Appendix A
Difference Operator

For the higher-order difference of the product of two functions we may write

$$\begin{aligned}\Delta^k(u_n v_n) &= (E-1)^k(u_n v_n)\\ &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}E^j(u_n v_n)\end{aligned} \tag{A.1}$$

Here

$$\begin{aligned}E^j(u_n v_n) &= E^j u_n E^j v_n\\ &= (1+\Delta)^j u_n (1+\Delta)^j v_n\end{aligned} \tag{A.2}$$

Thus

$$\Delta^k(u_n v_n) = (-1)^k \sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{n=0}^{j}\binom{j}{n}\Delta^n u_n \sum_{m=0}^{j}\binom{j}{m}\Delta^m v_n \tag{A.3}$$

We now wish to sum first over j. Interchanging j and n we have

$$\sum_{j=0}^{k}\sum_{n=0}^{j} = \sum_{n=0}^{k}\sum_{j=n}^{k} \tag{A.4}$$

from which

$$\Delta^k(u_n v_n) = (-1)^k \sum_{n=0}^{k}\sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\Delta^n u_n \sum_{m=0}^{j}\binom{j}{m}\Delta^m v_n \tag{A.5}$$

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We next interchange the summations over j and m, giving

$$\sum_{j=n}^{k}\sum_{m=0}^{j} = \sum_{m=0}^{n}\sum_{j=n}^{k} + \sum_{m=n+1}^{k}\sum_{j=m}^{k} \tag{A.6}$$

We have here, for the sums over j,

$$\sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m} \qquad 0 \le m \le n \le k \tag{A.7}$$

and

$$\sum_{j=m}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m} \qquad 0 \le n < m \le k \tag{A.8}$$

Thus

$$\begin{aligned}\Delta^k(u_n v_n) &= (-1)^k\sum_{n=0}^{k}\sum_{m=0}^{n}\sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m}\Delta^n u_n \Delta^m v_n \\ &\quad + (-1)^k\sum_{n=0}^{k}\sum_{m=n+1}^{k}\sum_{j=m}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m}\Delta^n u_n \Delta^m v_n \\ &= (-1)^k\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^n u_n \Delta^m v_n \sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m} \\ &\quad + (-1)^k\sum_{n=0}^{k}\sum_{m=n}^{k}\Delta^n u_n \Delta^m v_n \sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m} \\ &\quad - (-1)^k\sum_{n=0}^{k}\sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{n}\Delta^n u_n \Delta^n v_n \end{aligned} \tag{A.9}$$

Here

$$\sum_{n=0}^{k}\sum_{m=n}^{k} = \sum_{m=0}^{k}\sum_{n=0}^{m} \tag{A.10}$$

Thus

$$\Delta^k(u_n v_n) = (-1)^k\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^n u_n \Delta^m v_n \sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m}$$

$$+ (-1)^k \sum_{m=0}^{k} \sum_{n=0}^{m} \Delta^n u_n \Delta^m v_n \sum_{j=m}^{k} (-1)^j \binom{k}{j} \binom{j}{n} \binom{j}{m}$$
$$- (-1)^k \sum_{n=0}^{k} \Delta^n u_n \Delta^n v_n \sum_{j=n}^{k} (-1)^j \binom{k}{j} \binom{j}{n} \binom{j}{n} \tag{A.11}$$

Thus we need

$$\begin{aligned}
&\sum_{j=n}^{k} (-1)^j \binom{k}{j} \binom{j}{n} \binom{j}{m} \qquad \text{for} \quad 0 \le m \le n \le j \le k \\
&\sum_{j=m}^{k} (-1)^j \binom{k}{j} \binom{j}{n} \binom{j}{m} \qquad \text{for} \quad 0 \le n \le m \le j \le k \\
&\sum_{j=n}^{k} (-1)^j \binom{k}{j} \binom{j}{n} \binom{j}{n} \qquad \text{for} \quad 0 \le n \le j \le k
\end{aligned} \tag{A.12}$$

Here

$$\sum_{j=n}^{k} (-1)^j \binom{k}{j} \binom{j}{n} \binom{j}{m} = \frac{k!}{n!m!} \sum_{j=n}^{k} (-1)^j \frac{j!}{(k-j)!(j-n)!(j-m)!} \tag{A.13}$$

In the first sum in (A.12), for which $0 \le m \le n \le j$, we set $j' = j - n$, from which

$$\begin{aligned}
\sum_{j=n}^{k} (-1)^j \frac{j!}{(k-j)!(j-n)!(j-m)!} &= \sum_{j'=0}^{k-n} (-1)^{j'+n} \frac{(j'+n)!}{(k-n-j')!(j'+n-m)!j'!} \\
&= (-1)^n \frac{n!}{(n-m)!(k-n)!} \sum_{j'=0}^{k-n} \frac{(-k+n)_{j'}(n+1)_{j'}}{j'!(n+1-m)_{j'}}
\end{aligned} \tag{A.14}$$

where we have used

$$\frac{1}{(N-j)!} = \frac{(-1)^j(-N)_j}{N!} \tag{A.15}$$

and

$$(N+j)! = \Gamma(N+1+j) = (N+1)_j N! \tag{A.16}$$

We thus have, for $0 \le m \le n \le k$,

$$\sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m} = (-1)^n \frac{k!}{m!(n-m)!(k-n)!}\,{}_2F_1(-k+n, n+1; n+1-m; 1) \tag{A.17}$$

The well-known expression for the hypergeometric function with unit argument,

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{A.18}$$

is generally given with the condition $Re(c-a-b) > 0$. For the hypergeometric function directly above we have $c-a-b = k-n-m$ which might be a negative integer or zero, in which case the expression in (A.18) is ill-defined. However, analytic continuation of the integer values of the parameters to the complex plane make each gamma function well-behaved. Thus, since the hypergeometric polynomial is well-defined, a limit to integer arguments should exist. We thus let m be given a small imaginary component. Then, with $p = k-n$,

$$\Gamma((k-n)-m) = (p-1-m)(p-2-m)\cdots(p-p-m)\Gamma(-m) \tag{A.19}$$

is well-defined, and so is

$$\begin{aligned} F(-k+n, n+1; n+1-m; 1) &= \frac{\Gamma(n-m+1)}{\Gamma(k-m+1)}(-m)(-m+1)\cdots(-m+p-1) \\ &= \frac{\Gamma(n-m+1)}{\Gamma(k-m+1)}(-m)_p \\ &= \frac{(n-m)!}{(k-m)!}(-m)_{k-n} \end{aligned} \tag{A.20}$$

Using (A.15) we then have, for $0 \le m \le n \le k$,

$$F(-k+n, n+1; n+1-m; 1) = \frac{(-1)^{k-n}(n-m)!m!}{(k-m)!(m+n-k)!} \tag{A.21}$$

and from (A.17) and (A.21)

$$\sum_{j=n}^{k}(-1)^j\binom{k}{j}\binom{j}{n}\binom{j}{m} = \frac{(-1)^k k!}{(k-n)!(k-m)!(n+m-k)!} \tag{A.22}$$

Noting that the right-hand side of this equation is symmetric in n and m, it follows that the remaining sums in (A.11) give identical expressions. We thus have

$$\Delta^k(u_n v_n) = k! \sum_{n=0}^{k} \sum_{m=0}^{n} \frac{\Delta^n u_n \Delta^m v_n}{(k-n)!(k-m)!(n+m-k)!}$$
$$+ k! \sum_{m=0}^{k} \sum_{n=0}^{m} \frac{\Delta^n u_n \Delta^m v_n}{(k-n)!(k-m)!(n+m-k)!}$$
$$- k! \sum_{n=0}^{k} \frac{\Delta^n u_n \Delta^n v_n}{(k-n)!(k-n)!(2n-k)!} \tag{A.23}$$

These three sums cover the entire region $0 \le m \le k, \;\; 0 \le n \le k$, so that we may finally write

$$\Delta^k(u_n v_n) = k! \sum_{n=0}^{k} \sum_{m=0}^{k} \frac{\Delta^n u_n \Delta^m v_n}{(k-n)!(k-m)!(n+m-k)!} \tag{A.24}$$

in which it is understood that terms vanish when $k > n + m$. Alternatively, we may write

$$\Delta^k(u_n v_n) = k! \sum_{n=0}^{k} \sum_{m=0}^{k} \frac{\Delta^{k-n} u_n \Delta^{k-m} v_n}{n!m!(k-n-m)!} \tag{A.25}$$

in which terms vanish when $k < n + m$.

Appendix B
Notation

Throughout this work we encounter matrices of the form

$$\begin{vmatrix} u_1 & \cdots & u_{j-1} & 0 & u_{j+1} & \cdots & u_n \\ u_1^{(1)} & \cdots & u_{j-1}^{(1)} & 0 & u_{j+1}^{(1)} & \cdots & u_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-2)} & \cdots & u_{j-1}^{(n-2)} & 0 & u_{j+1}^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(n-1)} & \cdots & u_{j-1}^{(n-1)} & g_n & u_{j+1}^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \tag{B.1}$$

in which the jth column is distinguished, either by having elements which differ from those in the other columns or by being omitted. As written in the above matrix, the meaning is clear for $j = 2, 3, \ldots n-1$, but not for $j = 1$ or $j = n$. To clarify the intent, which is that the jth column is replaced by something else (or omitted), we give the above matrix for $j = 1$:

$$\begin{vmatrix} 0 & u_2 & \cdots & u_n \\ 0 & u_2^{(1)} & \cdots & u_n^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & u_2^{(n-2)} & \cdots & u_n^{(n-2)} \\ g_n & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \tag{B.2}$$

and for $j = n$:

$$\begin{vmatrix} u_1 & \cdots & u_{n-1} & 0 \\ u_1^{(1)} & \cdots & u_{n-1}^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ u_1^{(n-2)} & \cdots & u_{n-1}^{(n-2)} & 0 \\ u_1^{(n-1)} & \cdots & u_{n-1}^{(n-1)} & g_n \end{vmatrix} \tag{B.3}$$

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For an example in which the jth column is to be omitted, we have the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} u_1 & \cdots & u_{j-1} & u_{j+1} & \cdots & u_n \\ u_1^{(1)} & \cdots & u_{j-1}^{(1)} & u_{j+1}^{(1)} & \cdots & u_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-2)} & \cdots & u_{j-1}^{(n-2)} & u_{j+1}^{(n-2)} & \cdots & u_n^{(n-2)} \end{vmatrix} \tag{B.4}$$

For $j = 1$ this is

$$\begin{vmatrix} u_2 & \cdots & u_n \\ u_2^{(1)} & \cdots & u_n^{(1)} \\ \vdots & \vdots & \vdots \\ u_2^{(n-2)} & \cdots & u_n^{(n-2)} \end{vmatrix} \tag{B.5}$$

and for $j = n$ we have

$$\begin{vmatrix} u_1 & \cdots & u_{n-1} \\ u_1^{(1)} & \cdots & u_{n-1}^{(1)} \\ \vdots & \vdots & \vdots \\ u_1^{(n-2)} & \cdots & u_{n-1}^{(n-2)} \end{vmatrix} \tag{B.6}$$

Appendix C
Wronskian Determinant

The Wronskian determinant for the nth order homogeneous linear differential equation given in (2.1) is defined by the determinant

$$\mathcal{W}(x) = \begin{vmatrix} u_1(x) & u_2(x) & \dots & u_n(x) \\ u_1^{(1)}(x) & u_2^{(1)}(x) & \dots & u_n^{(1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \dots & u_n^{(n-1)}(x) \end{vmatrix} \tag{C.1}$$

where $u_k(x),\ k = 1, 2, \dots n$, are the n linearly independent solutions of (2.1).

The Wronskian determinant obeys the simple first order equation

$$\mathcal{W}'(x) = -\frac{a_{n-1}(x)}{a_n(x)}\mathcal{W}(x) \tag{C.2}$$

from which, on integrating, we have

$$\mathcal{W}(x) = \mathcal{W}(x_0)\exp\left(-\int_{x_0}^{x}\frac{a_{n-1}(x')}{a_n(x')}dx'\right), \tag{C.3}$$

known as Abel's theorem.

A variety of derivations of the first order equation for the Wronskian, Eq. (C.2), as well as its integral form, Eq. (C.3), known as Abel's identity, can be found in the literature.[1] We have chosen one that provides as well a derivation of the first derivative of a determinant of any order.[2] It starts with Leibnitz's formula for the expansion of an $n \times n$ determinant, expressed as the sum of $n!$ products of its elements. For the determinant

[1] See, e.g., Hartman [20].

[2] A derivation of the expression for the nth derivative of a $j \times j$ determinant has been given by Christiano and Hall, [9].

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$$A(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix} \tag{C.4}$$

the Leibnitz formula is

$$A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \tag{C.5}$$

Here S_n is the set of all permutations of the integers $\{1, 2, \dots n\}$, the sum $\sum_{\sigma \in S_n}$ is over all permutations σ, and $\operatorname{sgn}(\sigma)$ is $+1$ for even permutations σ, -1 for odd permutations. In $a_{i\sigma(i)}$, the subscript $\sigma(i)$ is the element in position i in the permutation σ.

Taking the first derivative of A, we have

$$\begin{aligned} \frac{d}{dx}A &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{k=1}^{n} a'_{k\sigma(k)} \prod_{\substack{i=1 \\ i \neq k}}^{n} a_{i\sigma(i)} \\ &= \sum_{k=1}^{n} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{\substack{i=1 \\ i \neq k}}^{n} a'_{k\sigma(k)} a_{i\sigma(i)} \end{aligned} \tag{C.6}$$

The derivative $\frac{d}{dx}A$ is thus the sum of n determinants, each obtained by replacing one row in A with the derivative of the elements of that row, leaving all other rows unchanged:

$$\frac{d}{dx}A(x) = \sum_{k=1}^{n} \begin{vmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a'_{k1}(x) & a'_{k2}(x) & \dots & a'_{kn}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix} \tag{C.7}$$

Applying Eq. (C.7) for the derivative of a determinant to the Wronskian, Eq. (C.1), we see that the terms in (C.7) for $k = 1, 2, \dots, n-1$ vanish since in each of these terms the rows k and $k+1$ are identical.[3] We are thus left with the term for which $k = n$, giving

[3] We use here the property of determinants that if two or more rows (or columns) are identical, then the value of the determinant is zero.

$$\mathcal{W}'(x) = \begin{vmatrix} u_1(x) & u_2(x) & \ldots & u_n(x) \\ u_1^{(1)}(x) & u_2^{(1)}(x) & \ldots & u_n^{(1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-2)}(x) & u_2^{(n-2)}(x) & \ldots & u_n^{(n-2)}(x) \\ u_1^{(n)}(x) & u_2^{(n)}(x) & \ldots & u_n^{(n)}(x) \end{vmatrix} \tag{C.8}$$

From the differential equation, Eq. (2.1), obeyed by each of the functions $u_i(x)$, $\sum_{i=0}^{n} a_i(x)u_k^{(i)}(x) = 0,\ k = 1, 2, \ldots, n$, we can express each of the terms in the last row of (C.8) by

$$u_k^{(n)}(x) = -\frac{1}{a_n(x)} \sum_{i=0}^{n-1} a_i(x)u_k^{(i)}(x). \tag{C.9}$$

Next, multiplying each of the rows corresponding to $i = 0, 1, \ldots, n-2$ in (C.9) by $a_i(x)$ and adding them to the last row in (C.8) then cancels all but the term in (C.9) for which $i = n-1$. The terms in the last row are then $-(a_i(x)/a_n(x))u_k^{(n-1)}(x)$, thus giving the first order differential equation for the Wronskian, Eq. (C.2).[4]

[4]Here we have used two other properties of determinants: (1), if one adds to any one row (or column) a linear combination of all other rows (or columns), then the value of a determinant is unchanged, and (2), multiplication of each element in any row (or column) by the same constant multiplies the determinant by that constant.

Appendix D
Casoratian Determinant

The Casoratian determinant for the Nth order homogeneous linear difference equation given in (2.4), analogous to the Wronskian for differential equations, is defined by the determinant

$$\mathcal{C}(n) = \begin{vmatrix} u_1(n) & u_2(n) & \dots & u_N(n) \\ u_1(n+1) & u_2(n+1) & \dots & u_N(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(n+N-1) & u_2(n+N-1) & \dots & u_N(n+N-1) \end{vmatrix} \tag{D.1}$$

where $u_k(n),\ k = 1, 2, \dots N$, are the N linearly independent solutions of (2.4).

The Casoratian obeys the simple first order equation

$$\mathcal{C}(n+1) = (-1)^N \frac{p_0(n)}{p_N(n)} \mathcal{C}(n) \tag{D.2}$$

and by iteration we obtain Abel's theorem

$$\mathcal{C}(n) = (-1)^{N(n-n_0)} \mathcal{C}(n_0) \prod_{j=n_0}^{n-1} \frac{p_0(j)}{p_N(j)} \tag{D.3}$$

From (D.1) we have

$$\mathcal{C}(n+1) = \begin{vmatrix} u_1(n+1) & u_2(n+1) & \dots & u_N(n+1) \\ u_1(n+2) & u_2(n+2) & \dots & u_N(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(n+N-1) & u_2(n+N-1) & \dots & u_N(n+N-1) \\ u_1(n+N) & u_2(n+N) & \dots & u_N(n+N) \end{vmatrix} \tag{D.4}$$

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From the differential equation, Eq. (2.4), obeyed by each of the functions $u_k(n)$, $\sum_{i=0}^{N} p_i(n)u_k(n+i) = 0,\ \ k = 1, 2, \ldots, N$, we can express each of the terms in the last row of (D.4) by

$$u_k(n+N) = -\frac{1}{p_N(n)} \sum_{i=0}^{N-1} p_i(n)u_k(n+i). \tag{D.5}$$

Next, multiplying each of the rows corresponding to $i = 1, 2, \ldots, N-1$ in (D.5) by $p_i(n)$ and adding them to the last row in (D.4) then cancels in (D.5) all but the term for which $i = 0$. The terms in the last row are then $-(p_0(n)/p_N(n))u_k(n)$. We now successively interchange the last row with the row above it, moving it finally to the position of first row. This involves $N-1$ interchanges, each of which introduces a factor of -1.[5] The determinant (D.4) is then $(-1)^N p_0(n)/p_N(n)$ times the determinant (D.1), thus giving the first order difference equation for the Casoratian, Eq. (D.2).

[5] In addition to the properties of determinants cited in the footnotes in the appendix on the Wronskian, we have used the property that if two adjacent rows (or columns) are interchanged, the determinant is multiplied by -1.

Appendix E
Cramer's Rule

Cramer's rule is an explicit formula for the solution of a system of n linear equations for n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{E.1}$$

Expressed as a matrix equation, we have

$$\mathbf{Ax} = \mathbf{b} \tag{E.2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \tag{E.3}$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \tag{E.4}$$

and

$$\mathbf{b} = (b_1, b_2, \ldots, b_n)^{\mathrm{T}} \tag{E.5}$$

Cramer's rule expresses the solution, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, in terms of the determinants of the square coefficient matrix $\mathbf{A}$ and of the matrices $\mathbf{A}_i$ formed by replacing the ith column of $\mathbf{A}$ by the column vector $\mathbf{b}$; it is valid and gives a unique solution provided $\det(\mathbf{A}) \neq 0$:

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$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})} \qquad i = 1, 2, \ldots, n \tag{E.6}$$

A simple proof of Cramer's rule follows from two properties of determinants: (1) that multiplication of each element in any row (or column) by the same constant multiplies the determinant by that constant, and (2) that adding a constant times any row (or column) to a given row (or column) leaves the value of the determinant unchanged. We illustrate this here for the case of a 3 by 3 matrix and derive the expression for the first unknown, x_1: Here $|\mathbf{A}|$ is the determinant of a 3 by 3 matrix:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{E.7}$$

From the first property we have

$$x_1\,|\mathbf{A}| = \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} \\ a_{21}x_1 & a_{22} & a_{23} \\ a_{31}x_1 & a_{32} & a_{33} \end{vmatrix} \tag{E.8}$$

and from the second property

$$x_1\,|\mathbf{A}| = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{12} & a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{22} & a_{23} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = |\mathbf{A}_1| \tag{E.9}$$

Appendix F
Green's Function and the Superposition Principle

In Chap. 6 on Green's function we considered a differential operator L and boundary conditions B_k, operating on a function $y(x)$. Both the differential operator and the boundary conditions are linear operators, from which we have the superposition principle, which simplifies greatly the solution of the differential equation, particularly in the case of general boundary conditions. Although this principle applies equally to the case of an nth order differential equation, we illustrate it here for a second order differential equation.

We consider a function $y_1(x)$ satisfying the inhomogeneous equation $Ly_1(x) = f_1(x)$ and boundary conditions $B_1 y_1 = \gamma_{11}$ and $B_2 y_1 = \gamma_{12}$, and a second function $y_2(x)$ satisfying the inhomogeneous equation $Ly_2(x) = f_2(x)$ and boundary conditions $B_1 y_2 = \gamma_{21}$ and $B_2 y_2 = \gamma_{22}$, with the same differential operator L. For the second order equation in which the function $y(x)$ is considered over the interval $a \leq x \leq b$ the general boundary conditions B_1 and B_2 are of the form

$$\begin{aligned} B_1 y &= \alpha_{11} y(a) + \alpha_{12} y'(a) + \beta_{11} y(b) + \beta_{12} y'(b) = \gamma_1 \\ B_2 y &= \alpha_{21} y(a) + \alpha_{22} y'(a) + \beta_{21} y(b) + \beta_{22} y'(b) = \gamma_2 \end{aligned} \tag{F.1}$$

in which α_{jk} and β_{jk} are given constants. Since the differential operator and the boundary conditions are linear, we can write $L(c_1 y_1 + c_2 y_2) = c_1 f_1 + c_2 f_2$ and $B_1(c_1 y_1 + c_2 y_2) = c_1 \gamma_{11} + c_2 \gamma_{12}$ and $B_2(c_1 y_1 + c_2 y_2) = c_1 \gamma_{21} + c_2 \gamma_{22}$, where c_1 and c_2 are constants. This leads directly to the question of the uniqueness of the solution of the inhomogeneous equation: If we consider two solutions, y_1 and y_2, of the equation $Ly = f$ with identical boundary conditions: $B_1 y_1 = \gamma_1$, $B_2 y_1 = \gamma_2$, and $B_1 y_2 = \gamma_1$, $B_2 y_2 = \gamma_2$, then with $c_1 = -1$ and $c_2 = 1$ we obtain the homogeneous equation $L(y_2 - y_1) = 0$ and homogeneous boundary conditions $B_1(y_2 - y_1) = 0$ and $B_2(y_2 - y_1) = 0$. Written more simply, with $Y = y_2 - y_1$ we have $LY = 0$ and $B_1 Y = 0$, $B_2 Y = 0$. Thus if $LY = 0$ has only the trivial solution $Y \equiv 0$ then $y_1 \equiv y_2$, that is, there is at most one unique solution to the equation $Ly = f$ with the boundary conditions $B_1 y = \gamma_1$ and $B_2 y = \gamma_2$. We derive this solution in Chap. 6 on Green's function. On the other hand, if $Y(x)$ has a non-trivial solution then there is either no

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solution to the inhomogeneous equation $Ly = f$ or there are many solutions. We will look into this situation after considering the condition which determines whether there is only the trivial solution $Y \equiv 0$:

Any solution of the equation $LY = 0$ may be written as a sum of two linearly independent solutions of this equation:

$$Y = c_1 u_1 + c_2 u_2$$

Applying the boundary conditions $B_1 Y = B_2 Y = 0$, we have

$$\begin{aligned} c_1 B_1 u_1 + c_2 B_1 u_2 &= 0 \\ c_1 B_2 u_1 + c_2 B_2 u_2 &= 0 \end{aligned}$$

Therefore, if the determinant

$$\Delta = \begin{vmatrix} B_1 u_1 & B_1 u_2 \\ B_2 u_1 & B_2 u_2 \end{vmatrix} \neq 0 \tag{F.2}$$

then $c_1 = c_2 = 0$ and there is only the trivial solution $Y \equiv 0$. The condition $\Delta \neq 0$ is illustrated if we choose $u_1(x)$ and $u_2(x)$ to be linearly independent solutions of $LY = 0$ such that $B_1 u_1 = B_2 u_2 = 0$. This can be done if we write u_1 and u_2 in terms of two arbitrary linearly independent solutions v_1 and v_2 of $LY = 0$: Defining u_1 and u_2 in terms of v_1 and v_2 by

$$\begin{aligned} u_1 &= (B_1 v_2) v_1 - (B_1 v_1) v_2 \\ u_2 &= (B_2 v_2) v_1 - (B_2 v_1) v_2 \end{aligned} \tag{F.3}$$

we then have $B_1 u_1 = B_2 u_2 = 0$ and $B_1 u_2 = -B_2 u_1$, from which $\Delta = (B_1 u_2)^2 \geq 0$.

As an example which shows when there is one solution ($\Delta \neq 0$) and when there is either no solution or many solutions ($\Delta = 0$), we consider the equation $y''(x) + \lambda^2 y(x) = c$ (where c is a constant), with the boundary conditions $B_1 y = y(0) = 0$ and $B_2 y = y(\pi) = 0$. Referring to (F.1), this implies $\alpha_{11} = 1, \ \alpha_{12} = \beta_{11} = \beta_{12} = 0$ and $\beta_{21} = 1, \ \alpha_{21} = \alpha_{22} = \beta_{22} = 0$. If we choose, as solutions of the homogeneous equation $y''(x) + \lambda^2 y(x) = 0$, the two linearly independent functions

$$\begin{aligned} u_1(x) &= \sin \lambda x \\ u_2(x) &= \cos \lambda\pi \sin \lambda x - \sin \lambda\pi \cos \lambda x \end{aligned} \tag{F.4}$$

we then have $B_1 u_1 = u_1(0) = 0, \ B_2 u_2 = u_2(\pi) = 0$ and $B_2 u_1 = \sin \lambda\pi, \ B_1 u_2 = -\sin \lambda\pi$, from which, from (F.2), $\Delta = \sin^2 \lambda\pi$. Thus if $\lambda \neq 1, 2, 3, \ldots$, then $\Delta > 0$ and there is only the trivial solution to $y''(x) + \lambda^2 y(x) = 0$ with boundary conditions $y(0) = y(\pi) = 0$. The inhomogeneous equation $y''(x) + \lambda^2 y(x) = c$ then has the unique solution

$$y(x) = \frac{c}{\lambda^2}\left[(1 - \cos\lambda x) - \frac{(1 - \cos\lambda\pi)}{\sin\lambda\pi}\sin\lambda x\right] \tag{F.5}$$

On the other hand, if $\lambda = 1, 2, 3, \ldots$ then $\Delta = 0$ and the homogeneous equation $y''(x) + \lambda^2 y(x) = 0$ with homogeneous boundary conditions $y(0) = y(\pi) = 0$ has the non-trivial solution $y(x) = A\sin\lambda x$; (λ is an eigenvalue of this equation and $A\sin\lambda x$ is the corresponding eigenfunction). The inhomogeneous equation then has the general solution

$$y(x) = \frac{c}{\lambda^2}\left[(1 - \cos\lambda x) + A\sin\lambda x\right] \tag{F.6}$$

where A is an arbitrary constant. Then if $\lambda = 1, 3, 5, \ldots,$ the boundary condition $y(\pi) = 0$ can not be satisfied, and there is no solution to the inhomogeneous equation. However, if $\lambda = 2, 4, 6, \ldots,$ the general solution just given to the inhomogeneous equation satisfies the boundary conditions with arbitrary A: we have many solutions.

For the nth order equation, the extension of (F.2) giving the necessary and sufficient condition in order that $Ly = f$ with boundary conditions $B_k y = 0 \quad (k = 1, 2, \ldots n)$ have a unique non-trivial solution is $\Delta \neq 0$ where

$$\Delta = \begin{vmatrix} B_1u_1 & B_1u_2 & \cdots & B_1u_n \\ B_2u_1 & B_2u_2 & \cdots & B_2u_n \\ \vdots & \vdots & \vdots & \vdots \\ B_nu_1 & B_nu_2 & \cdots & B_nu_n \end{vmatrix} \tag{F.7}$$

(See [33]).

Appendix G
Inverse Laplace Transforms and Inverse Generating Functions

As discussed in Chap. 7 on generating functions, z-transforms and Laplace transforms, the solution of linear differential and difference equations generally requires the inverse of the derived generating function or Laplace transform. In this appendix we derive the inverse of a few generating functions and Laplace transforms that are of particular use in the solution of second order linear differential and difference equations with linear coefficients.

(1) $G(\omega) = \sum_{n=0}^{\infty} y(n)\omega^n = (a-\omega)^{-\alpha}$, in which we assume that α is not a negative integer or zero.

Writing

$$(a-\omega)^{-\alpha} = a^{-\alpha}\left(1-\frac{\omega}{a}\right)^{-\alpha} = a^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\left(\frac{\omega}{a}\right)^n \tag{G.1}$$

we obtain the inverse of the generating function $G(w) = (a-\omega)^{-\alpha}$, namely

$$y(n) = \mathcal{G}^{-1}G(\omega) = a^{-\alpha}\frac{(\alpha)_n}{a^n n!} \tag{G.2}$$

For the particular case in which α is a positive integer: $\alpha = m+1$, where $m = 0, 1, 2, \ldots$, we have

$$\frac{1}{(a-\omega)^{m+1}} = \sum_{n=0}^{\infty}\frac{\binom{n+m}{m}}{a^{n+m+1}}\omega^n \tag{G.3}$$

and hence

$$y(n) = \mathcal{G}^{-1}\left\{\frac{1}{(a-\omega)^{m+1}}\right\} = \frac{1}{a^{n+m+1}}\binom{n+m}{m} \tag{G.4}$$

as given previously in Eq. (7.56).

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We consider next the generating function

(2) $G(\omega) = (a-\omega)^{-\alpha}(b-\omega)^{-\beta}$, in which we assume that neither α nor β is a negative integer.

We assume further that neither α nor β is zero, and that $a \neq b$, since any of these choices returns us to the simpler generating function just considered. We now have

$$G(\omega) = (a-\omega)^{-\alpha}(b-\omega)^{-\beta} = a^{-\alpha}b^{-\beta}\sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!}\left(\frac{\omega}{a}\right)^l \sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}\left(\frac{\omega}{b}\right)^m \quad \text{(G.5)}$$

We then let $n = l + m$, from which

$$(a-\omega)^{-\alpha}(b-\omega)^{-\beta} = a^{-\alpha}b^{-\beta}\sum_{n=0}^{\infty}\frac{\omega^n}{b^n}\sum_{l=0}^{n}\frac{(\alpha)_l(\beta)_{n-l}\,b^l}{l!(n-l)!a^l} \quad \text{(G.6)}$$

or, alternatively,

$$(a-\omega)^{-\alpha}(b-\omega)^{-\beta} = a^{-\alpha}b^{-\beta}\sum_{n=0}^{\infty}\frac{\omega^n}{a^n}\sum_{m=0}^{n}\frac{(\beta)_m(\alpha)_{n-m}\,a^m}{m!(n-m)!b^m} \quad \text{(G.7)}$$

Here, in (G.6) and (G.7), we write

$$\frac{1}{(n-l)!} = \frac{(-1)^l(-n)_l}{n!}$$
$$(\beta)_{n-l} = (-1)^l\frac{(\beta)_n}{(1-\beta-n)_l} \quad \text{(G.8)}$$

and

$$\frac{1}{(n-m)!} = \frac{(-1)^m(-n)_m}{n!}$$
$$(\alpha)_{n-m} = (-1)^m\frac{(\alpha)_n}{(1-\alpha-n)_m} \quad \text{(G.9)}$$

respectively.

From (G.6) we then have

$$(a-\omega)^{-\alpha}(b-\omega)^{-\beta} = a^{-\alpha}b^{-\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n\omega^n}{b^n n!}\sum_{l=0}^{n}\frac{(-n)_l(\alpha)_l}{l!(1-\beta-n)_l}\left(\frac{b}{a}\right)^l$$
$$= a^{-\alpha}b^{-\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n}{b^n n!}\,{}_2F_1\left(-n,\alpha;1-\beta-n;\tfrac{b}{a}\right)\omega^n \quad \text{(G.10)}$$

and from (G.7),

$$(a-\omega)^{-\alpha}(b-\omega)^{-\beta} = a^{-\alpha}b^{-\beta}\sum_{n=0}^{\infty}\frac{(\alpha)_n\omega^n}{a^n n!}\sum_{m=0}^{n}\frac{(-n)_m(\beta)_m}{m!(1-\alpha-n)_m}\left(\frac{a}{b}\right)^m$$
$$= a^{-\alpha}b^{-\beta}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{a^n n!}\,{}_2F_1\left(-n,\beta;1-\alpha-n;\tfrac{a}{b}\right)\omega^n \qquad \text{(G.11)}$$

The corresponding inverses of the generating function are

$$y(n) = \mathcal{G}^{-1}G(\omega) = a^{-\alpha}b^{-\beta}\frac{(\beta)_n}{b^n n!}\,{}_2F_1\left(-n,\alpha;1-\beta-n;\tfrac{b}{a}\right) \qquad \text{(G.12)}$$

from (G.10), and

$$y(n) = \mathcal{G}^{-1}G(\omega) = a^{-\alpha}b^{-\beta}\frac{(\alpha)_n}{a^n n!}\,{}_2F_1\left(-n,\beta;1-\alpha-n;\tfrac{a}{b}\right) \qquad \text{(G.13)}$$

from (G.11).

Here (G.12) is valid if either $\beta \neq$ integer or $\beta = 2, 3, \ldots,$ (i.e., a positive integer >1), and α is arbitrary; (G.13) is valid if either $\alpha \neq$ integer or $\alpha = 2, 3, \ldots,$ (i.e., a positive integer >1), and β is arbitrary. Thus for the following choices for α and β, we may choose either (G.12) or (G.13), or both, as solutions.

$$\begin{aligned}
\alpha \neq \text{integer},\ \beta \neq \text{integer}: &\quad \text{(G.12) or (G.13)}.\\
\alpha \neq \text{integer},\ \beta \text{ arbitary}: &\quad \text{(G.13)}.\\
\beta \neq \text{integer},\ \alpha \text{ arbitary}: &\quad \text{(G.12)}.\\
\alpha,\ \beta = 2,3,\ldots \text{(i.e., both positive integers} >1): &\quad \text{(G.12) or (G.13)}.\\
\alpha = 1,\ \beta = 2,3,\ldots \text{(i.e., a positive integer} >1): &\quad \text{(G.12)}.\\
\beta = 1,\ \alpha = 2,3,\ldots \text{(i.e., a positive integer} >1): &\quad \text{(G.13)}.
\end{aligned} \qquad \text{(G.14)}$$

We consider next the Laplace transform

(3) $F(s) = (s-\alpha)^{-\beta}$.

From [13, Sect. 1.1 (5)],

$$\int_0^\infty e^{-sx}x^{\beta-1}\,dx = \Gamma(\beta)s^{-\beta} \qquad \Re\,\beta > 0, \qquad \text{(G.15)}$$

so that

$$\frac{1}{\Gamma(\beta)}\int_0^\infty e^{-sx}e^{\alpha x}x^{\beta-1}\,dx = (s-\alpha)^{-\beta} \qquad \text{(G.16)}$$

We thus have the inverse transform of $F(s) = (s - \alpha)^{-\beta}$, namely

$$\mathfrak{L}^{-1} F(s) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sx}}{(s-\alpha)^{\beta}}\, ds = \frac{x^{\beta-1}}{\Gamma(\beta)} e^{\alpha x} \tag{G.17}$$

Next we consider the Laplace transform

(4) $F(s) = (s - \alpha_1)^{-\beta_1}(s - \alpha_2)^{-\beta_2}$.

The inverse Laplace transform is

$$\begin{aligned}
\mathfrak{L}^{-1} F(s) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sx}}{(s-\alpha_1)^{\beta_1}(s-\alpha_2)^{\beta_2}}\, ds \\
&= \frac{1}{2\pi i} e^{\alpha_1 x} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx} s^{-\beta_1}(s - (\alpha_2 - \alpha_1)^{-\beta_2}\, ds \\
&= \frac{1}{2\pi i} (\alpha_2 - \alpha_1)^{-\beta_1 - \beta_2 + 1} e^{\alpha_1 x} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{(\alpha_2 - \alpha_1)x\sigma} \sigma^{-\beta_1} (\sigma - 1)^{-\beta_2}\, d\sigma \\
&= \frac{1}{2\pi i} (\alpha_2 - \alpha_1)^{-\beta_1 - \beta_2 + 1} e^{\alpha_1 x} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{(\alpha_2 - \alpha_1)x\sigma} \sigma^{-\beta_1 - \beta_2} (1 - \sigma^{-1})^{-\beta_2}\, d\sigma
\end{aligned} \tag{G.18}$$

From [13, Sect. 6.10 (6)],

$$\int_0^{\infty} e^{-sx} x^{c-1} {}_1F_1(a; c; x)\, dx = \Gamma(c) s^{-c} (1 - s^{-1})^{-a} \qquad \Re\, c > 0, \quad \Re\, s > 1, \tag{G.19}$$

from which we have the inverse transform

$$\frac{1}{2\pi i} \Gamma(c) \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx} s^{-c} (1 - s^{-1})^{-a}\, ds = x^{c-1} {}_1F_1(a; c; x) \tag{G.20}$$

Thus,

$$\begin{aligned}
\mathfrak{L}^{-1} F(s) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sx}}{(s-\alpha_1)^{\beta_1}(s-\alpha_2)^{\beta_2}}\, ds \\
&= \frac{x^{\beta_1 + \beta_2 - 1}}{\Gamma(\beta_1 + \beta_2)} e^{\alpha_1 x} {}_1F_1(\beta_2; \beta_1 + \beta_2; (\alpha_2 - \alpha_1)x) \\
&= \frac{x^{\beta_1 + \beta_2 - 1}}{\Gamma(\beta_1 + \beta_2)} e^{\alpha_2 x} {}_1F_1(\beta_1; \beta_1 + \beta_2; (\alpha_1 - \alpha_2)x)
\end{aligned} \tag{G.21}$$

Appendix H
Hypergeometric Function

In this appendix we give a few of the transformations of the hypergeometric function ${}_2F_1(a, b; c; z)$ which have been useful in the analysis presented in this work.

$$ {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z) \tag{H.1} $$

$$ \begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \tfrac{z}{z-1}\right) \\ &= (1-z)^{-b}\,{}_2F_1\left(c-a, b; c; \tfrac{z}{z-1}\right) \end{aligned} \tag{H.2} $$

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z) \tag{H.3} $$

For (H.1) see [13, Sect. 2.1.2, p. 57]; for (H.2) and (H.3) see [13, Sect. 2.1.4, (22) and (23), p. 64].

$$ \begin{aligned} {}_2F_1(a, b; c; z) &= e^{i\pi a}\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a+b-c+1)\Gamma(c-a)} z^{-a}\,{}_2F_1\left(a, a-c+1; a+b-c+1; 1-\tfrac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)} z^{a-c}(1-z)^{c-a-b}\,{}_2F_1\left(1-a, c-a; b-a+1; \tfrac{1}{z}\right) \\ &= e^{i\pi b}\frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(c-b)} z^{-b}\,{}_2F_1\left(b, b-c+1; a+b-c+1; 1-\tfrac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a-c+1)}{\Gamma(b)\Gamma(a-b+1)} z^{b-c}(1-z)^{c-a-b}\,{}_2F_1\left(1-b, c-b; a-b+1; \tfrac{1}{z}\right) \end{aligned} \tag{H.4} $$

For (H.4) see [40, Sect. 4, (26), p. 447] or [13, Sect. 2.9, (26), p. 106].

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Appendix I
Confluent Hypergeometric Functions

In this appendix we list the confluent hypergeometric functions which result from different choices of integration path for the integrals given earlier in Eqs. (7.200)–(7.203). Here, from (7.192)–(7.194), (7.199), and (7.206),

$$\gamma_i = \frac{c_i}{d_i}, \qquad \beta_0 = \gamma_0, \qquad \beta_1 = \gamma_0 - \gamma_2, \qquad z = \gamma_0 - 2\gamma_1 + \gamma_2$$

From (7.200), we have the integral

$$\int s^{\beta_1-\beta_0-x-1}(1-s)^{\beta_0+x-1}e^{zs}\,ds = \int s^{-\gamma_2-x-1}(1-s)^{\gamma_0+x-1}e^{zs}\,ds$$

leading to the confluent hypergeometric function ${}_1F_1$:

$$\begin{aligned}\int_0^1 s^{-\gamma_2-x-1}(1-s)^{\gamma_0+x-1}e^{zs}\,ds &= \frac{\Gamma(-\gamma_2-x)\Gamma(\gamma_0+x)}{\Gamma(\beta_1)}\,{}_1F_1(-\gamma_2-x;\beta_1;z)\\ &= \frac{\Gamma(-\gamma_2-x)\Gamma(\gamma_0+x)}{\Gamma(\beta_1)}e^z\,{}_1F_1(\gamma_0+x;\beta_1;-z)\end{aligned} \tag{I.1}$$

$$\begin{aligned}\int_0^{(1+)} s^{-\gamma_2-x-1}(1-s)^{\beta_0+x-1}e^{zs}\,ds &= 2\pi i\, e^{(\gamma_0+x-1)\pi i}\frac{\Gamma(-\gamma_2-x)}{\Gamma(1-\gamma_0-x)\Gamma(\beta_1)}\,{}_1F_1(-\gamma_2-x;\beta_1;z)\\ &= 2\pi i\, e^{(\gamma_0+x-1)\pi i}\frac{\Gamma(-\gamma_2-x)}{\Gamma(1-\gamma_0-x)\Gamma(\beta_1)}e^z\,{}_1F_1(\gamma_0+x;\beta_1;-z)\end{aligned} \tag{I.2}$$

$$\begin{aligned}\int_1^{(0+)} s^{-\gamma_2-x-1}(1-s)^{\gamma_0+x-1}e^{zs}\,ds &= 2\pi i\, e^{(-\gamma_2-x)\pi i}\frac{\Gamma(\gamma_0+x)}{\Gamma(\gamma_2+x+1)\Gamma(\beta_1)}\,{}_1F_1(-\gamma_2-x;\beta_1;z)\\ &= 2\pi i\, e^{(\beta_1-\beta_0-x)\pi i}\frac{\Gamma(\gamma_0+x)}{\Gamma(\gamma_2+x+1)\Gamma(\beta_1)}e^z\,{}_1F_1(\gamma_0+x;\beta_1;-z)\end{aligned} \tag{I.3}$$

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$$\int_{\alpha}^{(0+,1+,0-,1-)} s^{-(\gamma_2+x+1)}(1-s)^{(\gamma_0+x-1)}e^{zs}\,ds = 4\pi^2 \frac{e^{\beta_1\pi i}\,{}_1F_1(-\gamma_2-x;\beta_1;z)}{\Gamma(\gamma_2+x+1)\Gamma(1-\gamma_0-x)\Gamma(\beta_1)}$$
$$= 4\pi^2 \frac{e^{\beta_1\pi i}e^{z}\,{}_1F_1(\gamma_0+x;\beta_1;-z)}{\Gamma(\gamma_2+x+1)\Gamma(1-\gamma_0-x)\Gamma(\beta_1)} \quad (I.4)$$

From (7.201), we have

$$\int_{-\infty}^{(0+,1+)} s^{-(2-\beta_1)}(1-\tfrac{1}{s})^{\beta_0+x-1}e^{zs}\,ds = 2\pi i\, z^{1-\beta_1}\frac{1}{\Gamma(2-\beta_1)}\,{}_1F_1(1-\gamma_0-x;2-\beta_1;z)$$
$$= 2\pi i\, z^{1-\beta_1}\frac{1}{\Gamma(2-\beta_1)}e^{z}\,{}_1F_1(\gamma_2+x+1;2-\beta_1;-z) \quad (I.5)$$

Here, in each of the last five equations, the second confluent hypergeometric function is obtained using the Kummer transformation ([36, Sect. 13.2(vii), Eq. 13.2.39]):

$${}_1F_1(a;c;z) = e^{z}\,{}_1F_1(c-a;c;-z) \quad (I.6)$$

From (7.202), we have, for the integral $\int s^{\beta_1-\beta_0-x-1}(1+s)^{\beta_0+x-1}e^{-zs}\,ds$ leading to confluent hypergeometric functions of the form $U(a;c;z)$,

$$\int_{0}^{\infty} s^{\beta_1-\beta_0-x-1}(1+s)^{\beta_0+x-1}e^{-zs}\,ds = \Gamma(-\gamma_2-x)U(-\gamma_2-x;\beta_1;z)$$
$$= \Gamma(-\gamma_2-x)z^{1-\beta_1}U(1-\gamma_0-x;2-\beta_1;z) \quad (I.7)$$

$$\int_{\infty}^{(0+)} s^{\beta_1-\beta_0-x-1}(1+s)^{\beta_0+x-1}e^{-zs}\,ds = 2\pi i\frac{e^{(-\gamma_2-x)\pi i}}{\Gamma(\gamma_2+x+1)}U(-\gamma_2-x;\beta_1;z)$$
$$= 2\pi i\frac{e^{(-\gamma_2-x)\pi i}}{\Gamma(\gamma_2+x+1)}z^{1-\beta_1}U(1-\gamma_0-x;2-\beta_1;z) \quad (I.8)$$

and from (7.203), we have, for the integral $\int s^{\beta_0+x-1}(1+s)^{\beta_1-\beta_0-x-1}e^{zs}\,ds$,

$$\int_{0}^{\infty} s^{\beta_0+x-1}(1+s)^{\beta_1-\beta_0-x-1}e^{zs}\,ds = \Gamma(\gamma_0+x)U(\gamma_0+x;\beta_1;-z)$$
$$= \Gamma(\gamma_0+x)(-z)^{1-\beta_1}U(\gamma_2+x+1;2-\beta_1;-z) \quad (I.9)$$

$$\int_{\infty}^{(0+)} s^{\beta_0+x-1}(1+s)^{\beta_1-\beta_0-x-1}e^{zs}\,ds = 2\pi i\frac{e^{(\gamma_0+x)\pi i}}{\Gamma(1-\gamma_0-x)}U(\gamma_0+x;\beta_1;-z)$$
$$= 2\pi i\frac{e^{(\gamma_0+x)\pi i}}{\Gamma(1-\gamma_0-x)}(-z)^{1-\beta_1}U(\gamma_2+x+1;2-\beta_1;-z) \quad (I.10)$$

Here, in each of the last four equations, the second confluent hypergeometric function is obtained using the Kummer transformation ([36, Sect. 13.2(vii), Eq. 13.2.40]):

$$U(a; c; z) = z^{1-c} U(a - c + 1; 2 - c; z) \tag{I.11}$$

In order to obtain a solution to the difference equation that satisfies arbitrary initial conditions $y(x_0)$ and $y(x_0 + 1)$, we require two linearly independent solutions of the difference equation, $y_1(x)$ and $y_2(x)$. We therefore wish to chose, from among the solutions listed above, pairs of solutions which are linearly independent. The particular pair of solutions chosen will depend on the value of the parameters $\gamma_2 + x + 1$ and $\gamma_0 + x - 1$. It is important to note, however, that the recursion defined by (7.205) fails if $\gamma_2 + x + 1 = 0$ if we assume to have two independent initial conditions, for example $w(0)$ and $w(1)$. The condition that the two solutions to (7.205) be linearly independent is that their Casoratian, $\mathcal{C}(x)$, be non-zero: $\mathcal{C}(x) = y_1(x)\, y_2(x + 1) - y_1(x + 1)\, y_2(x) \neq 0$. Noting that the Wronskian of the various solutions that we are considering is relatively well-known, (see, e.g., [36, Sect. 13.2(vi)]), we determine the Casoratian by expressing it in terms of the Wronskian by the use of raising and lowering operators; these relate the differential properties of the variable z in the solution to the discrete property of the parameter x.

From (I.1)–(I.10) we define the ten functions $F1 - F5t$ of the form ${}_1F_1(a; c; z)$ and the eight functions $U1 - U4t$ of the form $U(a; c; z)$, from which we obtain linearly independent pairs of solutions of the difference equation (7.153) in which $d_1^2 - 4d_0d_2 = 0$, each pair being valid for given values of the variable z and the parameters $\gamma_0 + x$ and $\gamma_2 + x + 1$, presented in detail in Table 7.1. As noted previously, in defining $F1 - F5t$ and $U1 - U4t$ we have neglected factors independent of x but retain the gamma functions $\Gamma(\beta_1)$ and $\Gamma(2 - \beta_1)$ in the denominators; the functions of the form ${}_1F_1(a; c; z)/\Gamma(c)$ then remain well-defined when the parameter c is zero or a negative integer. We then have, from (I.1)–(I.10),

$$F1 = \frac{\Gamma(-\gamma_2 - x)\Gamma(\gamma_0 + x)}{\Gamma(\beta_1)} e^{i\pi x}\, {}_1F_1(-\gamma_2 - x; \beta_1; z) \tag{I.12}$$

$$F1t = \frac{\Gamma(-\gamma_2 - x)\Gamma(\gamma_0 + x)}{\Gamma(\beta_1)} e^{i\pi x}\, {}_1F_1(\gamma_0 + x; \beta_1; -z) \tag{I.13}$$

$$F2 = \frac{\Gamma(-\gamma_2 - x)}{\Gamma(1 - \gamma_0 - x)\Gamma(\beta_1)}\, {}_1F_1(-\gamma_2 - x; \beta_1; z) \tag{I.14}$$

$$F2t = \frac{\Gamma(-\gamma_2 - x)}{\Gamma(1 - \gamma_0 - x)\Gamma(\beta_1)}\, {}_1F_1(\gamma_0 + x; \beta_1; -z) \tag{I.15}$$

$$F3 = \frac{\Gamma(\gamma_0 + x)}{\Gamma(\gamma_2 + x + 1)\Gamma(\beta_1)}\, {}_1F_1(-\gamma_2 - x; \beta_1; z) \tag{I.16}$$

$$F3t = \frac{\Gamma(\gamma_0 + x)}{\Gamma(\gamma_2 + x + 1)\Gamma(\beta_1)}\, {}_1F_1(\gamma_0 + x; \beta_1; -z) \tag{I.17}$$

$$F4 = \frac{e^{i\pi x}}{\Gamma(\gamma_2 + x + 1)\Gamma(1 - \gamma_0 - x)} \frac{{}_1F_1(-\gamma_2 - x; \beta_1; z)}{\Gamma(\beta_1)} \tag{I.18}$$

$$F4t = \frac{e^{i\pi x}}{\Gamma(\gamma_2 + x + 1)\Gamma(1 - \gamma_0 - x)} \frac{{}_1F_1(\gamma_0 + x; \beta_1; -z)}{\Gamma(\beta_1)} \tag{I.19}$$

$$F5 = \frac{{}_1F_1(1 - \gamma_0 - x; 2 - \beta_1; z)}{\Gamma(2 - \beta_1)} \tag{I.20}$$

$$F5t = \frac{{}_1F_1(\gamma_2 + x + 1; 2 - \beta_1; -z)}{\Gamma(2 - \beta_1)} \tag{I.21}$$

$$U1 = \Gamma(-\gamma_2 - x)U(-\gamma_2 - x; \beta_1; z) \tag{I.22}$$

$$U1t = \Gamma(-\gamma_2 - x)U(1 - \gamma_0 - x; 2 - \beta_1; z) \tag{I.23}$$

$$U2 = \frac{e^{i\pi x}}{\Gamma(\gamma_2 + x + 1)} U(-\gamma_2 - x; \beta_1; z) \tag{I.24}$$

$$U2t = \frac{e^{i\pi x}}{\Gamma(\gamma_2 + x + 1)} U(1 - \gamma_0 - x; 2 - \beta_1; z) \tag{I.25}$$

$$U3 = \Gamma(\gamma_0 + x)U(\gamma_0 + x; \beta_1; -z) \tag{I.26}$$

$$U3t = \Gamma(\gamma_0 + x)U(\gamma_2 + x + 1; 2 - \beta_1; -z) \tag{I.27}$$

$$U4 = \frac{e^{i\pi x}}{\Gamma(1 - \gamma_0 - x)} U(\gamma_0 + x; \beta_1; -z) \tag{I.28}$$

$$U4t = \frac{e^{i\pi x}}{\Gamma(1 - \gamma_0 - x)} U(\gamma_2 + x + 1; 2 - \beta_1; -z) \tag{I.29}$$

Appendix J
Solutions of the Second Kind

In this appendix we present two linearly independent solutions of the difference equation given in (7.205), namely

$$(\gamma_2 + x + 1)\, w(x+1) - 2\,(\gamma_1 + x)\, w(x) + (\gamma_0 + x - 1)\, w(x-1) = 0, \quad \text{(J.1)}$$

for the case in which $\gamma_2 + x + 1$ and $\gamma_0 + x$ are positive integers, from which $\beta_1 \equiv \gamma_0 - \gamma_2$ is an integer. Referring to Table 7.1, we consider first the case in which $\beta_1 \neq 0, -1, -2, \ldots$ (and hence $\beta_1 = 1, 2, 3, \ldots$). Then, for $z \equiv \gamma_0 - 2\gamma_1 + \gamma_2 < 0$, two linearly independent solutions are

$$U3 = \Gamma(\gamma_0 + x) U(\gamma_0 + x; \beta_1; -z) \quad \text{(J.2)}$$

and

$$F3t = \frac{\Gamma(\gamma_0 + x)}{\Gamma(\gamma_2 + x + 1)\Gamma(\beta_1)}\, {}_1F_1(\gamma_0 + x; \beta_1; -z) \quad \text{(J.3)}$$

Now from [36, Sect. 13.2(i), Eq. 13.2.9], with $\beta_1 = n+1$, $a = \gamma_0 + x$, and $a - n = \gamma_2 + x + 1$

$$\begin{aligned} U3 &= \Gamma(\gamma_0 + x) U(\gamma_0 + x; \beta_1; -z) \\ &= \frac{(-1)^{n+1}\Gamma(\gamma_0 + x)}{n!\Gamma(\gamma_2 + x + 1)} \sum_{k=0}^{\infty} \frac{(\gamma_0 + x)_k (-z)^k}{(n+1)_k k!} \big\{ \ln(-z) + \Psi(\gamma_0 + x + k) - \Psi(1+k) - \Psi(n+1+k) \big\} \\ &\quad + \sum_{k=1}^{n} \frac{(k-1)!(1+k-\gamma_0 - x)_{n-k}}{(n-k)!} (-z)^{-k} \\ &= (-1)^{n+1} F3t \ln(-z) \\ &\quad + \frac{(-1)^{n+1}\Gamma(\gamma_0 + x)}{n!\Gamma(\gamma_2 + x + 1)} \sum_{k=0}^{\infty} \frac{(\gamma_0 + x)_k (-z)^k}{(n+1)_k k!} \big\{ \Psi(\gamma_0 + x + k) - \Psi(1+k) - \Psi(n+1+k) \big\} \\ &\quad + \sum_{k=1}^{n} \frac{(k-1)!(1+k-\gamma_0 - x)_{n-k}}{(n-k)!} (-z)^{-k} \end{aligned} \quad \text{(J.4)}$$

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Here, since both U3 and F3t are solutions of the difference equation for $w(x)$, the remaining terms on the right hand side of the above equation also satisfy this equation, and give a linearly independent solution to that equation.

Furthermore, these terms give a valid solution for both $z < 0$ and $z > 0$. Thus, for $\beta_1 \equiv \gamma_0 - \gamma_2 = 1 + n = 1, 2, 3, \ldots,$ two linearly independent solutions, valid for all z, are F3t and

$$\frac{(-1)^{n+1}\Gamma(\gamma_0 + x)}{n!\Gamma(\gamma_2 + x + 1)} \sum_{k=0}^{\infty} \frac{(\gamma_0 + x)_k (-z)^k}{(n+1)_k k!} \big\{\Psi(\gamma_0 + x + k) - \Psi(1 + k) - \Psi(n + 1 + k)\big\}$$
$$+ \sum_{k=1}^{n} \frac{(k-1)!(1 + k - \gamma_0 - x)_{n-k}}{(n-k)!} (-z)^{-k} \tag{J.5}$$

The case in which $\beta_1 \neq 2, 3, 4, \ldots,$ (and hence $\beta_1 = 1, 0, -1, -2, \ldots$), may be treated in similar fashion. For $z < 0$, two linearly independent solutions are

$$U3t = \Gamma(\gamma_0 + x) U(\gamma_2 + x + 1; 2 - \beta_1; -z) \tag{J.6}$$

and

$$F5t = \frac{{}_1F_1(\gamma_2 + x + 1; 2 - \beta_1; -z)}{\Gamma(2 - \beta_1)} \tag{J.7}$$

Again from [36, Sect. 13.2(i), Eq. 13.2.9], now with $2 - \beta_1 = n + 1$, $a = \gamma_2 + x + 1$, and $a - n = \gamma_0 + x$,

$$\begin{aligned}
U3t &= \Gamma(\gamma_0 + x) U(\gamma_2 + x + 1; 2 - \beta_1; -z) \\
&= \frac{(-1)^{n+1}}{n!} \sum_{k=0}^{\infty} \frac{(\gamma_2 + x + 1)_k (-z)^k}{(n+1)_k k!} \big\{\ln(-z) + \Psi(\gamma_2 + x + 1 + k) - \Psi(1 + k) - \Psi(n + 1 + k)\big\} \\
&\quad + \frac{\Gamma(\gamma_0 + x)}{\Gamma(\gamma_2 + x + 1)} \sum_{k=1}^{n} \frac{(k-1)!(k - \gamma_2 - x)_{n-k}}{(n-k)!} (-z)^{-k} \\
&= (-1)^{n+1} F5t \ln(-z) \\
&\quad + \frac{(-1)^{n+1}}{n!} \sum_{k=0}^{\infty} \frac{(\gamma_2 + x + 1)_k (-z)^k}{(n+1)_k k!} \big\{\Psi(\gamma_2 + x + 1 + k) - \Psi(1 + k) - \Psi(n + 1 + k)\big\} \\
&\quad + \frac{\Gamma(\gamma_0 + x)}{\Gamma(\gamma_2 + x + 1)} \sum_{k=1}^{n} \frac{(k-1)!(k - \gamma_2 - x)_{n-k}}{(n-k)!} (-z)^{-k}
\end{aligned} \tag{J.8}$$

Here, since both U3t and F5t are solutions of the difference equation for $w(x)$, the remaining terms on the right hand side of the above equation also satisfy this equation, and give a linearly independent solution to that equation. Furthermore, these terms give a valid solution for both $z < 0$ and $z > 0$. Thus, for $\beta_1 \equiv \gamma_0 - \gamma_2 = 1 - n = 1, 0, -1, -2, \ldots,$ two linearly independent solutions, valid for all z, are F5t and

$$\frac{(-1)^{n+1}}{n!}\sum_{k=0}^{\infty}\frac{(\gamma_2+x+1)_k(-z)^k}{(n+1)_k k!}\left\{\Psi(\gamma_2+x+1+k)-\Psi(1+k)-\Psi(n+1+k)\right\}$$
$$+\frac{\Gamma(\gamma_0+x)}{\Gamma(\gamma_2+x+1)}\sum_{k=1}^{n}\frac{(k-1)!(k-\gamma_2-x)_{n-k}}{(n-k)!}(-z)^{-k} \tag{J.9}$$

An alternative approach which derives a second solution in the form of a polynomial in x is given in [37]. For $\beta_1 \neq 0, -1, -2, \ldots$, we write $\beta_1 = n+1 = 1, 2, \ldots$; $\gamma_2 + x = N = 0, 1, 2, \ldots$ and choose F3 as the first solution to the difference equation, writing

$$\begin{aligned} F3 &= \frac{\Gamma(\gamma_0+x)}{\Gamma(\gamma_2+x+1)\Gamma(\beta_1)}\,{}_1F_1(-\gamma_2-x;\beta_1;z) \\ &= \frac{(N+n)!}{n!N!}\,{}_1F_1(-N;n+1;z) \end{aligned} \tag{J.10}$$

We note that the function given here is the associated Laguerre polynomial (see [36, Sect. 13.6(v), Eq. 13.6.19 and Sect. 18.5(iii), Eq. 18.5.12]):

$$\frac{(N+n)!}{n!N!}\,{}_1F_1(-N;n+1;z) = L_N^{(n)}(z). \tag{J.11}$$

As shown in [34, Sect. 11, Eq. (4), p. 97], given a polynomial solution $y_n(z)$ to a differential equation of the hypergeometric type,[6] a solution of the second kind may be obtained as an extended Cauchy-integral:

$$Q_n(z) = \frac{1}{\rho(z)}\int_0^{\infty}\frac{y_n(s)\rho(s)}{s-z}\,ds, \tag{J.12}$$

where, for the confluent hypergeometric function ${}_1F_1(-N;n+1;z)$, satisfying the differential equation $zy''(z)+(n+1-z)y'(z)+Ny(z)=0$ considered here, $\rho(z)=z^n e^{-z}$ is a solution of $(z\rho(z))' = (n+1-z)\rho(z)$. This approach is developed in detail in [37], leading to a second linearly independent polynomial solution in closed form which satisfies both the differential equation and the difference equation (7.205) in each of the parameters. From [37, Eq. (2.12)], we then have two linearly independent polynomial solutions of the difference equation (7.205), in which $\gamma_2+x=N$; $n+1=\beta_1=\gamma_0-\gamma_2\neq 0,-1,-2,\ldots$ and $z=\gamma_0-2\gamma_1+\gamma_2$:

$$\frac{(N+n)!}{n!N!}\,{}_1F_1(-N;n+1;z) \tag{J.13}$$

and

[6] I.e., $\sigma(z)y''(z)+\tau(z)y'(z)+\lambda y(z)=0$. See [34, Sects. 2, 3, pp. 6–14].

$$\frac{(N+n)!}{n!N!}\sum_{k=0}^{N}\sum_{m=0}^{n+k-1}\frac{(-N)_k(n+k-1-m)!}{(n+1)_k\,k!}x^m\ . \tag{J.14}$$

For the case in which $\beta_1 \neq 2, 3, \ldots$ we write $2-\beta_1 = n+1 = 1, 2, \ldots;\ \gamma_0 + x - 1 = N = 0, 1, 2, \ldots$ and choose F5 as the first solution to the difference equation:

$$\begin{aligned} F5 &= \frac{{}_1F_1(1-\gamma_0-x; 2-\beta_1; z)}{\Gamma(2-\beta_1)} \\ &= \frac{{}_1F_1(-N; n+1; z)}{n!} \end{aligned} \tag{J.15}$$

The second, linearly independent solution of (7.205), is then

$$\frac{1}{n!}\sum_{k=0}^{N}\sum_{m=0}^{n+k-1}\frac{(-N)_k(n+k-1-m)!}{(n+1)_k\,k!}x^m. \tag{J.16}$$

Our second polynomial solution enables us to define a linearly independent associated Laguerre function of the second kind which satisfies the difference equations for the confluent hypergeometric function in each of its two parameters in terms of appropriately normalized polynomials. (See [37]).

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