\partial\mathscr{L}/\partial(\partial_\mu\psi)\big)_\mathrm{r} = \mathrm{i}\,\overline{\psi}\,\gamma^\mu$. Recall from (I.7) that $S^{\nu\lambda} = (\mathrm{i}/4)[\gamma^\nu, \gamma^\lambda]$. Hence from the definition of $\Omega^{\mu\nu\lambda}$ in (4.4.6), taking into account the adjoint field contribution,

$$\begin{aligned} \frac{1}{2}\Big(\Omega^{\mu\nu\lambda} + (\Omega^{\mu\nu\lambda})^\dagger\Big) &= -\,\frac{1}{4}\,\overline{\psi}\Big(\{\gamma^\mu, S^{\nu\lambda}\} + \{\gamma^\lambda, S^{\mu\nu}\} + \{\gamma^\nu, S^{\mu\lambda}\}\Big)\psi \\ &= -\,\frac{\mathrm{i}}{8}\,\overline{\psi}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\lambda\gamma^\nu\gamma^\mu\Big)\psi. \end{aligned}$$

Upon using the Dirac equations: $\partial_\lambda\overline{\psi}\,\gamma^\lambda = \mathrm{i}\,m\,\overline{\psi}$, $\gamma^\lambda\partial_\lambda\psi = -\mathrm{i}\,m\,\psi$, we obtain

$$\frac{1}{2}\,\partial_\lambda\Big(\Omega^{\mu\nu\lambda} + (\Omega^{\mu\nu\lambda})^\dagger\Big) = -\frac{\mathrm{i}}{4}\,\overline{\psi}\Big(\gamma^\mu\overleftrightarrow{\partial^\nu} - \gamma^\nu\overleftrightarrow{\partial^\mu}\Big)\psi,$$

$\mathscr{L} = 0$, and from (4.4.16)

$$\begin{aligned} T^{\mu\nu} &= -\,\frac{1}{2}\Big(\mathrm{i}\,\overline{\psi}\,\gamma^\mu\partial^\nu\psi + \big(\mathrm{i}\,\overline{\psi}\,\gamma^\mu\partial^\nu\psi\big)^\dagger\Big) + \frac{\mathrm{i}}{4}\,\overline{\psi}\Big(\gamma^\mu\overleftrightarrow{\partial^\nu} - \gamma^\nu\overleftrightarrow{\partial^\mu}\Big)\psi \\ &= \frac{1}{4\,\mathrm{i}}\,\overline{\psi}\Big(\gamma^\mu\overleftrightarrow{\partial}^\nu + \gamma^\nu\overleftrightarrow{\partial^\mu}\Big)\psi, \end{aligned}$$

which is obviously symmetric.

4.8. The general expression for the energy-momentum tensor, given in the previous problem is $T^{\mu\nu} = (1/4\,\mathrm{i})\,\overline{\psi}\Big(\gamma^{\mu}\overleftrightarrow{\partial}^{\nu} + \gamma^{\nu}\overleftrightarrow{\partial}^{\mu}\Big)\psi$. In particular,

$$T^{00} = \frac{1}{2\,\mathrm{i}}\,\psi^{\dagger}\overleftrightarrow{\partial}^{0}\psi, \qquad T^{0k} = \frac{1}{4\,\mathrm{i}}\,\overline{\psi}\big(\gamma^{0}\overleftrightarrow{\partial}^{k} + \gamma^{k}\overleftrightarrow{\partial}^{0}\big)\psi.$$

The Dirac equations give $\partial^{0}\psi = \gamma^{0}(\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + \mathrm{i}\,m)\psi$, $\partial^{0}\overline{\psi} = \overline{\psi}\big(\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - \mathrm{i}\,m\big)\gamma^{0}$. The right-hand sides of these two equations together with the identity $\gamma^{k}\gamma^{j} = (1/2)[\gamma^{k},\gamma^{j}] - \eta^{kj}$, lead to the following expressions

$$T^{00} = \overline{\psi}\Big(m + \frac{1}{2\,\mathrm{i}}\,\boldsymbol{\gamma}\cdot\overleftrightarrow{\boldsymbol{\nabla}}\Big)\psi, \quad T^{0k} = \frac{1}{8\,\mathrm{i}}\,\partial_{j}\Big(\psi^{\dagger}\,[\gamma^{j},\gamma^{k}]\,\psi\Big) + \frac{1}{2\,\mathrm{i}}\,\psi^{\dagger}\overleftrightarrow{\partial}^{k}\psi.$$

4.9. This commutation relation follows by the application of the general equal commutation rule of the commutator of bilinear forms:

$$\Big[\,[\psi_{a}^{\dagger}(x),\psi_{b}(x)],[\psi_{c}^{\dagger}(x'),\psi_{d}(x')]\,\Big]$$
$$= 2\,\delta_{bc}\,\delta^{3}(\mathbf{x}-\mathbf{x}')\,[\psi_{a}^{\dagger}(x),\psi_{d}(x')] - 2\,\delta_{ad}\,\delta^{3}(\mathbf{x}-\mathbf{x}')\,[\psi_{c}^{\dagger}(x'),\psi_{b}(x)]. \quad (*)$$

Note that the equal-time anti-commutation relations in (4.3.41) imply that since $\pi_{b}(x) = \mathrm{i}\,\overline{\psi}_{b}\gamma^{0}$, $\{\psi_{a}(x),\psi_{b}^{\dagger}(x')\} = \delta^{3}(\mathbf{x}-\mathbf{x}')$, $\{\psi_{a}(x),\psi_{b}(x')\} = 0$. To prove the identity (*), note that the following ones, in turn

$$\psi_{a}^{\dagger}(x)\psi_{b}(x)\psi_{c}^{\dagger}(x')\psi_{d}(x') = \psi_{c}^{\dagger}(x')\psi_{d}(x')\psi_{a}^{\dagger}(x)\psi_{b}(x)$$
$$- \delta_{ad}\,\delta^{3}(\mathbf{x}-\mathbf{x}')\,\psi_{c}^{\dagger}(x')\psi_{b}(x) + \delta_{bc}\,\delta^{3}(\mathbf{x}-\mathbf{x}')\,\psi_{a}^{\dagger}(x)\psi_{d}(x'),$$
$$\psi_{a}^{\dagger}(x)\psi_{d}(x')\psi_{b}^{\dagger}(x)\psi_{c}(x') = -\psi_{d}(x')\psi_{c}^{\dagger}(x')\psi_{a}^{\dagger}(x)\psi_{b}(x)$$
$$- \delta_{bc}\,\delta^{3}(\mathbf{x}-\mathbf{x}')\,\psi_{d}(x')\psi_{a}^{\dagger}(x) + \delta_{ad}\,\delta^{3}(\mathbf{x}-\mathbf{x}')\,\psi_{b}(x)\psi_{c}^{\dagger}(x'),$$

are sufficient to establish the identity in question, by a mere relabeling of the spinor indices, and by the fact that

$$\partial_{k}\delta^{3}(\mathbf{x}-\mathbf{x}')\Big[\psi_{c}^{\dagger}(x')\psi_{b}(x) + \psi_{c}^{\dagger}(x)\psi_{b}(x')\Big]$$
$$= \partial_{k}\delta^{3}(\mathbf{x}-\mathbf{x}')\Big[\psi_{c}^{\dagger}(x')\psi_{b}(x') + \psi_{c}^{\dagger}(x)\psi_{b}(x)\Big].$$

4.10. The right-hand side of (4.7.129) is given by:

$$-(p^{0}p^{i})/m^{2} + \mathrm{e}_{0}^{0*}\mathrm{e}_{0}^{i} = 0, \quad \text{for} \quad \mu = 0, \nu = i,$$

$$-p^{0}p^{0}/m^{2} + \mathrm{e}_{0}^{0*}\mathrm{e}_{0}^{0} = -(\mathbf{p}^{2} + m^{2})/m^{2} + \mathbf{p}^{2}/m^{2} = -1, \quad \text{for} \quad \mu = \nu = 0,$$

$$-p^{i}p^{j}/m^{2} + \mathrm{e}_{+}^{i*}\mathrm{e}_{+}^{j} + \mathrm{e}_{-}^{i*}\mathrm{e}_{-}^{j} + (p^{0})^{2}p^{i}p^{j}/(m^{2}|\mathbf{p}|^{2}) = \delta^{ij}, \text{ for } \mu = i, \nu = j,$$

using, in the process, the 3D completeness relation for the last relation.

4.11. By multiplying the given equation, in turn, by γ_μ and ∂_μ and by considering the two resulting equations *simultaneously*, we are led to the following equivalent two equations $(\gamma\partial \equiv \gamma^{\,\mu}\partial_\mu,\ \partial K = \partial_\mu K^\mu,\ \gamma K = \gamma_\mu K^\mu)$

$$m\,\partial\psi = (\gamma\partial\,\gamma K) + \partial K, \quad \gamma\psi = -2\,(\gamma\partial\,\gamma K \,+\, \partial K)/(\mathrm{i}m^2) - 3\,\gamma K/m,$$

which when are substituted back in the initial equation in the problem give

$$\Big(\frac{\gamma\partial}{\mathrm{i}} + m\Big)\psi^\mu \;= \Big[\eta^{\mu\nu} + \frac{1}{3}\Big(\gamma^\mu\gamma^\nu + \gamma^\mu\frac{\partial^\nu}{\mathrm{i}\,m} - \gamma^\nu\frac{\partial^\mu}{\mathrm{i}\,m} + \frac{2}{m^2}\,\frac{\partial^\mu}{\mathrm{i}}\,\frac{\partial^\nu}{\mathrm{i}}\Big)\Big]K_\nu$$
$$-\frac{2}{3\,m^2}\Big(\frac{\gamma\partial}{\mathrm{i}} + m\Big)\Big[\frac{\partial^\mu}{\mathrm{i}}\gamma^\nu - \frac{\partial^\nu}{\mathrm{i}}\gamma^\mu + \Big(\frac{\gamma\partial}{\mathrm{i}} - m\Big)\gamma^\mu\gamma^\nu\Big]K_\nu.$$

Upon multiplying this equation by $\big(-\gamma\partial/\mathrm{i}+m\big)$, the statement in the problem then follows from an analysis similar to the one leading to (4.7.143), and by making note that the first term in the above equation multiplied by $\big(-\gamma\partial/\mathrm{i}+m\big)$ is nothing but $\rho^{\mu\nu}K_\nu$, where $\rho^{\mu\nu}$ is given in (4.7.145) in the momentum description, while the coefficient of $(-\Box+m^2)$ of the second term is the non-propagating, non-singular term, mentioned in the statement of the problem. The equations in (4.7.137), (4.7.138) are also satisfied for $K^\mu = 0$.

4.12. Consider the equation for $\psi^{\,0}$ in (4.7.137) $(\gamma\partial/\mathrm{i} + m)\psi^{\,0} = 0$ for $K^\mu = 0$. It may be rewritten as

$$0 = \frac{\gamma^{\,0}\partial_0\psi^{\,0}}{\mathrm{i}} + \Big(\frac{\boldsymbol{\gamma}\cdot\nabla}{\mathrm{i}} + m\Big)\psi^{\,0} = -\gamma^{\,0}\frac{\partial_i\psi^{\,i}}{\mathrm{i}} + \Big(\frac{\boldsymbol{\gamma}\cdot\nabla}{\mathrm{i}} + m\Big)\gamma^{\,0}\gamma^{\,i}\psi^{\,i},$$

where we have used, in turn, the constraint in (4.7.137) and (4.7.138) to eliminate $\psi^{\,0}$. The constraint equation in (4.7.148) follows upon multiplying the above equation by $-\gamma^{\,0}$.

4.13. The following two equalities are explicitly verified

$$\gamma^{\,i}\beta^{ij} \;= \frac{\partial^j}{\mathrm{i}\,m} + \frac{\boldsymbol{\gamma}\cdot\nabla}{3\,\mathrm{i}\,m}\Big(\gamma^{\,j} + 2\frac{\partial^j}{\mathrm{i}\,m}\Big),$$
$$\frac{\partial^{\,i}}{\mathrm{i}}\beta^{ij} \;= \frac{\partial^j}{\mathrm{i}} + \frac{1}{3}\frac{\boldsymbol{\gamma}\cdot\nabla}{\mathrm{i}}\Big(\gamma^j + 2\frac{\partial^j}{\mathrm{i}\,m}\Big) - \frac{\boldsymbol{\gamma}\cdot\nabla}{\mathrm{i}}\frac{\partial^j}{\mathrm{i}\,m} - \frac{\nabla^2}{3\,m}\Big(\gamma^{\,j} + 2\frac{\partial^j}{\mathrm{i}\,m}\Big).$$

The equation in question follows directly from combining these two identities.

4.14. By working in the chiral representation of the gamma matrices, given in (2.3.3), $u(\pm 1)$ may be taken (see (I.25)) and conveniently normalized, as

$$u(+1) = \sqrt{|\mathbf{p}|}\begin{pmatrix}\xi_+\\ 0\end{pmatrix}, \quad u(-1) = \sqrt{|\mathbf{p}|}\begin{pmatrix}0\\ \xi_-\end{pmatrix}, \quad \sum_\sigma u(\sigma)\overline{u}(\sigma) = \gamma p,$$

where $\xi_{\pm}$ are defined in (I.13). We may then write

$$P_+{}^{ij} = -\frac{1}{2}\sum_{\lambda,\lambda'} \mathrm{e}^{\,i}_{\lambda}\,\mathrm{e}^{k*}_{\lambda}\gamma^{\,\ell}\Big(u(+)\overline{u}(+) + u(-)\overline{u}(-)\Big)\,\gamma^{\,k}\mathrm{e}^{\,\ell}_{\lambda'}\mathrm{e}^{j*}_{\lambda'}. \qquad (*)$$

By considering, for example, the vector $\mathbf{p}$ to be along the 3-axis, and $\mathbf{e}_+ = (1, -\mathrm{i}, 0)/\sqrt{2}$, $\mathbf{e}_- = (-1, -\mathrm{i}, 0)/\sqrt{2}$, the following are easily established

$$\sum_{\lambda'} \mathrm{e}^{j*}_{\lambda'}\,\mathbf{e}_{\lambda'}\cdot\boldsymbol{\gamma} \;=\; \delta^{j\,1}\,\gamma^1 + \delta^{j\,2}\,\gamma^{\,2},$$

$$\sum_{\lambda'} \mathrm{e}^{j*}_{\lambda'}\,\mathbf{e}_{\lambda'}\cdot\boldsymbol{\gamma}\, u(+) \;=\; \sqrt{2}\,\mathrm{e}^{\,j}_{-}\, u(-),$$

$$\overline{u}(+)\sum_{\lambda} \mathrm{e}^{\,i}_{\lambda}\,\mathbf{e}^{*}_{\lambda}\cdot\boldsymbol{\gamma} \;=\; -\sqrt{2}\,\mathrm{e}^{\,i}_{-}\overline{u}(-),$$

and with an almost identical analysis carried out for the $u(-)\overline{u}(-)$ in (*) completes the verification of (4.7.193).

4.15. In the presence of the external source, (4.7.103), for $\nu = 0$ becomes replaced by $-\partial_k F^{k\,0} + m^2 V^0 = K^{\,0}$, and with $F^{k\,0} = \pi^{\,k}$, denoting the canonical conjugate momenta of the components V^k, the *dependent* field $V^{\,0}$ satisfies the equation

$$V^0 = \frac{1}{m^2}\big(K^0 + \partial_k \pi^{\,k}\big).$$

By keeping the canonical conjugate momentum $\pi^{\,k}$ and its *space* derivative fixed, we have

$$\frac{\delta}{\delta K_\mu(x)} V^{\,0}(x\,') = \frac{1}{m^2}\,\eta^{\,0\,\mu}\,\delta^{(4)}(x\,' - x), \quad \text{or equivalently}$$

$$\frac{\delta}{\delta K_\mu(x)} V^{\,\nu}(x\,') = \frac{1}{m^2}\,\delta^{\,\nu}{}_{0}\,\eta^{\,0\,\mu}\,\delta^{(4)}(x\,' - x).$$

On the other hand, $(-\mathrm{i})\delta/\delta K_\nu(x')\langle\, 0_+ \,|\, 0_-\rangle = \langle\, 0_+ \,|\, V^\nu(x') \,|\, 0_-\rangle$, and hence from (4.6.38),

$$\begin{aligned}
&(-\mathrm{i})\frac{\delta}{\delta K_\mu(x)}(-\mathrm{i})\frac{\delta}{\delta K_\nu(x\,')}\langle\, 0_+ \,|\, 0_-\rangle = \langle\, 0_+ \,|\, \big(V^\mu(x) V^\nu(x\,')\big)_+ 0_-\rangle \\
&- \mathrm{i}\,\langle\, 0_+ \,|\, \frac{\delta}{\delta K_\mu(x)} V^{\,\nu}(x\,') \,|\, 0_-\rangle \\
&= \langle\, 0_+ \,|\, \big(V^\mu(x) V^\nu(x\,')\big)_+ \,|0_-\rangle - \frac{\mathrm{i}}{m^2}\,\delta^{\,\nu}{}_{0}\,\eta^{\,0\,\mu}\,\delta^{(4)}(x\,' - x)\langle 0_+ \,|\, 0_-\rangle,
\end{aligned}$$

from which the statement in the problem follows upon multiplying by i, and is a consequence of the presence of a dependent field.

4.16. Although $V_0(x)$ is a dependent field we note that

$$\lambda\,(-\mathrm{i})\,\frac{\delta}{\delta K^\mu(x)}\,(\mathrm{i})\,\frac{\delta}{\delta\eta(x)}\,\gamma^\mu\,(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\,\langle\, 0_+\,|\, 0_-\rangle$$

$$=\lambda\,(-\mathrm{i})\frac{\delta}{\delta K^\mu(x)}\,\langle\, 0_+\,|\,\big(\overline{\psi}(x)\gamma^\mu\psi(x)\big)_+\,|\, 0_-\rangle$$

$$=\lambda\,\langle\, 0_+\,|\,\big(\overline{\psi}(x)\gamma^\mu\psi(x)V_\mu(x)\big)_+\,|\, 0_-\rangle = \langle\, 0_+\,|\,\mathscr{L}_I(x)\,|\, 0_=\rangle$$

where the functional derivative $\delta/\delta K^0(x)$ does not generate an additional term as in (4.6.38) because $\overline{\psi}(x)\gamma^\mu\psi(x)$ in the first equality consists only of independent fields.

Chapter 5

5.1. The solution immediately follows by writing down the rotation matrix $[R^{ik}]$ explicitly and the polarization vectors as 3×1 (column) matrices. The expression for $[R^{ik}]$ follows from (5.2.6) to be

$$[R^{ik}] = \begin{pmatrix} \cos^2\phi\,\cos\theta+\sin^2\phi & \sin\phi\,\cos\phi\,(\cos\theta-1) & \cos\phi\,\sin\theta \\ \sin\phi\,\cos\phi\,(\cos\theta-1) & \sin^2\phi\,\cos\theta+\cos^2\phi & \sin\phi\,\sin\theta \\ -\cos\phi\,\sin\theta & -\sin\phi\,\sin\theta & \cos\theta \end{pmatrix}.$$

5.2. $\partial_\mu\int(\mathrm{d}x')D^{\mu\nu}(x-x')\,J_\nu(x') = \lambda\int(\mathrm{d}x')D_+(x-x')\partial'_\nu J^\nu(x') = \lambda\chi(x)$.

5.3. Going through these various steps, we have: $E_n(K,R)\exp[-\mathrm{i}E_n(K,R)T]$

$$=\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}T}\exp[-\mathrm{i}\,E_n(K,R)T] = -\frac{\mathrm{d}^2}{\mathrm{d}T^2}\Big(\frac{\exp[-\mathrm{i}\,E_n(K,R)T]}{E_n(K,R)}\Big),$$

$$\frac{\exp[-\mathrm{i}\,E_n(K,R)T]}{E_n(K,R)} = \frac{2\,\mathrm{i}}{T}\,\frac{\partial}{\partial\mathbf{K}^2}\exp\Big[-\mathrm{i}\sqrt{\mathbf{K}^2+\frac{n^2\pi^2}{R^2}}\;T\Big].$$

Therefore

$$E_n(K,R)\exp[-\mathrm{i}\,E_n(K,R)T] = -2\,\mathrm{i}\,\frac{\mathrm{d}^2}{\mathrm{d}T^2}\Big\{\frac{1}{T}\,\frac{\partial}{\partial\mathbf{K}^2}\exp\Big[-\mathrm{i}\sqrt{\mathbf{K}^2+\frac{n^2\pi^2}{R^2}}\;T\Big]\Big\}.$$

An elementary application of $\partial/\partial a$ to the above equation gives the stated result, where recall that R is expressed in terms of a.

5.4. Clearly

$$f_{ka}(\mathbf{n}_1,\mathbf{n}_2) = \frac{1}{C}\Big|\,\mathrm{e}^{-\mathrm{i}\,|\mathbf{k}|\,\mathbf{n}_1\cdot\mathbf{R}_1}\,\mathrm{e}^{-\mathrm{i}\,|\mathbf{k}|\,\mathbf{n}_2\cdot\mathbf{R}_2}+\mathrm{e}^{-\mathrm{i}\,|\mathbf{k}|\,\mathbf{n}_1\cdot\mathbf{R}_2}\,\mathrm{e}^{-\mathrm{i}\,|\mathbf{k}|\,\mathbf{n}_2\cdot\mathbf{R}_1}\Big|^2.$$

independently of $f(|\mathbf{k}|)$ where

$$C = \int \mathrm{d}\Omega_1 \int \mathrm{d}\Omega_2 \Big| \mathrm{e}^{-\mathrm{i}|\mathbf{k}|\mathbf{n}_1\cdot\mathbf{R}_1}\, \mathrm{e}^{-\mathrm{i}|\mathbf{k}|\mathbf{n}_2\cdot\mathbf{R}_2} + \mathrm{e}^{-\mathrm{i}|\mathbf{k}|\mathbf{n}_1\cdot\mathbf{R}_2}\, \mathrm{e}^{-\mathrm{i}|\mathbf{k}|\mathbf{n}_2\cdot\mathbf{R}_1} \Big|^2.$$

Upon using the elementary integrals,

$$\int_{-1}^{1} \mathrm{d}x\, x\, \mathrm{e}^{-\mathrm{i}kax} = 2\,\mathrm{i}\Big[\frac{\cos ka}{ka} - \frac{\sin ka}{k^2a^2}\Big], \qquad C = 32\pi\Big[1 + \frac{\sin^2 ka}{k^2a^2}\Big],$$

we obtain the following expression

$$\langle c \rangle = \Big(\frac{\cos ka}{ka} - \frac{\sin ka}{k^2a^2}\Big)^2 \Big/ \Big(1 + \frac{\sin^2 ka}{k^2a^2}\Big).$$

5.5. For a matrix $S = 1/[A - \mathrm{e}_0 B]$, we have

$$\frac{\partial}{\partial \mathrm{e}_0}\frac{1}{A - \mathrm{e}_0 B} = \frac{1}{\delta\mathrm{e}_0}\Big[\frac{1}{A - (\mathrm{e}_0 + \delta\mathrm{e}_0)B} - \frac{1}{A - \mathrm{e}_0 B}\Big], \quad \delta\mathrm{e}_0 \to 0$$

$$= \frac{1}{A - (\mathrm{e}_0 + \delta\mathrm{e}_0)B}\Big[\frac{\big(A - \mathrm{e}_0 B\big) - \big(A - (\mathrm{e}_0 + \delta\mathrm{e}_0)B\big)}{\delta\mathrm{e}_0}\Big]\frac{1}{A - \mathrm{e}_0 B}$$

$$\to \frac{1}{A - \mathrm{e}_0 B}\, B\, \frac{1}{A - \mathrm{e}_0 B},$$

and the result follows by matrix multiplication with matrix elements indices specified by spacetime variables.

5.6. From (5.7.3) and (5.7.4), we have

$$\partial_\mu \langle\, 0_+ | j^\mu \mid 0_- \rangle = \mathrm{i}\,\mathrm{e}_0 \big[\langle\, 0_+ | \overline{\psi} \mid 0_- \rangle\, \eta - \overline{\eta}\, \langle\, 0_+ | \psi \mid 0_- \rangle \big],$$

upon taking matrix element between vacuum states. The first result then follows from the explicit equations for $\langle\, 0_+ | \psi(x) \mid 0_- \rangle$ in (5.7.13), $\langle\, 0_+ | \overline{\psi}(x) \mid 0_- \rangle$ in (5.7.14), and (5.7.7). Now write $\langle\, 0_+ \mid 0_- \rangle = F[\delta/\delta J]\langle\, 0_+ \mid 0_- \rangle_{0\gamma}$ in (5.7.27). Then

$$\partial_\mu \langle\, 0_+ | A^\mu(x) \mid 0_- \rangle = \lambda\, F[\delta/\delta J] \int (\mathrm{d}x')D_+(x - x')\partial'^\nu J_\nu(x')\langle\, 0_+ \mid 0_- \rangle_{0\gamma}.$$

Upon using the identity: $F[\delta/\delta J]J_\nu(x) = \big[J_\nu(x) + \delta F[T]/\delta T^\nu(x)\big]\big|_{T^\nu = \delta/\delta J_\nu}$, the expansion

$$S_+(y, y'; \mathrm{e}_0 T) = S_+(x - x') - \mathrm{i}\mathrm{e}_0 \int (\mathrm{d}y)\, S_+(y - y_1)\, \gamma_\mu\, T^\mu(y_1)\, S_+(y_1 - y') + \cdots,$$

and the identity

$$\int (\mathrm{d}x')(\mathrm{d}y_1) D_+(x-x')\partial^{x'}_{\mu} \frac{\delta}{\delta T_\mu(x')} S_+(y-y_1)\,\gamma_\mu\, T^\mu(y_1)\, S_+(y_1-y')$$
$$= -\mathrm{i}\,[\,D_+(x-y)\, S_+(y-y') - D_+(x-y')\, S_+(y-y')\,],$$

which applied to the $\exp[\,\mathrm{i}\,\overline{\eta} S_+(.\,;\mathrm{e}_0\hat{A})\,\eta]$ factor leads to the stated result. By the same method just developed, the other factor $\exp[\int \mathrm{d}e' \,\mathrm{Tr}[\,\gamma\hat{A}(.)S_+(.\,;e'\hat{A})\,]$ gives no contribution (see also Appendix IV).

5.7. Write the exponential in (5.8.2) involving $(-\mathrm{i})\delta/\delta J^\mu$, acting on $\langle\, 0_+ \mid 0_-\rangle_{0\gamma}$, as

$$\exp \mathrm{i}\,[A_0 + \mathrm{e}_0 A_1 + \mathrm{e}_0^2(A_2 + \mathrm{i}B_2) + \mathrm{e}_0^3 A_3 + \cdots] =$$
$$\Big[1 + \mathrm{i}\,\mathrm{e}_0 A_1 + \mathrm{i}\,\mathrm{e}_0^2\Big(A_2 + \mathrm{i}B_2 + \frac{\mathrm{i}}{2}A_1^2\Big) + \mathrm{i}\,\mathrm{e}_0^3\Big(A_3 - \frac{1}{3!}A_1^3 + \mathrm{i}A_1\big(A_2 + \mathrm{i}\,B_2\big)\Big) + \cdots\Big]\,\mathrm{e}^{\mathrm{i}A_0},$$

up to third order, and where B_2 is the coefficient of e_0^2 in (5.8.6). Obviously, $a_0 = A_0$ as the zeroth order involving no functional differentiations. Also $a_1 = [\,\overline{\eta} S\gamma^\mu S\eta\,]D_{\mu\nu}J^\nu$ in a matrix notation. Clearly, A_1^3 involves only disconnected parts as it involves only three photon propagators since to *third* order they cannot connect together three different expressions of the form $[\,\overline{\eta} S\gamma^\mu S\eta\,]$. B_2 leads to $(1/2)\,[\,(-\mathrm{i})\,D_{\mu_1\mu_2} + D_{\mu_1\nu_1}J^{\nu_1}D_{\mu_2\nu_2}J^{\nu_2}\,]\,K^{\mu_1\mu_2}$, and the first term in the latter $(1/2)(-\mathrm{i})D_{\mu_1\mu_2}K^{\mu_1\mu_2}$ involves no external lines and should be omitted due to the normalization condition of $\langle\, 0_+ \mid 0_-\rangle$. For A_1^2, the only connected term is $[\,\overline{\eta} S\gamma^{\mu_1} S\eta\,][\,\overline{\eta} S\gamma^{\mu_2} S\eta\,](-\mathrm{i})D_{\mu_1\mu_2}$. The application of A_2 is straightforward and leads only to connected parts, and so is for A_3. The application of $\mathrm{i}A_1A_2$ leads to

$$\mathrm{i}\,[\,\overline{\eta} S\gamma^{\mu_1} S\eta][\overline{\eta} S\gamma^{\mu_2} S\gamma^{\mu_3}\eta\,] \times \Big((-\mathrm{i})D_{\mu_1\mu_3}D_{\mu_2\nu_2}J^{\nu_2} +$$
$$+ D_{\mu_1\nu_1}J^{\nu_1}D_{\mu_2\nu_2}J^{\nu_2}D_{\mu_3\nu_3}J^{\nu_3} + (-\mathrm{i})D_{\mu_1\mu_2}D_{\mu_3\nu_3}J^{\nu_3} + (-\mathrm{i})D_{\mu_2\mu_3}D_{\mu_1\nu_1}J^{\nu_1}\Big).$$

Clearly the second and the fourth terms within the round brackets give rise to disconnected parts and are to be omitted. Finally, $-A_1B_2$ leads to

$$\frac{\mathrm{i}}{2}[\,[\,\overline{\eta} S\gamma^{\mu_3} S\eta\,]K^{\mu_1\mu_2}\Big[\,D_{\mu_1\mu_3}D_{\mu_2\nu_2}J^{\nu_2} + D_{\mu_2\mu_3}D_{\mu_1\nu_1}J^{\nu_1}\,\Big],$$

retaining only connected parts.

5.8. This involves four terms. The term obtained by multiplying the first term in (5.9.23) by its complex conjugate is given by

$$\frac{1}{(p_2-p'_2)^4}[\overline{u}(\mathbf{p}'_2,\sigma'_2)\,\gamma^{\mu}\,u(\mathbf{p}_2,\sigma_2)\overline{u}(\mathbf{p}'_1,\sigma'_1)\gamma_{\mu}u(\mathbf{p}_1,\sigma_1)$$
$$\times\,\overline{u}(\mathbf{p}_1,\sigma_1)\gamma^{\sigma}u(\mathbf{p}'_1,\sigma'_1)\overline{u}(\mathbf{p}_2,\sigma_2)\gamma_{\sigma}u(\mathbf{p}'_2,\sigma'_2)]$$
$$=\frac{1}{(2m)^4}\frac{1}{(p_2-p'_2)^4}\mathrm{Tr}\,[\,\gamma^{\mu}(-\gamma p'_1+m)\gamma^{\sigma}(-\gamma p_1+m)\,]$$
$$\times\,\mathrm{Tr}\,[\,\gamma_{\mu}(-\gamma p'_2+m)\gamma_{\sigma}(-\gamma p_2+m)\,],$$

where we have used the relation $\sum_{\sigma}u_a(\mathbf{p},\sigma)\overline{u}_b(\mathbf{p},\sigma)=(-\gamma p+m)_{ab}/2m$. Application of the properties of the gamma matrices readily gives the first term in (5.9.32). It is easy to see that the two cross terms are identical, and direct applications of the method just given for the first term leads to the other two.

5.9. The four-spinors of the ingoing electrons are given by

$$u(\mathbf{p}_1,+)=\sqrt{\frac{p^0+m}{2m}}\begin{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ \mathrm{i}\rho\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix},\;u(\mathbf{p}_2,-)=\sqrt{\frac{p^0+m}{2m}}\begin{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\ \mathrm{i}\rho\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix},$$

where $\rho=\gamma\beta/(\gamma+1)=\beta/(1+\sqrt{1-\beta^2})$. The four spinors of the outgoing electrons may be written as

$$u(\mathbf{p}'_1,\sigma'_1)=\sqrt{\frac{p^0+m}{2m}}\begin{pmatrix}\xi_1\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'_1}{p^0+m}\xi_1\end{pmatrix},\;u(\mathbf{p}'_2,\sigma'_2)=\sqrt{\frac{p^0+m}{2m}}\begin{pmatrix}\xi_2\\ -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'_1}{p^0+m}\xi_2\end{pmatrix},$$

where

$$\mathbf{n}_1=(0,\sin\chi_1,\cos\chi_1),\;\mathbf{n}_2=(0,\sin\chi_2,\cos\chi_2),\;\mathbf{p}'_1=\gamma\,m\,\beta\,(1,0,0)=-\mathbf{p}'_2,$$
$$\xi_j=(\mathrm{e}^{-\mathrm{i}\pi/4}\cos(\chi_j/2)\;\;\mathrm{e}^{\mathrm{i}\pi/4}\sin(\chi_j/2)^{\top},\;\;j=1,2.$$

From (5.9.23), we have for the amplitude of the process

$$A\propto\xi_1^{\dagger}\,\xi^{\dagger}{}_2\Big[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\Big].$$

This leads to $|A|^2/2 = \mathrm{P}[\chi_1, \chi_2]$ as given in the problem, satisfying the completeness relation:

$$\mathrm{P}[\chi_1, \chi_2] + \mathrm{P}[\chi_1 + \pi, \chi_2] + \mathrm{P}[\chi_1, \chi_2 + \pi] + \mathrm{P}[\chi_1 + \pi, \chi_2 + \pi] = 1.$$

If only one spin is measured: $\mathrm{P}[\chi_1, -] = \mathrm{P}[\chi_1, \chi_2] + \mathrm{P}[\chi_1, \chi_2 + \pi] = 1/2$, and similarly $\mathrm{P}[-, \chi_2] = 1/2$. The obvious correlation between the two electron spins, that is the measurement of the spin of one is correlated with the spin value of the other, is expressed by the fundamental general relation $\mathrm{P}[\chi_1, \chi_2] \neq \mathrm{P}[\chi_1, -]\,\mathrm{P}[-, \chi_2]$. It is interesting to note that the above probabilities are independent of the speed β of the particles. This is due to the special choice of axes of measurements and spin orientations chosen in the problem. More generally polarization correlations depend on the speed of the underlying particles (see, e.g., Sect. 5.9.2 and references therein).

5.10. $(\gamma p + m)^2 = -p^2 + 2\,m\,\gamma p + m^2 = -(p^2 + m^2) + 2m(\gamma p + m)$ from which the identity follows.

5.11. From the method of the Feynman parameter representation, as given in (II.34), in Appendix I at the end of the book,

$$\int (\mathrm{d}k) \frac{1}{(k^2 + m^2)^2} \frac{\Lambda^2}{(k^2 + \Lambda^2)} = 2\,\Lambda^2 \int_0^1 x\,\mathrm{d}x \int \frac{(\mathrm{d}k)}{\big[k^2 + \Lambda^2 + (m^2 - \Lambda^2)x\big]^3}$$

$$= \mathrm{i}\pi^2 \int_0^1 \frac{\Lambda^2\,x\,\mathrm{d}x}{\big(\Lambda^2 + (m^2 - \Lambda^2)\,x\big)} = \frac{\Lambda^2}{m^2 - \Lambda^2} \int_0^1 \mathrm{d}x \Big[1 - \frac{\Lambda^2}{\Lambda^2 + (m^2 - \Lambda^2)x}\Big],$$

using the integral (II.8). The stated result then follows upon carrying out the elementary x-integral, and then considering the limit $\Lambda^2 \to \infty$.

5.12. Using the integral

$$\int \frac{x\,\mathrm{d}x}{x^2 - ax + a} = \frac{1}{2}\ln\big[x^2 - ax + a\big] + \frac{a}{2}\frac{1}{\sqrt{a - (a/2)^2}}\arctan\Big(\frac{x - a/2}{\sqrt{a - (a/2)^2}}\Big),$$

for $a > 0$, we obtain with $a = \mu^2/m^2 \to 0$, $C_{\mathrm{ir}} = -(1/2)\ln\big(\mu^2/m^2\big) + (\mu/2m) \times [(\pi/2) + \mathscr{O}(\mu/m)]$. On the other hand $\mu^2 \partial C_{\mathrm{ir}}/\partial\mu^2 = -C_{\mathrm{ir}} + D_{\mathrm{ir}}$.

5.13.

$$I \equiv \int \frac{(\mathrm{d}k)}{[\,(p - k)^2 + m^2\,][\,k^2 + \mu^2\,]} = \int (\mathrm{d}k) \int_0^1 \frac{\mathrm{d}x}{F(k^2, x)},$$

where $F(k^2, x) = \big[\,k^2 + (p^2 + m^2)x(1 - x) + m^2x^2 + \mu^2(1 - x)\,\big]^2$, after having made a shift of the (integration) variable $k \to k + px$. Upon integration over x *by parts*, and using the elementary integral (II.8), we obtain $I =$

$\mathrm{i}\pi^2[\,C_{\mathrm{uv}} + I_2\,]$, where C_{uv} is given in (5.10.21), and

$$I_2 = (p^2 + m^2)\int_0^1 \frac{x\,(1-2x)\;\mathrm{d}x}{[\,(p^2+m^2)\,x(1-x) + m^2x^2 + \mu^2(1-x)\,]}$$
$$+ \int_0^1 \frac{x\,(2\,m^2x - \mu^2)\;\mathrm{d}x}{[\,(p^2+m^2)\,x(1-x) + m^2x^2 + \mu^2(1-x)\,]}.$$

In the first integral $I_2^{(1)}$, we may set $(p^2 + m^2) = 0$, in the denominator, obtaining, $I_2^{(1)} = [(p^2 + m^2)/m^2]\,[\,C_{\mathrm{ir}} - 2\,]$, for $\mu^2/m^2 \to 0$, where C_{ir} is given in (5.10.22). The second integral may be written, up to first order in $(p^2 + m^2)$, as

$$I_2^{(2)} = -(p^2+m^2)\int_0^1 \frac{x^2(1-x)(2\,m^2x-\mu^2)\,\mathrm{d}x}{[\,m^2x^2+\mu^2(1-x)\,]^2} + \int_0^1 \frac{x\,(2\,m^2x-\mu^2)\,\mathrm{d}x}{[\,m^2x^2+\mu^2(1-x)\,]}.$$

Both integrals are elementary except, perhaps, the integral

$$\mu^2\int_0^1 \frac{x^2(1-x)\,\mathrm{d}x}{[\,m^2x^2+\mu^2(1-x)\,]^2} \to \mu^2\int_0^1 \frac{x^2\,\mathrm{d}x}{[\,m^2x^2+\mu^2(1-x)\,]^2},$$

where we have used, in the process, the value of the second integral D_{ir} in Problem 5.12, to set the second term to zero. The above integral, multiplied by μ^2, vanishes like $\mathscr{O}(\mu/m)$. Accordingly, $I_2^{(2)} = \big(2 + [\,(p^2+m^2)/m^2\,][\,3 - 2\,C_{\mathrm{ir}}\,]\big)$. Hence $I_2 = 2 + [\,(p^2+m^2)/m^2\,][\,C_{\mathrm{ir}} - 2 + 3 - 2\,C_{\mathrm{ir}}\,]$,

$$\int\frac{(\mathrm{d}k)}{[\,(p-k)^2+m^2\,][k^2+\mu^2\,]} = \mathrm{i}\,\pi^2\Big[(C_{\mathrm{uv}}+2) - 2\,\frac{\gamma p+m}{m}(C_{\mathrm{ir}}-1) + \mathscr{O}\big((\gamma p+m)^2\big)\Big],$$

where we have used the identity in Problem 5.10. The statement of the problem follows upon multiplying this integral by $-\mathrm{i}e^2[-(\lambda+1)(\gamma p+m) - 2m]/(2\pi)^4$.

5.14. From (5.10.57), the modified Coulomb potential, to second order may be written as $U'(\mathbf{x}) = q^2\big(Z_3 + (\alpha/3\pi)\,\Delta U'(\mathbf{x})\big)/(4\pi\,|\mathbf{x}|)$, where

$$\Delta U'(\mathbf{x}) = \int_{(2m)^2}^{\infty}\frac{\mathrm{d}M^2}{M^2}\,\mathrm{e}^{-M\,|\mathbf{x}|} + \int_{(2m)^2}^{\infty}\frac{\mathrm{d}M^2}{M^2}\bigg[\Big(1+\frac{2\,m^2}{M^2}\Big)\sqrt{1-\frac{4\,m^2}{M^2}} - 1\bigg]\mathrm{e}^{-M\,|\mathbf{x}|}.$$

The first integral is readily expressed in terms of the exponential integral function,[6] and it is given by the equivalent integral with asymptotic

[6]See, e.g., I. S. Gradshteyn and I. M. Ryzhik (2000). Tables of Integrals, Series and Products (6th ed.). San Diego/San Francisco: Academic Press. pp. 875–877.

behavior

$$I_1 = 2\int_{2m|\mathbf{x}|}^{\infty} \mathrm{d}t\, \frac{\mathrm{e}^{-t}}{t} = -2\,\mathrm{Ei}(-2m|\mathbf{x}|) \simeq -2\,\gamma_{\mathrm{E}} + 2\ln\Big(\frac{1}{2m|\mathbf{x}|}\Big) + \mathscr{O}(m|\mathbf{x}|),$$

where $\gamma_{\mathrm{E}} = 0.5772157\ldots$ denotes Euler's constant. On the other hand, it is justifiable to take the limit $m|\mathbf{x}| \to 0$ inside the second integral. This amounts to evaluate the integral

$$I_2' \equiv \int_{(2m)^2}^{\infty} \frac{\mathrm{d}M^2}{M^2}\Big[\Big(1+\frac{2\,m^2}{M^2}\Big)\sqrt{1-\frac{4\,m^2}{M^2}} - 1\Big].$$

Upon introducing the integration variable $z = 1-(1-4m^2/M^2)^{1/2}$, the above integral simplifies to

$$I_2' = \int_0^1 \mathrm{d}z\, \Big[\, -1 - 2z + z^2 - \frac{2}{z-2}\,\Big] = -2\Big[\ln\Big(\frac{1}{2}\Big) + \frac{5}{6}\Big].$$

All told, we obtain for $m|\mathbf{x}| \ll 1$,

$$U'(\mathbf{x}) = \frac{q^2}{4\pi|\mathbf{x}|}\Big(Z_3 + \frac{2\alpha}{3\pi}\Big[\ln\Big(\frac{1}{m|\mathbf{x}|}\Big) - \gamma_{\mathrm{E}} - \frac{5}{6}\Big]\Big).$$

5.15. From (5.10.42), $\overline{u}(\mathbf{p}', \sigma)\gamma^{\mu}\Pi_{\mu\nu}(k)\,(1/k^2)u(\mathbf{p},\sigma)$ is equal to

$$\overline{u}(\mathbf{p}',\sigma)\,\gamma_\nu\, u(\mathbf{p},\sigma)\,\Pi(k^2) - \overline{u}(\mathbf{p}',\sigma)(\gamma p' - \gamma p)u(\mathbf{p},\sigma)\frac{k_\nu}{k^2}.$$

The Dirac spinors satisfy the equations $(\gamma p + m)u(\mathbf{p},\sigma) = 0$, $\overline{u}(\mathbf{p}',\sigma)(\gamma p' + m) = 0$. That is the last expression above is zero. The statement of the problem then follows from (5.10.44), (5.10.49), by taking the limit $k^2 \to 0$ in this order, where $\Pi(0) = 1 - Z_3$.

5.16. Using the normalization condition of $\rho(\mathbf{x})$, the change in potential energy is

$$\Delta U(\mathbf{x}) = -\alpha\Big[\int \mathrm{d}^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}|}\Big] = -\alpha\int \mathrm{d}^3\mathbf{x}'\rho(\mathbf{x}')\Big[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}|}\Big].$$

leading to an approximate energy shift

$$\delta E = -\alpha\,|\varphi_{n0}(0)|^2\int \mathrm{d}^3\mathbf{x}'\rho(\mathbf{x}')\int \mathrm{d}^3\mathbf{x}\Big[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}|}\Big].$$

The $\mathbf{x}'$ - integrand, multiplying $\rho(\mathbf{x}')$, is easily worked out, for example, by a Legendre polynomial expansion,[7] to be

$$\int \mathrm{d}^3\mathbf{x}\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}-\frac{1}{|\mathbf{x}|}\right] = 2\pi \int_{|\mathbf{x}|<|\mathbf{x}'|} \mathrm{d}|\mathbf{x}|\ |\mathbf{x}|^2\left[\frac{2}{|\mathbf{x}'|}-\frac{2}{|\mathbf{x}|}\right] + 0 = -2\pi|\mathbf{x}'|^2/3,$$

and $\int \mathrm{d}^3\mathbf{x}'\rho(\mathbf{x}')\,|\mathbf{x}'|^2 = 12\,\gamma^2$. This gives

$$\delta E = \frac{8\,\alpha^4\,m}{n^3}\,m^2\gamma^2.$$

5.17. $F^{k\nu}A_\nu = F^{k0}A_0 + F^{ki}A_i$. Using the notation $(-\mathrm{i})\delta/\delta J^\mu \equiv \widehat{A}_\mu$, note that $\widehat{A}_0(y)\,\widehat{F}^{k0}(x)\langle\, 0_+ \mid 0_-\rangle$ is equal to

$$\widehat{A}_0(y)\langle\, 0_+|\,F^{k\,0}(x)\ \mid 0_-\rangle = \langle\, 0_+|\Big(F^{k\,0}(x)A_0(y)\Big)_+ \mid 0_-\rangle + \mathrm{i}\frac{\partial^k}{\boldsymbol{\nabla}^2}\,\delta^{(4)}(x-y)\langle\, 0_+ \mid 0_-\rangle,$$

where we have used (5.14.27) and (4.6.38). Hence

$$\langle\, 0_+|\big(F^{k\nu}(x)\,A_\nu(y)\big)_+ \mid 0_-\rangle = \left[\,\widehat{F}^{k\nu}(x)\,\widehat{A}_\nu(y) - \mathrm{i}\frac{\partial^k}{\boldsymbol{\nabla}^2}\,\delta^{(4)}(x-y)\,\right]\langle\, 0_+ \mid 0_-\rangle.$$

5.18. We note from the just mentioned equation that

$$\pi^{\,a} = \partial_3^{-1}\partial^a\partial^0 A^3 - \partial^0 A^a, \qquad a = 1,2,$$
$$\partial_{\,a}\pi^{\,a} = \partial_3^{-1}\partial^0(\partial_a\partial^a A^3 - \partial^3\partial_a A^a) = \partial_3^{-1}\partial^0\boldsymbol{\nabla}^2 A^3.$$

where we have used the relation $-\partial^a A^a = \partial_3 A^3$, in writing the last equality. From the equal-time commutation relations:

$$[A^a(x^0,\mathbf{x}'),\pi^{\,b}(x^0,\mathbf{x})] = \mathrm{i}\,\delta^{ab}\delta^{(3)}(\mathbf{x}'-\mathbf{x}), \qquad a,b = 1,2,$$

and the above two equations in turn, lead to

$$[A^a(x^0,\mathbf{x}'),\partial_3^{-1}\partial^b\partial^0 A^3(x^0,\mathbf{x})] - [A^a(x^0,\mathbf{x}'),\partial^0 A^b(x^0,\mathbf{x})] = \mathrm{i}\,\delta^{ab}\delta^{(3)}(\mathbf{x}'-\mathbf{x}),$$
$$[A^a(x^0,\mathbf{x}'),\partial_3^{-1}\partial^0 A^3(x^0,\mathbf{x})] = \mathrm{i}\,\frac{\partial^a}{\boldsymbol{\nabla}^2}\,\delta^{(3)}(\mathbf{x}'-\mathbf{x}),$$

[7] Recall: $(|\mathbf{x}-\mathbf{x}'|)^{-1} = (1/r_>)\sum_{n=0}^{\infty}(r_</r_>)^n P_n(\cos\theta)$, in a standard notation, $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $\int_{-1}^{1}\mathrm{d}\cos\theta\ P_n(\cos\theta)\,P_{n'}(\cos\theta) = 2\,\delta_{nn'}/(2n+1)$.

from which the following key equal-time commutation relations emerges

$$[A^a(x^0,\mathbf{x}'),\partial_0 A^b(x^0,\mathbf{x})] = \mathrm{i}\Big(\delta^{ab} - \frac{\partial^a\partial^b}{\boldsymbol{\nabla}^2}\Big)\,\delta^{(3)}(\mathbf{x}'-\mathbf{x}), \qquad a,b = 1,2.$$

The expression $A^3 = -(\partial^a/\partial_3)A^a$ then leads to the equal time commutation relations in question.

5.19. The Fourier transform of the left-hand side of the identity reads $\big[\eta^{\alpha\beta} - (\eta^{\alpha j}k^j k^\beta/\mathbf{k}^2) - (\eta^{\beta j}k^j k^\alpha/\mathbf{k}^2) + (k^\alpha k^\beta/\mathbf{k}^2)\big]/k^2$, and coincides with $D_{\mathrm{C}}^{\alpha\beta}(k)$ (see (5.14.13).

5.20. Denote the left-hand side by K. This gives

$$(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}K = \exp\Big(-\mathrm{i}\,\mathrm{e}_0\frac{\delta}{\delta\eta}\,\gamma^{\,\mu}\frac{\delta}{\delta\overline{\eta}}\,\partial_\mu\Lambda\Big)\Big[S_+(x-.)\,\eta(.)\Big]\exp\big[\,\mathrm{i}\,\overline{\eta}\,S_+\,\eta\,\big]$$
$$= S_+(x-.)\Big[\eta(.) + \mathrm{i}\,\mathrm{e}_0\gamma^\mu\frac{\delta}{\delta\overline{\eta}(.)}\partial_\mu\Lambda(.)\Big]K,$$

which upon multiplying by $S_+^{-1}(z-x)$ and integrating over x, gives

$$\int(\mathrm{d}x)\,S_+^{-1}(z-x)(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}K = \Big[\eta(z) + \mathrm{i}\,\mathrm{e}_0\gamma^\mu\,\partial^z_\mu\Lambda(z)\frac{\delta}{\delta\overline{\eta}(z)}\Big]K$$

which, after multiplying it by $\exp[\,\mathrm{i}\,\mathrm{e}_0\,\Lambda(z)\,]$, may be rewritten as

$$\int(\mathrm{d}x)\exp[\,\mathrm{i}\,\mathrm{e}_0\,\Lambda(z)\,]\Big[\,S_+^{-1}(z-x) + \delta^{(4)}(z-x)\,\mathrm{e}_0\gamma^{\,\mu}\,\partial_\mu\Lambda(x)\,\Big](-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}K$$
$$= \exp[\,\mathrm{i}\,\mathrm{e}_0\,\Lambda(z)\,]\,\eta(z)K.$$

But

$$\exp[\,\mathrm{i}\,\mathrm{e}_0\,\Lambda(z)\,][S_+^{-1}(z-x) \;+\; \delta^{(4)}(z-x)\,\mathrm{e}_0\gamma^{\,\mu}\big(\partial_\mu\Lambda(x)\,]$$
$$=\; S_+^{-1}(z-x)\exp[\,\mathrm{i}\,\mathrm{e}_0\,\Lambda(x)\,],$$

(see Problem 3.15). Hence upon multiplying the former equation by: $\exp[-\mathrm{i}\,\mathrm{e}_0\,\Lambda(y)\,]\,S_+(y-z)$, and integrating over z, we obtain

$$(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(y)}K = \int(\mathrm{d}z)\;\exp[-\mathrm{i}\,\mathrm{e}_0\,\Lambda(y)\,]\;S_+(y-z)\;\exp[\,\mathrm{i}\,\mathrm{e}_0\,\Lambda(z)\,]\,\eta(z)\,K.$$

Functionally integrating over $\overline{\eta}(y)$ leads to the right-hand side of the equation stated in the problem, and incidentally satisfies the appropriate boundary condition for $\mathrm{e}_0 \to 0$.

5.21. Upon setting $\mathrm{e}_0\,\big(\delta/\delta\rho(z)\big)\gamma^{\mu}\big(\delta/\delta\overline{\rho}(z)\big)=\hat{f}^{\mu}(z)$, with $\eta\to\rho$, $\overline{\eta}\to\overline{\rho}$, $J^{\mu}\to K^{\mu}$, we have from (5.15.1),

$$\exp\Big[\int(\mathrm{d}x)\big[\big(\eta^{\mu\nu}-\frac{\partial^{\mu}\partial^{\nu}}{\Box}\big)J_{\nu}(x)\big]\frac{\delta}{\delta K^{\mu}(x)}\Big]\mathrm{F}[\rho,\overline{\rho},K^{\mu},\lambda]\Big|_{K^{\mu}=0}$$
$$=\exp\Big[\frac{\mathrm{i}}{2}(\hat{f}^{\mu}+J^{\mu})D_{\mu\nu}(\lambda)(\hat{f}^{\nu}+J^{\nu})\Big]\exp\Big[-\frac{\mathrm{i}}{2}(\partial_{\mu}J^{\mu})\,G\,(\partial_{\nu}J^{\nu})\Big]$$
$$\times\exp\Big[-\mathrm{i}\hat{f}^{\mu}\partial_{\mu}\Big(\frac{\partial_{\alpha}}{\Box^2}J^{\alpha}\Big)\Big]\exp\big[\mathrm{i}\,\overline{\rho}\,S_{+}\,\rho\big],$$

where G is defined in (5.15.25). From Problem 5.20, we also have, with

$$\Lambda=\mathrm{e}_0\Big[\frac{\partial_{\alpha}}{\Box^2}J^{\alpha}\Big],$$

$$\exp\Big[-\mathrm{i}\hat{f}^{\mu}\partial_{\mu}\Big(\frac{\partial_{\alpha}}{\Box^2}J^{\alpha}\Big)\Big]\exp\big[\mathrm{i}\,\overline{\rho}\,S_{+}\,\rho\big]=\exp\big[\mathrm{i}\big(\overline{\rho}\,\mathrm{e}^{-\mathrm{i}\Lambda}\big)\,S_{+}\,\big(\mathrm{e}^{\mathrm{i}\Lambda}\rho\big)\big].$$

Since we eventually have to set the external Fermi sources to zero, we may make a change of these source variables,

$$\rho\to\mathrm{e}^{-\mathrm{i}\Lambda}\rho,\quad \overline{\rho}\to\mathrm{e}^{\mathrm{i}\Lambda}\overline{\rho},$$

and use the invariance of $\hat{f}^{\mu}$, under such a transformation, to reach the statement made in the problem by finally using, in the process, (5.15.20)–(5.15.22).

5.22. Set $\mathrm{e}_0\,\big(\delta/\delta\rho(z)\big)\gamma^{\mu}\big(\delta/\delta\overline{\rho}(z)\big)=\hat{f}^{\mu}(z)$. Then from (5.15.20),

$$(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(\mathrm{i})\frac{\delta}{\delta\eta(y)}\mathrm{F}[\eta,\overline{\eta},J^{\mu},\lambda=0]\Big|_{\eta=0,\,\overline{\eta}=0}$$
$$=(-\mathrm{i})\frac{\delta}{\delta\overline{\rho}(x)}(\mathrm{i})\frac{\delta}{\delta\rho(y)}\,\exp\big[\hat{Q}\big]\,\mathrm{F}[\rho,\overline{\rho},K^{\mu},\lambda]\Big|_{\rho=0,\,\overline{\rho}=0,\,K^{\mu}=0},\qquad(*)$$
$$\hat{Q}=\mathrm{e}_0\Big[\tilde{a}^{\mu}_{y}\frac{\delta}{\delta K^{\mu}(y)}-\tilde{a}^{\mu}_{x}\frac{\delta}{\delta K^{\mu}(x)}\Big]+\int(\mathrm{d}x')\Big[\big(\eta^{\mu\nu}-\tilde{a}'^{\mu}\partial'^{\nu}\big)J_{\nu}(x')\Big]\frac{\delta}{\delta K^{\mu}(x')}.$$

The first term in $\hat{Q}$ may be more conveniently rewritten as

$$\mathrm{e}_0\int(\mathrm{d}x')\,\tilde{a}'^{\mu}\big[\delta^{(4)}(x'-x)-\delta^{(4)}(x'-y)\big]\frac{\delta}{\delta K^{\mu}(x')}.$$

With $\exp[\hat{Q}]$ generating translations in K^ν, the right-hand side of the former equation $(*)$ is given, in matrix multiplication notation in spacetime, by

$$\exp[\mathrm{i}\,\phi(J^\mu)](-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(\mathrm{i})\frac{\delta}{\delta\eta(y)}\exp\Big[\frac{\mathrm{i}}{2}(\hat{f}^\mu + J^\mu)D_{\mu\nu}(\hat{f}^\nu + J^\nu)\Big]$$

$$\times\ \exp\big[-\mathrm{i}\hat{f}^\mu\partial_\mu\Lambda\big]\exp\big[\mathrm{i}\,\overline{\rho}\,S_+\,\rho\big]\Big|_{\rho=0,\,\overline{\rho}=0,\,K^\mu=0},$$

$$\Lambda(z\,;x,y) = \int(\mathrm{d}z\,')\,G(z-z\,')\Big[\mathrm{e}_0\Big(\delta^{(4)}(z\,'-x)-\delta^{(4)}(z\,'-y)\Big)-\partial^\mu J_\mu(z\,')\Big],$$

$$\phi(J^\mu) = -J^\mu(.)\,\partial^{\boldsymbol{\cdot}}_\mu\,\Lambda(\,.\,;x,y)+\frac{1}{2}\,g(\,.\,;x,y)\,G(.-.)\,g(\,.\,;x,y),$$

$$g(z\,;x,y) = \mathrm{e}_0\,\Big(\delta^{(4)}(z-x)-\delta^{(4)}(z-y)\Big)-\partial^z_\mu J^\mu(z).$$

where $G(z-z\,')$ is defined in (5.15.25) with an ultraviolet cut-off. From Problem 5.20: $\exp\big[-\mathrm{i}\hat{f}^\mu\partial_\mu\Lambda\big]\exp\big[\mathrm{i}\,\overline{\rho}\,S_+\,\rho\big] = \exp\big[\mathrm{i}(\overline{\rho}\mathrm{e}^{-\mathrm{i}\Lambda})\,S_+\,(\mathrm{e}^{\mathrm{i}\Lambda}\rho)\big]$. Hence upon defining sources

$$T = \mathrm{e}^{\mathrm{i}\Lambda}\rho,\quad \overline{T} = \overline{\rho}\,\mathrm{e}^{\mathrm{i}\Lambda},$$

using the chain rule: $(\delta/\delta\rho) = \mathrm{e}^{\mathrm{i}\Lambda}(\delta/\delta T)$, and simply evaluating the functional $\phi(J^\mu)$ above, we obtain

$$(\mathrm{i})\,(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(\mathrm{i})\frac{\delta}{\delta\eta(y)}\mathrm{F}[\,\eta,\overline{\eta},J^\mu,\lambda=0]\Big|_{\eta=0,\,\overline{\eta}=0,}$$

$$=\mathrm{e}^{\mathrm{i}\Psi[J^\mu]}\mathrm{e}^{-\mathrm{i}\,\mathrm{e}_0^2\big[G(0)-G(x-y)\big]}(\mathrm{i})\,(-\mathrm{i})\frac{\delta}{\delta\overline{T}(x)}(\mathrm{i})\frac{\delta}{\delta T(y)}\mathrm{F}[T,\overline{T},J^\mu,\lambda]\Big|_{T=0,\overline{T}=0,},\quad(**)$$

$$\Psi[J^\mu] = -\frac{1}{2}(\partial^\mu J_\mu)\,G\,(\partial^\nu J_\nu)-\mathrm{e}_0\int(\mathrm{d}z)J^\mu(z)\partial^z_\mu[\,G(z-x)-G(z-y)\,].$$

Upon dividing $(**)$ by $\mathrm{F}[0,0,J^\mu,\lambda=0]$, as given in (5.15.24), the statement of the problem follows.

Chapter 6

6.1. For infinitesimal transformations $V(x)\simeq I+\mathrm{i}g_\mathrm{o}\,\theta_c(x)t_c$, and the transformation rule $A^\mu\to\big(VA^\mu V^{-1}+\mathrm{i}\,(V)\partial^\mu V^{-1}/g_\mathrm{o}\big)$, defined in (6.2.4), gives $A^\mu \to A^\mu + \mathrm{i}g_\mathrm{o}[\,t_c,t_b]\theta_c A^\mu_b + t_c\partial^\mu\theta_c$, where we have used the relation $A^\mu = t_b A^\mu_b$. This leads to the following infinitesimal transformation, upon using the anti-symmetric nature of the structure constants,

$$A^\mu_a\to A^\mu_a+\nabla^\mu_{ac}\,\theta_c,\qquad \nabla^\mu_{ac}=\delta_{ac}\,\partial^\mu+g_\mathrm{o}f_{abc}A^\mu_b.$$

6.2. We explicitly have

$$[\nabla^\mu, \nabla^\nu]_{cb} = g_o f_{cab}(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) + g_o^2\big(f_{cda}f_{aeb} - f_{cea}f_{adb}\big)A^\mu_d A^\nu_e.$$

Using the identity $f_{cda}f_{aeb} - f_{cea}f_{adb} = -f_{cba}f_{ade} = f_{cab}f_{ade}$, the result follows upon factoring out $g_o f_{cab}$, and using the definition of $G^{\mu\nu}_a$.

6.3. From the commutation relation in (6.2.7), established in the previous problem, we have

$$\begin{aligned}\nabla_{ab\mu}\,\nabla_{bc\nu}\,G^{\mu\nu}_c &= \nabla_{ab\nu}\,\nabla_{bc\mu}\,G^{\mu\nu}_c + g_o f_{abc}G_{b\mu\nu}G^{\mu\nu}_c \\ &= \nabla_{ab\nu}\,\nabla_{bc\mu}\,G^{\mu\nu}_c = -\nabla_{ab\mu}\,\nabla_{bc\nu}\,G^{\mu\nu}_c,\end{aligned}$$

where in going from the first line to the second we have used the anti-symmetry of f_{abc}. In the last equality we have used the fact that $G^{\mu\nu}_c = -G^{\nu\mu}_c$, and relabeled $\mu \leftrightarrow \nu$, thus establishing the equality.

6.4. By an integral representation of the delta functional, up to an unimportant multiplicative constant, the left-hand side becomes

$$\begin{aligned}\int \Pi_{bx}\,\mathscr{D}\phi_b(x)\exp \mathrm{i}\,\Big[\int (\mathrm{d}x)\Big(\phi_a(x)\partial_\mu\mathscr{A}^\mu_a + \frac{\lambda}{2}\phi_a(x)\phi_a(x)\Big)\Big] = \int \Pi_{bx}\mathscr{D}\phi_b(x) \\ \times\exp\frac{\mathrm{i}\,\lambda}{2}\Big[\int (\mathrm{d}x)\phi_a(x)\phi_a(x)\Big]\exp -\frac{\mathrm{i}}{2\lambda}\Big[\int (\mathrm{d}x)\partial_\mu\mathscr{A}^\mu_a(x)\,\partial_\nu\mathscr{A}^\nu_a(x)\Big],\end{aligned}$$

upon completing the squares in the exponential, and shifting the variable ϕ_a. The result follows after integration over the latter variable.

6.5. From Problem 6.1: $A^{(\theta)\mu}_a \simeq A^\mu_a + (\delta_{ab}\,\partial^\mu + g_o f_{acb}A^\mu_c)\theta_b$. The constraint: $\partial_k A^{(\theta)k}_a = 0$ gives $\theta_a \simeq (-\partial_k/\boldsymbol{\partial}^2)A^k_a + \mathscr{O}(A^2)$. When the latter is substituted back in the expression for $A^{(\theta)\mu}_a$, we obtain

$$A^{(\theta)\mu}_a \simeq \big(\eta^{\mu\nu} - [(\partial^\mu\eta^{\nu k}\partial_k)/\boldsymbol{\partial}^2]\big)A_{a\nu} + \mathscr{O}(A^2).$$

6.6. This expression is obtained from the corresponding differential cross section for $\mathrm{e}^-\mu^- \to \mathrm{e}^-\mu^-$ in Sect. 5.9.3, by replacing the expression

$$\frac{1}{(p^0/M)(p'^0/M)}\,M^2\sum_{\text{spins}}\mathrm{Tr}\big[\,\big(\overline{u}(\mathbf{p}',\sigma')\gamma^\mu u(\mathbf{p},\sigma)\big)\big(\overline{u}(\mathbf{p},\sigma)\gamma^\nu u(\mathbf{p}',\sigma')\big)\,\big]\ \text{in it by}$$

$$\frac{1}{(p^0/M_p)(p'^0/M_p)}\,M_p^2\sum_{\text{spins}}\mathrm{Tr}\big[\,\langle p',\sigma'|j^\mu(0)|p,\sigma\rangle\langle p\sigma|j^\nu(0)|p',\sigma'\rangle\,\big], \quad \text{where}$$

$$\langle p',\sigma'|j^\mu(0)|p\sigma\rangle = \overline{u}(\mathbf{p}',\sigma')\Big[\gamma^\mu F_1(Q^2) + \frac{[\gamma^\mu,\gamma^\alpha]}{4M_p}Q_\alpha\,\kappa F_2(Q^2)\Big]u(\mathbf{p},\sigma),$$

as readily follows by invariance arguments and application of the Dirac equation $(\gamma p + M_p)u(\mathbf{p},\sigma) = 0$. M_p denotes the mass of the proton. This gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\bigg|_{\mathrm{TF}} = \frac{\alpha^2}{4E^2\sin^4\frac{\vartheta}{2}}\frac{E'}{E}\left[\left(F_1^2 + \frac{Q^2}{4M_p^2}\kappa^2 F_2^2\right)\cos^2\frac{\vartheta}{2} + \frac{Q^2}{2M_p^2}\left(F_1 + \kappa F_2\right)^2\sin^2\frac{\theta}{2}\right].$$

The result follows upon setting: $G_{\mathrm{E}} = [F_1 - (Q^2/4M_p^2)\kappa F_2]$, $G_{\mathrm{M}} = [F_1 + \kappa F_2]$.

6.7. Let (k,k'), denote the momenta of (e^-,e^+), and (p,p') denote the momenta of $q,\bar{q}$. Using the fact that

$$m_e^2 \sum_{\text{spins}} \mathrm{Tr}\Big[\big(\overline{v}(\mathbf{k}',\sigma')\gamma^\mu u(\mathbf{k},\sigma)\big)\big(\overline{u}(\mathbf{k},\sigma)\gamma^\nu v(\mathbf{k}',\sigma')\big)\Big] \to [k'^\mu k^\nu + k'^\nu k^\mu - \eta^{\mu\nu} kk'],$$

for $m_e \to 0$. The corresponding expression for the quarks is then

$$[p'^\mu p^\nu + p'^\nu p^\mu - \eta^{\mu\nu} pp'] \quad \text{and hence,}$$

$$\begin{aligned}&[k'^\mu k^\nu + k'^\nu k^\mu - \eta^{\mu\nu} kk'][p'_\mu p_\nu + p'_\nu p_\mu - \eta_{\mu\nu} pp'] \\ &= 2[kp' \, kp + kp \, kp'] \propto (1 + \cos^2\vartheta),\end{aligned}$$

in the CM frame, where $\mathbf{k}\cdot\mathbf{p}/|\mathbf{k}\cdot\mathbf{p}| = \cos\vartheta$. That is, ϑ is the angle made by the momentum of an emerging quark q relative to that of the electron. Hence

$$\mathrm{d}\sigma/\mathrm{d}\Omega \propto 3\,\mathrm{e}^4 \sum_f (\mathrm{e}_f^2/\mathrm{e}^2)(1 + \cos^2\vartheta),$$

where e_f is the charge of the quark of a given flavor, and the factor 3 is for the three different colors. The cross section then works out to be $\sigma \propto 3\,\mathrm{e}^4 \sum_f (\mathrm{e}_f^2/\mathrm{e}^2)(16\,\pi)/3$. Upon comparison of this expression with the cross section for $e^+e^- \to \mu^+\mu^-$, with masses set equal to zero in (6.5.7), we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \alpha^2 \frac{(1 + \cos^2\vartheta)}{4s} \, 3 \sum_f (\mathrm{e}_f^2/\mathrm{e}^2), \qquad \sqrt{s} = \text{CM energy}.$$

6.8. (i) By using a Feynman parameter representation and shifting the variable of integration k, the integrand becomes replaced by:

$$2\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}z \, 1/[k^2 + Q^2(1-x)z]^3,$$

which by using the integral representation over k in (II.7) in Appendix II, at the end of the book, gives

$$\frac{\mathrm{i}}{(4\pi)^{D'/2}}\Big(\frac{1}{Q^2}\Big)^{(1-\delta/2)}\frac{\Gamma\big(1-\frac{\delta}{2}\big)}{\Gamma(3)}\,2\int_0^1 \mathrm{d}x\,(1-x)^{(-1+\delta/2)}\int_0^x \mathrm{d}z\,z^{(-1+\delta/2)}.$$

Finally carrying out the z-integral, followed by the use of the integral (III.12), involving gamma functions, the stated result follows.

(ii) As in part (i), except the $x{-}z$ integrands become simply multiplied by $[p_1^\mu(1-x)+p_2^\mu z]$, after the shift of the integration variables k and setting, in the process, an odd integral in k equal to zero. Finally the $x{-}z$ – integrals are readily carried out as above leading to the stated result.

6.9. By using the Feynman parameter representation in Problem 6.8 above, and shifting the k-integration variable again, the denominator of the integrand becomes simply

$$k^\mu k^\nu + (p_1^\mu(1-x)+p_2^\mu z)(p_1^\nu(1-x)+p_2^\nu z),$$

after setting an odd integral in k equal to zero. The integral involving the $k^\mu k^\nu$ part may be ultraviolet-regularized using the integral in (III.8), while the integral involving $(p_1^\mu(1-x)+p_2^\mu z)(p_1^\nu(1-x)+p_2^\nu z)$ may be infrared-regularized as in Problem 6.8. Finally, the (x,z) – integrations yield the stated result in a straightforward manner as in Problem 6.8.

6.10. It is sufficient to spell out, the general infra-red singular structure of the function $h_{\mathrm{IR}}(\mu_{D'}^2/Q^2,\delta)$. To this end, we refer to the right-hand sides of the integrals in Box 6.2 of the regularized integrals in Sect. 6.6. If an integral depends on p_1^μ, then multiplying it, by p_1^μ gives zero, and if multiplied by either, $p_{2\mu}$, or Q_μ, give a factor Q^2 which cancels out the factor $1/Q^2$ multiplying $(\mu_{D'}^2/Q^2)^{-\delta/2}$. Similar statements follow if the integral depends on p_2^μ. On the other hand, if we multiply the first integral by either p_1p_2 or Q^2, these terms cancel out again the $1/Q^2$ factor just mentioned. That is, in all the terms contributing to the fermion-gluon vertex, the $1/Q^2$ factor multiplying $(\mu_{D'}^2/Q^2)^{-\delta/2}$ is canceled out in the infra-red regularized part. The most infra-red singular part in evaluating the vertex function comes from the first integral in the Table now involving the factor $\big(\mu_{D'}^2/Q^2\big)^{-\delta/2}\Gamma^2\big(\delta/2\big)$. Therefore the infra-red singular structure of $h_{\mathrm{IR}}(\mu_{D'}^2/Q^2,\delta)$ is given by a linear combination of the following terms : $1/\delta^2$, $1/\delta$, $(1/\delta)\ln(Q^2/\mu_{D'}^2)$, $\ln^2(Q^2/\mu_{D'}^2)$.

6.11.

$$\sum_{nP_n}(2\pi)^4\,\delta^4(p+Q-P_n)\langle\,P,\sigma|\,j_\mu(0)|nP_n\rangle\langle nP_n|\,j_\mu(0)|P,\sigma\rangle$$

$$=\sum_{nP_n}\int(\mathrm{d}y)\;\mathrm{e}^{\mathrm{i}(p_n-P-Q)y}\,\langle\,P,\sigma|\,j_\mu(0)|nP_n\rangle\langle nP_n|\,j_\mu(0)|P,\sigma\rangle$$

$$=\int(\mathrm{d}y)\;\mathrm{e}^{-\mathrm{i}Qy}\,\langle\,P,\sigma|\,j_\mu(y/2)j_\mu(-y/2)|P,\sigma\rangle$$

where we have used the fact $\langle\,P,\sigma|\,j_\mu(0)|nP_n\rangle$

$$=\mathrm{e}^{\mathrm{i}(y/2)P}\mathrm{e}^{-\mathrm{i}(y/2)P_n}\langle\,P,\sigma|\mathrm{e}^{[-\mathrm{i}(y/2)\mathrm{Mom.Op.}]}j_\mu(0)\,\mathrm{e}^{[\mathrm{i}(y/2)\mathrm{Mom.Op.}]}|P_n\rangle,$$

and a similar expression for the other factor, and we finally summed over (n, P_n).

6.12. Since $P_i = 0, Q_i = Q_3\delta_{i3}$, we explicitly have $W_{11} = W_{22} = W_1$. Hence for a transversal photon

$$\sigma_{\mathrm{T}}\sim\epsilon^\mu_\lambda\,W_{\mu\sigma}\,\epsilon^\sigma_\lambda\;=\;W_1\;\geq 0,$$

for $\lambda = 1, 2$. On the other hand, $\epsilon^\mu_0\,W_{\mu\sigma}\,\epsilon^\sigma_0$, involves the following three terms:

$$[(Q^2+\nu^2)/Q^2]^2[(Q^2+\nu^2)W_2/Q^2\;-\;W_1],$$
$$[2\nu^2(Q^2+\nu^2)/Q^2][W_1-(Q^2+\nu^2)W_2/Q^2],$$
$$(\nu^2/Q^2)^2[(Q^2+\nu^2)W_2/Q^2-W_1].$$

Their sum gives

$$\sigma_{\mathrm{L}}\sim\epsilon^\mu_0\,W_{\mu\sigma}\,\epsilon^\sigma_0=[W_2(Q^2+\nu^2)/Q^2\;-\;W_1]\;\geq 0,$$

which establishes (6.9.13). Equations (6.9.14), (6.9.16) follow upon multiplying (6.9.13) by $2xM$, with $x = Q^2/2M\nu$, and finally using, in the process, the definitions in (6.9.15).

6.13. We explicitly have (see also (6.9.7))

$$W_i^{\mu\nu}=\frac{1}{2\pi\xi M}\,\frac{\mathrm{e}_i^2}{\mathrm{e}^2}\int\frac{\mathrm{d}^3\mathbf{p}\,'}{2p'^{\,0}\,(2\pi)^3}\,[\,.\,]^{\mu\nu}\,(2\pi)^4\,\delta^{(4)}(\xi P+Q-p'),$$

where $[\,.\,]^{\mu\nu}$ is obtained from (6.9.5) by making the substitutions: $k\to\xi P$, $k'\to p'$, and finally using the conservation law $p' = \xi P + Q$.

6.14. The integral on the left-hand side is equal to

$$\int (\mathrm{d}p')\,\Theta(p'^{\,0})\,\delta(p'^{\,2} + \xi^2 M^2)\,\delta^{(4)}(\xi P + Q - p')$$
$$= \Theta(\xi P^0 + Q^0)\delta\big((\xi P + Q)^2 + \xi^2 M^2\big) = \delta(2\xi PQ + Q^2).$$

The result in question then follows upon taking $2QP$ outside the argument of $\delta(2\xi PQ + Q^2)$.

6.15. The vertex function V^μ for a spin 0 boson going from momentum p to p' after interacting with the virtual photon must be of the form $p^\mu + p'^\mu$, with equal coefficients due to gauge invariance: $Q^\mu(p_\mu + p'_\mu) = 0$, where $Q = p' - p$. On the other hand, from the definition of Q, we may rewrite

$$(p^\mu + p'^\mu) = 2(p^\mu + Q^\mu/2).$$

Also $pQ = -Q^2/2$, i.e., $Q^\mu/2 = -p\,Q\,Q^\mu/Q^2$. Thus the vertex function for the spin 0 boson, consistent with gauge invariance, is simply proportional to $(p^\mu - pQ\,Q^\mu/Q^2)$. This in turn gives rise to a structure function contribution proportional to

$$(p^\mu - pQ\,Q^\mu/Q^2)(p^\nu - p\,Q\,Q^\nu/Q^2),$$

involving *no* $(\eta^{\mu\nu} - Q^\mu Q^\nu/Q^2)$ term. This leads to $W_1 = 0$ and the results stated in the problem follow.

6.16. We may write

$$\sum_i \mathrm{e}_i^2\, x f_i(x) = x f(x) \sum_i \mathrm{e}_i^2 = (2/3)\, x f(x) = (2/3)(1/3) \sum_i x f_i(x),$$

where we have used the fact that $\sum_i \mathrm{e}_i^2 = (4/9) + (1/9) + (1/9) = 2/3$. Upon integration over x, this gives the relation stated in the problem.

6.17. For A^n_{qG}, we note that $n \geq 3$ implies that $n(n+1) \geq 3 \times 4 = 12$, and hence $0 < 1/n(n+1) \leq 1/12$. Also, we may write

$$\sum_{j=2}^{n} 1/j = 1/2 + 1/3 + \sum_{j=4}^{n} 1/j.$$

The inequality for A^n_{qG}, then follows. The lower bound is easy to obtain, just omit the positive part. The same reasoning leads to the inequalities in (6.11.18) for A^n_{GG}, and A^n_{Gq}, $A^n_{q\bar{q}}$ in (6.11.17), where note, for example, that A^n_{Gq} in (6.11.12) may be rewritten as: $(4/3)[\big(1/(n-1)\big) + \big(2/n(n^2-1)\big)]$.

6.18. Let $t = \ln(Q^2/\Lambda^2)$, then $\tau = (1/b_0)\ln\big[\ln(Q^2/\Lambda^2)/\ln(Q_o^2/\Lambda^2)\big]$, or $\tau = (1/b_0)\ln(t/t_o)$. This gives $(\mathrm{d}\tau/\mathrm{d}t) = (1/b_0\, t) = \alpha_s(Q^2)/(2\,\pi)$, where we have used the relation $\alpha_s(Q^2) = 1/(\beta_0\, t)$, to lowest order. From the chain rule $\mathrm{d}/\mathrm{d}t = (\mathrm{d}\tau/\mathrm{d}t)(\mathrm{d}/\mathrm{d}\tau)$, the relation follows.

6.19. (i) This directly follows by noting that $(A_n D_n - B_n C_n) = \lambda_n^+ \lambda_n^-$, and that $A_n + D_n = \lambda_n^+ + \lambda_n^-$, upon carrying out the multiplication of the three matrices on the left-hand side of (6.11.32). (ii) From (6.11.17), $0 < 8n_f A^n_{q\bar{q}} A^n_{Gq} \equiv 4\,B_n C_n \leq (98/135)n_f$. Thus $\lambda_n^+ - \lambda_n^- = \sqrt{(A_n - D_n)^2 + 4B_n C_n}$ is real and positive. Also from (6.11.16), (6.11.18) we establish the positivity of $(A_n - D_n) \equiv (A^n_{qG} - A^n_{GG})$:

$$\frac{59}{45} + \frac{n_f}{3} + \frac{10}{3}\vartheta_4^n < (A_n - D_n) < \frac{49}{18} + \frac{n_f}{3} + \frac{10}{3}\vartheta_4^n.$$

Hence using the fact that for any two positive numbers a, b: $\sqrt{a^2 + b^2} \leq (a + b)$, we obtain from (6.11.34), (6.11.16)

$$\lambda_n^+ \leq \frac{1}{2}\big[A_n + D_n + (A_n - D_n) + \sqrt{(98/135)n_f}\,\big],$$
$$= \frac{1}{2}\big[2\,A_n + \sqrt{(98/135)n_f}\,\big] < \frac{1}{2}\big[-\frac{50}{9} + \sqrt{(98/135)n_f}\,\big] < 0,$$

with the upper bound, as shown, is strictly negative for unusually large $n_f < 43$. On the other hand $(A_n - D_n) > 4B_n C_n$, and $(A_n + D_n) > \sqrt{2}(A_n + D_n)$ with the latter being negative. Hence

$$\lambda_n^- = (1/2)[A_n + D_n - \sqrt{(A_n - D_n)^2 + 4B_n C_n}\,]$$
$$> (\sqrt{2}/2)\,[2D_n] > -\sqrt{2}\,[\,(11/2) + (n_f/3) + 6\,\vartheta_4^n\,],$$

as follows from (6.11.18). (iv) Finally

$$(\lambda_n^+ - A_n) = (1/2)[-(A_n - D_n) + \sqrt{(A_n - D_n)^2 + 4B_n C_n}\,] > 0,$$
$$(\lambda_n^- - A_n) = -(1/2)[(A_n - D_n) + \sqrt{(A_n - D_n)^2 + 4B_n C_n}\,] < 0.$$

6.20. For $x^0 > y^0$, $\Big(\exp[+\mathrm{i}g\int_{y^0}^{x^0} \mathrm{d}\xi\, A_0(\xi, \mathbf{x})]\Big)_+$

$$= 1 + (\mathrm{i}g)\int_{y^0}^{x^0} \mathrm{d}\xi_1\, A_0(\xi_1, \mathbf{x}) + (\mathrm{i}g)^2 \int_{y^0}^{x^0} \mathrm{d}\xi_2 \int_{x'^0}^{\xi_2} \mathrm{d}\xi_1\, A_0(\xi_2, \mathbf{x}) A_0(\xi_1, \mathbf{x}) + \cdots,$$

and $\partial_0(\,.\,)_+ = \mathrm{i}gA_0(x^0, \mathbf{x})\,[1 + (\mathrm{i}g)\int_{y^0}^{x^0} \mathrm{d}\xi_1\, A_0(\xi_1, \mathbf{x}) + \cdots]$.

6.21. For $T > 0$, consider the following expression, as a function of $t \geq 0$:

$$Q_-(t) = h_-(t, a, 0, a) V^{-1}(t, a)$$

with condition $Q_-(0) = V^{-1}(0, a)$. Using the fact that

$$\mathrm{d}h_-(t, a, 0, a)/\mathrm{d}t = -\mathrm{i}gh_-(t, a, 0, a)A_0(t, a),$$

we obtain $\mathrm{d}Q_-(t)/\mathrm{d}t$

$$= -\mathrm{i}gh_-(t, a; 0, a)V^{-1}(t, a)V(t, a)\big[A_0(t, a) - (1/\mathrm{i}g)(\mathrm{d}/\mathrm{d}t)\big]V^{-1}(t, a),$$

which is just $-\mathrm{i}\, g\, Q_-(t)A_0^V(t, a)$. Upon integration from 0 to T, gives:

$$h_-(T, a; 0, a)V^{-1}(T, a) = V^{-1}(0, a)h_-^V(T, a; 0, a),$$

from which the transformation rule in question follows. The transformation rules of the last two are almost identical to the first two by simply exchanging time variables with space variables.

6.22. If a priori ϕ^+ is zero, then by a specific choice of the transformation in (6.14.23), as a phase factor, we may remove any phase that ϕ^0 may have upon the transformation in. Otherwise, suppose that $\phi^+ \neq 0$. Using the identity

$$\exp[\mathrm{i}\varphi\mathbf{n}\cdot\boldsymbol{\sigma}/2] = \cos(\varphi/2) + \mathrm{i}\,\mathbf{n}\cdot\boldsymbol{\sigma}\,\sin(\varphi/2),$$

the transformation in (6.14.22) gives

$$\begin{pmatrix} \cos(\varphi/2) + \mathrm{i}\, n_3 \sin(\varphi/2) & (\mathrm{i}\, n_1 + n_2)\sin(\varphi/2) \\ (\mathrm{i}\, n_1 - n_2)\sin(\varphi/2) & \cos(\varphi/2) - \mathrm{i}\, n_3 \sin(\varphi/2) \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}.$$

Upon considering the expression ϕ^0/ϕ^+, and writing the latter as $(\phi^0/\phi^+) = (a + \mathrm{i}\, b)$, with a and b real, it is easily checked, by equating the resulting upper entry to zero, i.e., by setting

$$\big(\cos(\varphi/2) + \mathrm{i}\, n_3 \sin(\varphi/2)\big) + (\mathrm{i}\, n_1 + n_2)\sin(\varphi/2)(a + \mathrm{i}\, b) = 0,$$

that this equation has always a solution for all real a and b, by appropriate choices of n_1, n_2, n_3, and φ. This makes the resulting upper entry equal to zero in vacuum expectation value. Any phase that may arise from the second row of the transformation above may be removed by the appropriate choice of the transformation in (6.14.23), giving finally a real non-negative field for the lower entry in vacuum expectation value. Thus such transformations give rise to a field as given on the right-hand side of (6.14.26), with its components satisfying Eq. (6.14.27).

6.23. Using the anti-commutativity of the fermion fields, we have

$$[\bar{e}_\mathrm{L}\gamma^{\rho}\nu_\mathrm{L}\,[[\bar{\nu}_\mathrm{L}\gamma_{\rho}e_\mathrm{L}] = -(1/4)\bar{e}_a e_D \bar{\nu}_c \nu_B\big[[\gamma^{\rho}(1-\gamma^5)]_{aB}\,[\gamma_\rho(1-\gamma^5)]_{cD}\big].$$

Multiplying the Fierz identity[8]

$$(\gamma^{\rho})_{ab}(\gamma_{\rho})_{cd} = -\delta_{ad}\delta_{cb} - \frac{1}{2}(\gamma^{\rho})_{ad}(\gamma_{\rho})_{cb} - \frac{1}{2}(\gamma^{5}\gamma^{\rho})_{ad}(\gamma^{5}\gamma_{\rho})_{cb} + (\gamma^{5})_{ad}(\gamma^{5})_{cb},$$

by $(1-\gamma^5)_{bB}(1-\gamma^5)_{dD}$ and using the identities $\{\gamma^5, \gamma^{\mu}\} = 0, (\gamma^5)^2 = 1$, give $-[\gamma^{\rho}(1-\gamma^5)]_{aD}[\gamma_{\rho}(1-\gamma^5)]_{cB}$, and the identity immediately follows.

6.24. Following the method in Sects. 5.9.3, 5.9.1, and averaging over the spin of the muon and summing over the spins of the product particles, we have

$$(2M_{\mu})(2m_e)(2m_{\nu_e})(2m_{\nu_{\mu}})\frac{1}{2}\sum_{\text{spins}}|\mathscr{A}|^2\Big|_{m_e, m_{\nu_e}, m_{\nu_{\mu}} \to 0} = 64\,\mathrm{G_F}^2\,(p_{\mu}\,k_{\nu})(k_1^{\mu}\,k_2^{\nu}),$$

which we conveniently denote by X. The decay rate is then given by $(p = (M_{\mu}, \mathbf{0}))$

$$\mathrm{d}\Gamma = \frac{1}{16M_{\mu}}\int X \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|}\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3|\mathbf{k}_1|}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3|\mathbf{k}_2|}(2\pi)^4\delta^{(4)}(p-k-k_1-k_2).$$

Also note that

$$p_{\mu}k_{\nu}\int\big(\mathrm{d}^3\mathbf{k}_1\mathrm{d}^3\mathbf{k}_2/|\mathbf{k}_1||\mathbf{k}_2|\big)\,k_1^{\mu}k_2^{\nu}\delta^{(4)}(p-k-k_1-k_2) = (\pi M_{\mu}^2/6)[3M_{\mu}|\mathbf{k}|-4|\mathbf{k}|^2].$$

Accordingly, using the fact that $\mathrm{d}^3\mathbf{k}/|\mathbf{k}| = E_e\,\mathrm{d}E_e\mathrm{d}\Omega$, and noting by conservation of energy and momentum that the maximum value of the energy E_e attained by the electron corresponds to the neutrinos moving in the same direction leading to $E_e|_{max} = M_{\mu}/2$, we readily obtain, upon carrying the **k**-integration, the decay rate stated in the problem.

[8]For many details on Fierz identity and some of its generalizations, see Appendix A to Chapter 2 in Volume II.

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