# Appendices

These appendices are intended to provide a convenient reference for some prerequisite background material. Use only as needed!

# Appendix A
# Advanced Calculus

## A.1 Differentiation in $\mathbb{R}^N$

Here we'll review those parts of the theory of differentiation of vector-valued functions of several variables needed for our proof of the Brouwer Fixed-Point Theorem. For more details see, e.g., [101, Chap. 9], [2, Chap. 6] or the freely available online textbooks [110] and [118].

**Definition and Basic Properties.** We'll think of $\mathbb{R}^N$ as a space of column vectors, with $\mathbb{R}^N$-valued functions on a subset $V$ of $\mathbb{R}^N$ to be thought of as column vectors with each component a real-valued function on $V$. If $V$ is an open subset of $\mathbb{R}^N$, to say that a function $f\colon V \to \mathbb{R}^N$ is *differentiable* at a point $x_0 \in V$ means that there is a linear transformation on $\mathbb{R}^N$ (which we denote by $f'(x_0)$) such that

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - f'(x_0)h|}{|h|} = 0. \tag{A.1}$$

Suppose $f$ is differentiable at $x_0$. Then it's clear from the definition that $f$ is continuous at $x_0$. Furthermore, upon letting $h = te_j$ where $e_j$ is the $j$-th standard basis vector for $\mathbb{R}^N$ (the vector with 1 in the $j$-th position and zeros elsewhere) and $t$ is real, we have from the definition above:

$$\lim_{t\to 0} \frac{f(x_0+te_j) - f(x_0)}{t} = f'(x_0)e_j\,.$$

Thus $f'(x_0)e_j$, the $j$-th column of our matrix, is the partial derivative $(\partial f/\partial x_j)(x_0)$ of the vector-valued function $f$ with respect to the $j$-th variable. Consequently the derivative $f'(x_0)$ is uniquely defined by (A.1), each coordinate function $f_i$ is differentiable at $x_0$, and with respect to the standard basis of $\mathbb{R}^N$ the matrix of $f'(x_0)$ has:

- As its $j$-th column the partial derivative of $f$ with respect to its $j$-th variable,
- As its $i$-th row the gradient of the coordinate function $f_i$, and
- As its $(i,j)$-th element the partial derivative $(\partial f_i/\partial x_j)(x_0)$.



J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7

There is a partial converse[1]:

> Suppose $f$ is an $\mathbb{R}^N$-valued function defined on a neighborhood $V$ of a point $x_0 \in \mathbb{R}^N$, and that each partial derivative $(\partial f_i/\partial x_j)(x_0)$ exists $(i, j = 1, 2, \ldots, N)$ *and is continuous on* $V$. Then $f$ is differentiable at $x_0$, and $f'$ is continuous on $V$.

Here the continuity of $f'$ at a point $x \in V$ can be interpreted in several equivalent ways. Perhaps easiest is to demand for every vector $h \in \mathbb{R}^N$ that $f'(x_k)h \to f'(x)h$ in $\mathbb{R}^N$ whenever $x_k \to x$ in $V$. Equivalently, we may, for $x \in V$, identify $f'(x)$ with its matrix with respect to the standard basis and demand that each matrix entry be continuous on $V$, or equivalently that $f'$, viewed as a mapping $V \to \mathbb{R}^{N^2}$, be continuous.

**The Chain Rule and Some Consequences**

**Theorem A.1** (The Chain Rule)**.** *Suppose $f$ and $g$ are $\mathbb{R}^N$-valued functions, with $f$ defined on a neighborhood of $x_0 \in \mathbb{R}^N$ and $g$ defined on a neighborhood of $f(x_0)$. If $f$ is differentiable at $x_0$ and $g$ is differentiable at $f(x_0)$ then $g \circ f$ is differentiable at $x_0$ and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$, where on the right-hand side we see a composition of linear transformations.*

The proof is almost identical with that of the one-variable case. For the details see [101, Theorem 9.15, p. 214].

**Theorem A.2** (The Mean-Value Inequality)**.** *Suppose $f$ is an $\mathbb{R}^N$-valued function defined on an open subset $V$ of $\mathbb{R}^N$, that $f \in C^1(V)$, and that $K$ is a compact, convex subset of $V$. Then there exists a positive constant $M$ such that*

$$|f(y) - f(x)| \leq M|y - x|$$

*for every pair $x, y$ of points of $K$.*

*Proof.* Let $M = \max_{x \in K} \|f'(x)\|$, where the norm of $f'(x)$ is defined to be

$$\|f'(x)\| = \left[\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}(x)\right)^2\right]^{\frac{1}{2}},$$

the norm that results when the matrix of $f'(x)$ is viewed as a vector in $\mathbb{R}^{N^2}$.

Define $\gamma\colon [0,1] \to \mathbb{R}^N$ by $\gamma(t) = (1-t)x + ty$ for $0 \leq t \leq 1$. Thus $\gamma(0) = x, \gamma(1) = y$, and $\gamma' \equiv y - x$. The convexity of $K$ insures that $\gamma([0,1]) \subset K$, so $g = f \circ \gamma$ maps $[0,1]$ into $K$, hence

[1] See, e.g., [101, Theorem 9.21, p. 219].

$$|f(y)-f(x)| = |g(1)-g(0)| = \left|\int_0^1 g'(t)\,dt\right|$$
$$\leq \int_0^1 |g'(t)|\,dt = \int_0^1 |f'(\gamma(t))\,\gamma'(t)|\,dt$$
$$\leq \int_0^1 |f'(\gamma(t))|\,|\gamma'(t)|\,dt \leq M|y-x| \qquad \square$$

Some explanation is needed in the calculation above. In the first line the functions being integrated are $\mathbb{R}^N$-valued; the integrals are vectors obtained by integrating each coordinate of the integrand. In the second line the inequality obtained by passing the norm through the integral sign is Exercise A.1 below, while the following equality comes from the Chain Rule. The first inequality in the third line follows from our definition of matrix norm and the Cauchy–Schwarz inequality, while the final inequality comes from the definition of the constant $M$ and the fact that $\gamma' \equiv y-x$.

*Exercise* A.1. Suppose $h\colon [0,1] \to \mathbb{R}^N$ is continuous. Show that $\left|\int_0^1 h(t)\,dt\right| \leq \int_0^1 |h(t)|\,dt$.

*Suggestion:* For each vector $x \in \mathbb{R}^N\backslash\{0\}$ we have (trivially) that $|x| = \langle x, w\rangle$ where $w = \frac{x}{|x|}$. Use this with $x = \int_0^1 h(t)\,dt$, which can be assumed to be non-zero.

**Theorem A.3** (The Inverse-Function Theorem)**.** *Suppose $V$ is an open subset of $\mathbb{R}^N$, $x_0 \in V$, and $f : V \to \mathbb{R}^N$ is a $C^1$-map for which the derivative $f'(x_0)$ is invertible (i.e., for which $\det f'(x_0) \neq 0$). Then there is a neighborhood of $x_0$ upon which the restriction of $f$ is a homeomorphism with $C^1$ inverse.*

For the proof, see, e.g., [101, Theorem 9.24, p. 221].

## A.2 Approximation by Smooth Functions

In Sect. 4.3 our proof of the Brouwer Fixed-Point Theorem required that continuous, real-valued functions on the unit ball $B$ of $\mathbb{R}^N$ be uniformly approximated by functions having continuous derivatives. Here is a proof of this fact.

Suppose $f\colon B \to \mathbb{R}$ is continuous. By Exercise A.2 below it's enough to assume that $f$ extends to a function continuous on all of $\mathbb{R}^N$, with compact support.

The rest of the proof begins with a $C^1$ "bump function." Let $\varphi$ be a non-negative $C^1$ function on $\mathbb{R}^N$ supported in $B$, with $\int \varphi = 1$ (here unadorned integrals extend over all of $\mathbb{R}^N$). For example, take $\varphi(x)$ to be $\frac{2}{\mathrm{vol}(B)}\cos^2(\frac{\pi}{2}|x|)$ when $|x| \leq 1$, and 0 when $|x| > 1$.

Now fix $\delta > 0$ (to be later specified precisely) and set $\varphi_\delta(x) = \delta^{-N}\varphi(x/\delta)$.Then $\varphi_\delta$ has the same properties as $\varphi$ ($C^1$, non-negative, compact support, integral = 1), but now its support lies in $\delta B = \{x \in \mathbb{R}^N : |x| \leq \delta\}$. Define

$$g(x) = \int f(t)\varphi_\delta(x-t)\,dt \qquad (x \in \mathbb{R}^N), \tag{A.2}$$

where the integral on the right (and all further integrals in this proof) are understood to extend over all of $\mathbb{R}^N$. "Differentiation under the integral sign" (cf. [101, Theorem 9.42, pp. 236–237]) shows that $g$ is a $C^1$ function on $\mathbb{R}^N$. Since $\int \varphi_\delta = 1$ the same is true of the $t$-integral of $\varphi_\delta(x-t)$, so we have for $x \in K$:

$$|f(x)-g(x)| = \left|\int [f(x)-f(t)]\varphi_\delta(x-t)\,dt\right| \leq \int |f(x)-f(t)|\varphi_\delta(x-t)\,dt \quad \text{(A.3)}$$

In the estimate above the integrands are supported in the ball

$$B_{x,\delta} := x+\delta B = \{t \in \mathbb{R}^N : |x-t| < \delta\}$$

so the integrals extend only over that ball, hence

$$\int |f(x)-f(t)|\varphi_\delta(x-t)\,dt \leq \omega(f,\delta) := \max\{|f(x)-f(t)| : x,t \in \mathbb{R}^N\}. \quad \text{(A.4)}$$

Because $f$ has compact support it is uniformly continuous on $\mathbb{R}^N$, so $\omega(f,\delta)$ is finite for each $\delta > 0$ and $\to 0$ as $\delta \to 0$. Thus for each $x \in \mathbb{R}^N$ we obtain from the estimates (A.3) and (A.4) above:

$$|f(x)-g(x)| \leq \omega(f,\delta)\int \varphi_\delta(x-t)\,dt = \omega(f,\delta)$$

which, upon choosing $\delta$ so that $\omega(f,\delta) < \varepsilon$, insures that $g$ provides the desired uniform approximation to $f$, even on all of $\mathbb{R}^N$. □

*Exercise* A.2. Show that: given $f\colon B \to \mathbb{R}$ continuous and $\varepsilon > 0$, there exists a continuous, compactly supported function $f_0$ on $\mathbb{R}^N$ such that $|f(x)-f_0(x)| < \varepsilon$ for every $x \in B$.

*Suggestion:* First show that for $r > 1$, but sufficiently close to 1, the function $f_1\colon x \to f(x/r)$ approximates $f$ to within $\varepsilon$ and is continuous on the ball $rB$. Next, create a function $\psi$, continuous on $\mathbb{R}^N$ and supported on $rB$ such that $\psi \equiv 1$ on $B$ and $\psi \equiv 0$ off $rB$. Let $f_0 = \psi f_1$.

## A.3 Change-of-Variables in Integrals

Here is the fundamental result about changing variables in Riemann integration of functions of several variables.

**Theorem A.4** (The Change-of-Variable Theorem)**.** *Suppose $\varphi$ is an $\mathbb{R}^N$-valued $C^1$ mapping of an open subset of $\mathbb{R}^N$ that contains a compact, connected subset $K$ whose boundary has volume zero. Suppose further that on $K$ the mapping $\varphi$ is one-to-one and that* $\det\varphi'$ *is never zero. Then for every continuous, real-valued function $f$ defined on $\varphi(K)$:*

$$\int_{\varphi(K)} f(y)\,dy = \int_K f(\varphi(x))|\det\varphi'(x)|\,dx$$

The "volume-zero" condition in the hypotheses means that for every $\varepsilon > 0$ the boundary of $K$ can be covered by open "boxes" with sides parallel to the coordinate axes (i.e., $N$-fold cartesian products of intervals), the sum of whose volumes is $< \varepsilon$. This is precisely the condition needed to insure that every real-valued continuous function on $K$ is Riemann integrable.

## Notes

*Approximation by smooth functions.* The integral defining the smooth approximation $g$ in (A.2) is called the *convolution* of $f$ and $\varphi_\delta$, written $g = f * \varphi_\delta$. Exercise 9.5 also featured a convolution integral in the context of topological groups.

*Regarding Exercise A.2.* Thanks to the Tietze Extension Theorem (see, e.g., [102, Sect. 20.4, p. 389]) the extension promised by this exercise exists with $B$ replaced by any compact subset of $\mathbb{R}^N$.

*The change-of-variable formula.* A proof of the theorem as stated above can be found in many places, e.g., Apostol's classic text [2, Theorem 10.30, p. 271], or the textbooks of Shurman [110, Theorem 6.7.1, p. 313], and Trench [118, Theorem 7.3.8, p. 496], which are freely available online.

Some authors finesse the hypothesis of zero boundary-volume by demanding that the function $f$ have compact support and that the integrals on both sides of the formula extend over all of $\mathbb{R}^N$; see, e.g., [101, Theorem 10.9, p. 252] for this point of view.

# Appendix B
# Compact Metric Spaces

We introduced the definition of metric space in Sect. 3.1, p. 27. In what follows we'll be working in a metric space $(X,d)$, which we'll usually just call "$X$".

*Notation.* For a point $x_0 \in X$, and a positive real number $r$, let $B(x_0,r)$ denote the open ball of radius $r$ in $X$ centered at $x_0$, i.e., $B(x_0,r) = \{x \in X : d(x,x_0) < r\}$.

## B.1 $\varepsilon$-Nets and Total Boundedness

**Definition B.1** ($\varepsilon$-net)**.** For $\varepsilon > 0$ and $E \subset X$, an $\varepsilon$*-net* is a finite subset $F$ of $X$ with the property that $E \subset \bigcup_{c \in F} B(c,\varepsilon)$.

In other words: An $\varepsilon$-net is a finite subset $F \subset X$ that is "$\varepsilon$-dense" in $E$ in the sense that each point of $E$ lies within $\varepsilon$ of some point of $F$.

**Definition B.2.** To say a subset $E$ of a metric space is *relatively compact* means that its closure is compact.

**Proposition B.3.** *If a subset of a metric space is relatively compact then it has, for every* $\varepsilon > 0$*, an* $\varepsilon$*-net.*

*Proof.* Suppose $E$ is a relatively compact subset of $X$, so that $\overline{E}$, the closure of $E$ in $X$, is compact. Let $\varepsilon > 0$ be given. Each point of $\overline{E}$ lies within $\varepsilon$ of some point of $E$, i.e., the collection of balls $\{B(e,\varepsilon) : e \in E\}$ is an open cover of $\overline{E}$, so by compactness there is a finite subcover. The centers of the balls in this subcover form the desired $\varepsilon$-net for $E$. □

**Definition B.4.** To say a metric space is *totally bounded* means that it has an $\varepsilon$-net for every $\varepsilon > 0$.

With this definition Proposition B.3 can be rephrased:

*If a subset of a metric space is relatively compact then it is totally bounded.*



J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7

The converse of Corollary B.3 is not true in general. *Example*: the set of rationals in the closed unit interval is closed in itself, not compact, but still totally bounded. However, as the Exercise below shows, the converse *does* hold for *complete* metric spaces.

*Exercise* B.1. In a *complete* metric space, if a subset is totally bounded, then it is relatively compact.

Suggestion: Suppose our subset $S$ is totally bounded, so that for each positive integer $n$ there is a $1/n$-net $S_n$ in $S$. Fix a sequence of elements in $S$; we wish to find a subsequence that is convergent in the ambient metric space $X$. Show by a diagonal argument that there is a subsequence of the original which, for each $n$, lies eventually within $1/n$ of some element of $S_n$. Show that this subsequence is Cauchy, hence by completeness, convergent.

## B.2 Continuous Functions on Compact Spaces

**Definition B.5** (Partition of unity)**.** For an $n$-tuple $\mathscr{U} = (U_1, U_2, \ldots U_n)$ of open sets that cover a metric space $X$, a *partition of unity subordinate to* $\mathscr{U}$ is an $n$-tuple $(p_1, p_2, \ldots p_n)$ of continuous functions $X \to [0,1]$, the $j$-th one vanishing off $U_j$, for which the totality sums to 1 on $X$.

**Proposition B.6.** *Every finite open cover of a compact metric space has a subordinate partition of unity.*

*Proof.* As usual, denote our metric space by $X$, and its metric by $d$. Note that $d\colon X \times X \to [0,\infty)$ is continuous, and—because of the compactness of $X$—bounded (this is our only use of compactness here). Suppose $\mathscr{U} = (U_1, U_2, \ldots U_n)$ is our finite open cover of $X$. Define $d_j\colon X \to [0,\infty]$ by

$$d_j(x) = \operatorname{dist}(x, X \backslash U_j) := \inf_{\xi \notin U_j} d(x,\xi) \qquad (x \in X)$$

The boundedness and continuity of $d$ insures that $d_j$ is a continuous function $X \to [0,\infty)$, and $d_j$ vanishes on $U_j$ (note that if we're working with intervals of the real line, then the graph of $d_j$ is a "tent" over $U_j$). Thus the collection of functions

$$p_j := \frac{d_j}{\sum_{k=1}^{n} d_k} \qquad (j = 1, 2, \ldots n)$$

is easily seen to have the desired properties (the denominator above never vanishes because $d_k > 0$ on $V_k$, and the $V_k$'s cover X). □

**Proposition B.7** (Separability)**.** *If $X$ is a compact metric space then both $X$ and $C(X)$ are separable.*

*Proof.* To see that $X$ is separable, for each positive integer $n$ cover $X$ by the collection of all open balls of radius $1/n$; by compactness this open cover has a finite subcover $\mathscr{B}_n$. Let $S_n$ be the collection of centers of the balls in $\mathscr{B}_n$; this is a finite set with the property that each point of $X$ lies within $1/n$ of one of its points. Thus $\cup_n S_n$ is a countable dense subset of $X$.

As for the separability of $C(X)$, we know from Proposition B.6 that there is a partition of unity $\mathscr{P}_n = \{p_1, p_2, \dots, p_N\}$ on $X$ subordinate to the covering $\mathscr{B}_n$. The countable dense subset of $C(X)$ we seek is going to be the collection of rational linear combinations of vectors in $\bigcup_n \mathscr{P}_n$. To see why this is true, fix $f \in C(X)$ and let $\varepsilon > 0$ be given. Since $f$ is continuous on the compact metric space $G$ it is *uniformly continuous* so there exists $\delta > 0$ such that if $x, y \in X$ with $d(x,y) < \delta$ then $|f(x) - f(y)| < \varepsilon$. Choose a positive integer $n > 1/\delta$ and let $x_j$ be the center of the ball $U_j \in \mathscr{B}_n$. Define $g \in C(G)$ by $g = \sum_{j=1}^N f(x_j) p_j$. Then for $x \in G$

$$|f(x) - g(x)| = \left| \sum_{j=1}^N [f(x) - f(x_j)] p_j(x) \right| \leq \sum_{j=1}^N \underbrace{|f(x) - f(x_j)|}_{<\varepsilon \text{ on } U_j} \underbrace{p_j(x)}_{\equiv 0 \text{ off } U_j} < \varepsilon,$$

i.e., $\|f - g\| < \varepsilon$. To finish the proof, go back to the linear combination defining $g$ and replace each coefficient $f(x_j)$ by a rational number sufficiently close that the new $g$ still lies within $\varepsilon$ of $f$ in $C(X)$. □

**Equicontinuity.** Suppose $X$ is a metric space with metric $d$. To say a subset $E$ of $C(X)$ is *equicontinuous* means that: for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends *only* on $\varepsilon$) such that for $x, y \in X$:

$$d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon \quad \forall f \in E.$$

**Theorem B.8** (Arzela–Ascoli, 1883–1885)**.** *If $X$ is a compact metric space then every bounded, equicontinuous subset of $C(X)$ is relatively compact.*

*Proof.* Let $X$ be a compact metric space and $B$ a bounded, equicontinuous subset of $C(X)$. Suppose $(f_n)_1^\infty$ is a sequence in $B$. We desire to show that there is a subsequence that converges in $C(X)$, i.e., uniformly on $X$.

To this end let $S$ be a countable dense subset of $X$ (which we know exists by Proposition B.7) and enumerate its elements as $(s_1, s_2, \dots)$. The first order of business is to find a subsequence of $(f_n)$ that converges pointwise on $S$. This follows from a standard diagonal argument. By the boundedness of $B$ in $C(I)$ the numerical sequence $(f_n(s_1))_{n=1}^\infty$ is bounded, so by Bolzano-Weierstrass it has a convergent subsequence, which we'll write using double subscripts: $(f_{1,n}(s_1))_{n=1}^\infty$. Now the numerical sequence $(f_{1,n}(s_2))_{n=1}^\infty$ is bounded, so it has a convergent subsequence $(f_{2,n}(s_2))_{n=1}^\infty$. Note that the sequence of functions $(f_{2,n})_{n=1}^\infty$, since it is a subsequence of $(f_{1,n})_{n=1}^\infty$, converges at both $s_1$ and $s_2$. Proceeding in this fashion we obtain a countable collection of subsequences of our original sequence:

$$\begin{matrix} f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

where the sequence in the $n$th row converges at the points $s_1, \ldots, s_n$, and each row is a subsequence of the one above it. Thus the *diagonal sequence* $(f_{n,n})$ is a subsequence of the original sequence $(f_n)$, and it converges at each point of $S$.

For simplicity of notation let $g_n = f_{n,n}$. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ by the equicontinuity of the set $B$ to which these functions belong. Thus $d(x,y) < \delta$ implies $|g_n(x) - g_n(y)| < \varepsilon/3$ for each $x, y \in X$ and each positive integer $n$. Since the sequence $(g_n)$ converges at every point of $S$ there exists for each $s \in S$ a positive integer $N(s)$ such that

$$m,n > N(s) \implies |g_n(s) - g_m(s)| < \varepsilon/3. \tag{*}$$

The open balls in $X$ with centers in $S$ and radius $\delta$ cover $X$, so there is a finite subcover with centers in a finite subset $S_\delta$ of $S$. Let $N = \max\{N(s) \colon s \in S_\delta\}$. Fix $x \in X$. Then $x$ lies within $\delta$ of some $s \in S_\delta$, so if $n, m > N$:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(s)| + |g_n(s) - g_m(s)| + |g_m(s) - g_m(x)|.$$

The first and last terms on the right are $< \varepsilon/3$ by our choice of $\delta$ (which was possible because of the equicontinuity of the original sequence), and the same estimate holds for the middle term by our choice of $N$ in (*). In summary: given $\varepsilon > 0$ we have produced $N$ so that for each $x \in X$,

$$m,n > N \implies |g_n(x) - g_m(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus the subsequence $(g_n)$ of $(f_n)$ is Cauchy in $C(X)$, hence convergent there. □

## Notes

*Arzela–Ascoli converse.* If $X$ is a compact metric space and $B$ is a relatively compact subset of $C(X)$ then $B$ is bounded and equicontinuous (exercise).

# Appendix C
# Convex Sets and Normed Spaces

## C.1 Convex Sets

Suppose $V$ is a real vector space. In Sect. 1.6 we defined a subset $C$ of $V$ to be *convex* provided that whenever $x$ and $y$ are points of $C$ then so is $tx + (1-t)y$ for each real number $t$ with $0 \le t \le 1$ (Definition 1.4, p. 10). The empty set is trivially convex, as is every singleton. It's an easy exercise to check that the intersection of a family of convex sets is convex.

**Definition C.1.** A *convex combination* of vectors $x_1, x_2, \ldots, x_N$ in $V$ is a linear combination $\sum_{j=1}^{N} \lambda_j x_j$ where $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ is an $N$-tuple of non-negative real numbers that sum to 1 (i.e., a vector the *standard simplex* $\Pi_N$ of $\mathbb{R}^N$, introduced in Sect. 1.7).

**Proposition C.2.** *A subset of $V$ is convex if and only if it contains every convex combination of its vectors.*

*Proof.* Since for $0 \le t \le 1$ the sum $tx + (1-t)y$ is a convex combination of the vectors $x$ and $y$, a set that contains *every* convex combination of its vectors is surely convex. Conversely, suppose $C$ is a convex subset of $V$ and $x_1, x_2, \ldots, x_N \in C$. Consider a convex combination $x = \sum_{j=1}^{N} \lambda_j x_j$ of these vectors. *To show*: $x \in C$.

We'll prove this by induction on $N$. It's trivial for $N = 1$, and the case $N = 2$ follows from the definition of convexity. So suppose $N > 2$ and we know that every convex combination of $N-1$ vectors in $C$ also belongs to $C$. Let $t = \sum_{j=1}^{N-1} \lambda_j$ so that $0 \le t \le 1$ and $\lambda_N = 1 - t$. Suppose $t \neq 0$ (else $x = x_N$, so trivially $x \in C$). Then $x = ty + (1-t)x_N$ where $y = \sum_{j=1}^{N-1} (\lambda_j / t) x_j$ is a convex combination of $N-1$ vectors in $C$, hence belongs to $C$ by our induction hypothesis. Thus $x \in C$. □

**Definition C.3** (Convex Hull). If $S$ is a subset of $V$ then the *convex hull* of $S$, denoted $\operatorname{conv} S$, is the intersection of all the convex sets that contains $S$.

The collection of such convex sets contains $V$ itself, so is nonempty, and we've noted above that the intersection of convex sets is convex. Thus $\operatorname{conv} S$ is the smallest

 
J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7

convex subset of $V$ that contains $S$. With this definition, Proposition C.2 can be rephrased: *A subset $C$ of $V$ is convex if and only if $C = \operatorname{conv} C$.*

**Proposition C.4.** *The convex hull of a subset $S$ of $V$ is the collection of all convex combinations of vectors in $S$.*

*Proof.* Let $\hat{S}$ denote the collection of all convex combinations of vectors in $S$. Clearly $\hat{S}$ contains $S$. To see why $\hat{S}$ is convex, suppose $x$ and $y$ are vectors therein. We may assume each is a convex combination of vectors $x_1, x_2, \ldots, x_N$ in $S$, say $x = \sum_{j=1}^N \lambda_j x_j$ and $y = \sum_{j=1}^N \mu_j x_j$. Suppose $0 < t < 1$. Then $tx + (1-t)y = \sum_{j=1}^N \eta_j x_j$ where for each index $j$ we have $\eta_j = t\lambda_j + (1-t)\mu_j$. Thus $(\eta_1, \eta_2, \ldots, \eta_n) \in \Pi_N$ so $tx + (1-t)y \in \hat{S}$, proving the convexity of $\hat{S}$. To prove that $\hat{S}$ is the *smallest* convex set containing $S$, suppose $C$ is convex and $C \supset S$. Then by Proposition C.2 we know that $C$ contains all the convex combinations of its vectors, hence in particular all the vectors in $\operatorname{conv} S$. □

*Example.* The standard simplex $\Pi_N$ (see Definition 1.7, page 11) is the convex hull of the standard unit vectors $e_1, \ldots, e_N$ in $\mathbb{R}^N$.

## C.2 Normed Linear Spaces

**Norms.** A normed linear space is a real or complex vector space $X$ upon which is defined a function $\|\cdot\| : X \to [0, \infty)$ that is

(a) Subadditive: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,
(b) Positively homogeneous: $\|tx\| = |t|\|x\|$ for every $x \in X$ and scalar $t$, and for which
(c) $\|x\| = 0$ if and only if $x = 0$.

Examples of norms are:

- The *Euclidean norm* on $\mathbb{R}^N$ or $\mathbb{C}^N$.
- More generally, the norm induced on any real or complex vector space by an inner product $\langle \cdot, \cdot \rangle$: $\|x\| = \sqrt{\langle x, x \rangle}$.
- The one-norm introduced on $\mathbb{R}^N$ by Eq. (1.5) of Sect. 1.7 (p. 11).

**Norm-Induced Metric.** Any norm $\|\cdot\|$ on a vector space $X$ induces a metric $d$ thereon via the equation: $d(x,y) := \|x - y\|,\ (x, y \in X)$. The metric $d$ so defined is *translation-invariant:*

$$d(x+h, y+h) = d(x,y) \qquad (x, y, h \in X).$$

If $X$ is complete in this metric it's called a *Banach space*.

### Convex Sets in Normed Linear Spaces

**Proposition C.5.** *In any normed linear space the convex hull of a finite set of vectors is compact.*

*Proof.* Suppose $E := \{x_1, x_2, \dots, x_N\}$ is our finite set of vectors. By Proposition C.4, $\operatorname{conv} E$ is the set of convex combinations of these vectors, i.e., the set of vectors $x_\lambda = \sum_1^N \lambda_j x_j$ where $\lambda = (\lambda_j)_1^N$ is a vector in the standard simplex $\Pi_N$ of $\mathbb{R}^N$. Suppose $(y_n)_1^\infty$ is a sequence of vectors in $\operatorname{conv} E$. Then $y_n = x_{\eta_n}$ for some sequence $(\eta_n)_1^\infty$ of vectors in $\Pi_N$. Since $\Pi_N$ is a compact subset of $\mathbb{R}^N$ we can extract a subsequence of $(\eta_n)$ that converges to an element $\eta \in \Pi_N$. The corresponding subsequence of $(y_n)$ therefore converges to $x_\eta \in \operatorname{conv} E$, thus establishing the compactness of $\operatorname{conv} E$. □

Recall that a subset of a metric (or topological) space is called *relatively compact* if its closure is compact. With this terminology we have the following generalization of the previous result:

**Proposition C.6.** *The convex hull of a relatively compact subset of a Banach space is relatively compact.*

*Proof.* In our Banach space $X$ let $B_r$ denote the open ball of radius $r$ centered at the origin. Given subsets $A$ and $B$ of that space, we'll denote by $A+B$ the collection of sums $a+b$ where $a$ ranges through $A$ and $b$ through $B$. Thanks to the completeness of $X$, a subset is relatively compact if and only if it is totally bounded (Proposition B.3 and Exercise B.1 of Appendix B). Thus if $A \subset X$ is totally bounded, we wish to show that $\operatorname{conv}(A)$ is totally bounded. To this end, fix $\varepsilon > 0$. Then $A$ has an $\varepsilon/2$-net $F$, i.e., $A \subset F + B_{\varepsilon/2}$. Then $\operatorname{conv}(F)$, being the convex hull of a finite set, is compact, hence totally bounded, and so possesses an $\varepsilon/2$-net $G$, i.e., $\operatorname{conv}(F) \subset G + B_{\varepsilon/2}$. Thus

$$A \subset F + B_{\varepsilon/2} \subset \operatorname{conv}(F) + B_{\varepsilon/2}$$

and since the latter set, being the algebraic sum of two convex sets, is convex, we have $\operatorname{conv}(A) \subset \operatorname{conv}(F) + B_{\varepsilon/2}$. Putting it all together:

$$\operatorname{conv}(A) \subset \operatorname{conv}(F) + B_{\varepsilon/2} \subset G + B_{\varepsilon/2} + B_{\varepsilon/2} = G + B_\varepsilon .$$

Thus $G$ is an $\varepsilon$-net for $\operatorname{conv}(A)$, so $\operatorname{conv}(A)$ is totally bounded. □

**Remark C.7.** Proposition C.6 is *not true* in the generality of normed linear spaces. For an example, in the sequence space $\ell^2$ let $X$ be the dense subspace that consists of real sequences with only finitely many non-zero terms. This is an incomplete normed linear space. Let $(e_n)_1^\infty$ be the standard orthonormal basis of $\ell^2$, and for each positive integer $n$ set $x_n = n^{-1} e_n$. Let $E = \{x_n\}_1^\infty \cup \{0\}$, a compact subset of $X$. However $\operatorname{conv} E$ is *not* compact; it contains each partial sum of the series $\sum_1^\infty 2^{-n} x_n$, which converges in $\ell^2$ to a sum that does not belong to $X$. □

The proof of Proposition C.6 really shows that in a normed linear space, convex hulls inherit *total boundedness*. The example just presented shows that one needs to assume completeness in order to assert that convex hulls inherit *compactness*.

**Operators on Normed Spaces.** Let $X$ and $Y$ denote normed linear spaces. We'll denote the norm in either space by $\|\cdot\|$, letting context will determine the space to which the notation applies.

**Proposition C.8.** *Suppose $X$ and $Y$ are normed linear spaces and $T\colon X \to Y$ is a linear map. Then $T$ is continuous on $X$ if and only if it is bounded on some (equivalently: on every) ball therein.*

*Proof.* Let $B(x_0,r)$ denote the open ball in $X$ of radius $r>0$, with center $x_0$. Let $B = B(0,1)$, the open unit ball of $X$. Suppose $T$ is bounded on some ball in $X$, say with center $x_0$. Then $T$ is bounded on some open ball $B(x_0,r)$, i.e., there exists $R>0$ such that $T(B(x_0,r)) \subset B(0,R)$. Thus by the linearity of $T$:

$$RB = B(0,R) \supset T(B(x_0,r)) = T(rB + x_0) = rT(B) + Tx_0$$

hence $T(B) \subset (R/r)B - (1/r)Tx_0$ which implies, upon letting $M = \frac{R}{r} + \frac{1}{r}\|Tx_0\|$ and using the triangle inequality that $T(B) \subset MB$, i.e., that $\|Tx\| \le M$ whenever $x \in B$. If $x \in X\backslash\{0\}$ then $\xi = x/(\|x\|) \in B$, so $\|T\xi\| \le M\|\xi\|$, which translates—thanks to the linearity of $T$—into the inequality $\|Tx\| \le M\|x\|$, now valid for every $x \in X$. The continuity of $T$ on $X$ follows from this and the fact that if $x,y \in X$ then

$$\|Tx - Ty\| = \|T(x-y)\| \le M\|x-y\|.$$

Conversely, if $T$ is continuous on $X$ then the inverse image of the open unit ball in $Y$ is an open subset of $X$ that contains the origin, and so contains, for some $r>0$, the open ball of radius $r$ in $X$ centered at the origin. Thus the values of $T$ on this ball are bounded in norm by 1. By an argument similar to the one of the last paragraph, the linearity of $T$ insures its boundedness on any ball in $X$. □

*Terminology.* Continuous linear maps between normed spaces are often called *operators* or, thanks to the above Proposition, *bounded operators*. Here's an application of the previous results to convex sets.

*Another proof of Proposition C.5.* Suppose $F = \{x_1, \dots x_N\}$ is a finite set of points in the normed linear space $X$. Define the linear transformation $T\colon \mathbb{R}^N \to X$ by

$$Tx = \sum_{j=1}^{N} \xi_j x_j \text{ where } x := (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N.$$

By the Cauchy–Schwarz inequality:

$$\|Tx\| \le \sum_j |\xi_j|\|x_j\| \le M\left(\sum_j \xi_j^2\right)^{1/2} = M\|x\|,$$

where $M = (\sum_j \|x_j\|^2)^{1/2}$. This shows that $T$ is bounded on the (euclidean) unit ball of $\mathbb{R}^N$, and so by Proposition C.8 is continuous on $\mathbb{R}^N$.

Let $C$ denote the convex hull of the original set $F$. Then $C = T(\Pi_N)$, where $\Pi_N$ is the standard $N$-simplex. Since $\Pi_N$ is compact in $\mathbb{R}^N$ and $T$ is continuous there, $C$ is compact. $\square$

*Exercise* C.1. For a normed linear space $X$ let $\mathscr{L}(X)$ denote collection of continuous linear transformations $X \to X$.

(a) Show that $\mathscr{L}(X)$ is an algebra under the usual algebraic operations on linear transformations.

(b) Define the *norm* of $T \in \mathscr{L}(X)$ to be: $\|T\| := \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$ ($\|T\| < \infty$ by Proposition C.8). Show that $\|\cdot\|$ is a norm on $\mathscr{L}(X)$ that makes it into a Banach space (note that it's not required here that $X$ itself be complete).

(c) Show that $\mathscr{L}(X)$, in the norm defined above, is a *Banach algebra,* i.e., that operator multiplication is a continuous map $\mathscr{L}(X) \times \mathscr{L}(X) \to \mathscr{L}(X)$.

## C.3 Finite Dimensional Normed Linear Spaces

Here we'll work in normed linear spaces over the complex field $\mathbb{C}$. However everything we do will apply equally well to normed spaces over the reals.

**Proposition C.9.** *If $X$ is a normed linear space and* $\dim(X) = N < \infty$*, then $X$ is linearly homeomorphic to $\mathbb{C}^N$.*

*Proof.* Let $(e_j)_1^N$ be the standard unit vector basis in $\mathbb{C}^N$ (so $e_j$ is the vector with 1 in the $j$-th coordinate and zeros elsewhere), and let $(x_j)_1^N$ be any basis for $X$. Define the map $T\colon \mathbb{C}^N \to X$ by $Tv = \sum_j \lambda_j(v) x_j$ where $\lambda_j(v)$ is the $j$-th coordinate of the vector $v \in \mathbb{C}^N$ (i.e., $T$ is the linear map that takes $e_j$ to $x_j$ for $1 \leq j \leq N$). Thus $T$ is a linear isomorphism taking $\mathbb{C}^N$ onto $X$. We've already seen in the course of proving Theorem C.5 (page 195) that $T$ is continuous on $\mathbb{C}^N$ (actually, this proof was carried there out for $\mathbb{R}^N$, but it's the same for $\mathbb{C}^N$). Left to prove is the continuity of $T^{-1}$. If $X$ were a Banach space this would follow immediately from the Open Mapping Theorem (see [103, Theorem 2.11, pp. 48–49], for example).

Here is a more elementary argument that does not require $X$ to be complete. Let $B$ denote the closed unit ball of $\mathbb{C}^N$ and $\partial B$ its boundary—the unit sphere, a compact subset of $\mathbb{C}^N$ that does not contain the origin. Thus $T(\partial B)$ is, thanks to the continuity and injectivity of $T$, a compact subset of $X$ that does not contain the origin. Consequently $T(\partial B)$ is disjoint from some open ball $W$ in $X$ that is centered at the origin.

*Claim.* $\Omega := T^{-1}(W)$ *is contained in* $B^\circ$*, the open unit ball of* $\mathbb{C}^N$.

This will show that $T^{-1}$ is bounded on $W$, hence continuous on $X$ (Proposition C.8).

*Proof of Claim.* Note that:

(a) $\Omega$ is convex (thanks to the linearity of $T^{-1}$), hence arcwise connected.
(b) $\Omega$ contains the origin, and
(c) $\Omega$ does not intersect $\partial B$ (thanks to the injectivity of $T^{-1}$).

That does it! If $\Omega$ were not contained entirely in $B^\circ$ its connectedness would force it to pass through $\partial B$, which it does not. □

**Corollary C.10.** *Every finite dimensional normed linear space is complete.*

*Proof.* Suppose $X$ is a finite dimensional normed linear space. By Proposition C.9, for some positive integer $N$ there is a linear homeomorphism $T$ taking $X$ onto $\mathbb{C}^N$. To see that $X$ inherits the completeness of $\mathbb{C}^N$ recall from the proof of Proposition C.8 that the continuity of $T$ is equivalent to the existence of a positive constant $M$ such that $\|Tx\| \leq M\|x\|$ for every $x \in X$. Thus if $(x_n)$ is a Cauchy sequence in $X$:

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq M\|x_n - x_m\|,$$

so the image sequence $(Tx_n)$ is Cauchy in $\mathbb{C}^N$, hence convergent there. Thus $(x_n)$ is the image of a convergent sequence under the continuous map $T^{-1}$, so it converges in $X$. □

*Warning:* It's crucial here that our homeomorphism of $X$ onto $\mathbb{C}^N$ is *linear*. In general, a metric space homeomorphic to a complete one need not be complete. For an example, consider the arctangent function, which effects a homeomorphism of the (complete) real line onto the (incomplete) open interval $(\pi/2, \pi/2)$.

Thanks to Proposition C.9 and the exercise below: *Every linear transformation on a finite dimensional normed linear space is continuous.*

*Exercise* C.2. Show that every linear transformation on $\mathbb{C}^N$ is continuous.

Consequently, when working on a finite dimensional normed linear space one often uses the words "operator" and "linear transformation" interchangeably, and assumes as familiar the connection between operators and matrices.

## Notes

*Proposition C.9.* The proof given here is taken from [103, Theorem 1.21, pp. 17–18], where it is proved in the setting of topological vector spaces. Consequently, every $N$ dimensional subspace of a topological vector space is linearly homeomorphic to Euclidean space $N$-space.

*Proposition C.5.* The two proofs given for this result work as well for topological vector spaces.

# Appendix D
# Euclidean Isometries

This appendix concerns the group of isometric transformations of $\mathbb{R}^N$ with particular emphasis on the case $N = 3$, for which we'll show that the subgroup of rotations about the origin is isomorphic to the matrix group $\mathrm{SO}(3)$ of $3 \times 3$ real matrices with columns orthonormal in $\mathbb{R}^3$ and determinant 1.

## D.1 Isometries and Orthogonal Matrices

By an *isometry* of a metric space we mean a mapping of the space into itself that preserves distances. In Euclidean space, translations and rotations are isometries. For $\mathbb{R}^N$ the isometries are easily characterized in terms of orthogonal matrices, whose definition and basic properties we'll now review.

*Notation.* For $\mathbb{R}^N$ we will denote the inner product by $\langle \cdot, \cdot \rangle$ and the standard unit vector basis by $(e_j : 1 \leq j \leq N)$; $e_j$ is the vector with 1 in the $j$-th coordinate and zeros elsewhere. We'll think of $\mathbb{R}^N$ as a space of column vectors. For any matrix $A$ we'll denote its transpose by $A^t$.

There is a fundamental connection between the inner product in $\mathbb{R}^N$ and the matrix transpose.

**Proposition D.1.** *For any $N \times N$ real matrix $A$,*

$$\langle Av, w \rangle = \langle v, A^t w \rangle \qquad (v, w \in \mathbb{R}^N).$$

*Proof.* It's enough to prove the result for vectors in the standard basis, so let $v = e_i$ and $w = e_j$. Then the left-hand side of the identity is just the $(i, j)$-element of the matrix $A$, while the right-hand side is, by the symmetry of the real inner product, $\langle A^t e_j, e_i \rangle$, the $(j, i)$-element of the transpose of $A$. Thus the right-hand side equals the left-hand side for these vectors hence, by the bilinearity of the inner product, for all vectors. □



J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7

**Definition D.2** (Orthogonal Matrices)**.** To say a square matrix with real entries is *orthogonal* means that its transpose is its inverse.

More precisely: an $N \times N$ matrix $A$ orthogonal if and only if $AA^t = A^tA = I$ where $I$ is the $n \times n$ identity matrix. We'll use $\mathrm{O}(N)$ to denote the collection of all $N \times N$ orthogonal matrices.

*Exercise* D.1. $\mathrm{O}(N)$ is, for each positive integer $N$, a group under matrix multiplication.

**Proposition D.3.** *An $N \times N$ real matrix is orthogonal if and only if its columns form an orthonormal basis for $\mathbb{R}^N$.*

*Proof.* For an $N$-tuple of vectors in $\mathbb{R}^N$, orthonormality implies linear independence, and hence "basis-ness." Suppose $A$ is an $n \times n$ real matrix. Its $j$-th column is $Ae_j$, so by Proposition D.1 the inner product of the $j$-th and $k$-th columns is

$$\langle Ae_j, Ae_k \rangle = \langle A^t Ae_j, e_k \rangle = \text{the } (k, j)\text{-element of } A^tA$$

Thus the $N$-tuple of vectors $(f_j : 1 \le j \le N)$ is orthonormal if and only if $A^tA = I$. Linear algebra (or the argument above, with $A$ replaced by $A^t$) shows that this happens if and only if $A^t$ and $A$ are inverse to each other, i.e., if and only if $A$ is orthogonal. □

**Proposition D.4.** *If $A$ is an $N \times N$ orthogonal matrix, then:*

*(a)* $\langle Av, Aw \rangle = \langle v, w \rangle$ *for any pair $v, w$ of vectors in $\mathbb{R}^N$.*
*(b)* *The linear transformation $v \to Av$ is an isometry taking $\mathbb{R}^N$ onto itself.*

*Proof.* (a) Using successively Proposition D.1 and the definition of orthogonality:

$$\langle Av, Aw \rangle = \langle A^t Av, w \rangle = \langle v, w \rangle.$$

(b) Upon setting $v = w$ in part (a) we obtain

$$\|Av\|^2 = \langle Av, Av \rangle = \langle v, v \rangle = \|v\|^2,$$

so the mapping induced on $\mathbb{R}^N$ by $A$ is an isometry of $\mathbb{R}^N$ into itself. Being an isometry this map is one-to-one, hence the matrix $A$ is nonsingular, thus the induced map itself is surjective. □

Now for the converse direction: "isometry implies linearity."

**Lemma D.5.** *If $T: \mathbb{R}^N \to \mathbb{R}^N$ is an isometry with $T(0) = 0$, then*

$$\langle T(u), T(v) \rangle = \langle u, v \rangle$$

*for every pair $u, v$ of vectors in $\mathbb{R}^N$.*

*Proof.* This follows immediately from the relationship between norms of differences and inner products. For $u, v \in \mathbb{R}^N$:

$$\|u - v\|^2 = \langle u - v, u - v\rangle = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2. \tag{D.1}$$

Upon replacing $u$ and $v$ in the above calculation with $T(u)$ and $T(v)$, respectively (being careful not to inadvertently assume linearity for $T$):

$$\begin{aligned} \|T(u) - T(v)\|^2 &= \|T(u)\|^2 - 2\langle T(u), T(v)\rangle + \|T(v)\|^2 \\ &= \|u\|^2 - 2\langle T(u), T(v)\rangle + \|v\|^2 \end{aligned} \tag{D.2}$$

where the second equality arises from the fact that the distance from the vector 0 to $v$ is the same as that from $0 = T(0)$ to $Tv$. Similarly the distance from $u$ to $v$ is the same as that from $Tu$ to $Tv$, so the left-hand sides of Eqs. (D.1) and (D.2) are equal, hence so are the right-hand sides, and this yields the desired identity. □

**Proposition D.6.** *If $T$ is an isometry taking $\mathbb{R}^N$ into $\mathbb{R}^N$ with $T(0) = 0$, then there exists $A \in O(N)$ for which $T(v) = Av$ for every $v \in \mathbb{R}^N$.*

*Proof.* Let $(e_1, e_2, \ldots e_N)$ denote the standard orthonormal basis for $\mathbb{R}^N$. Let $f_j = Te_j$ for $1 \le j \le N$. Since $T$ preserves inner products (Lemma D.5 above) there results another orthonormal basis $(f_1, f_2, \ldots f_N)$ for $\mathbb{R}^N$. Let $A$ be the matrix that has as its $j$-th column the coefficients of $f_j$ with respect to the original basis $(e_j)$. Then $A \in O(N)$ by Proposition D.3, and $T(e_j) = Ae_j$ for each index $j$. Thus for every $v \in \mathbb{R}^N$:

$$T(v) = \sum_{j=1}^{N} \langle T(v), f_j\rangle f_j = \sum_{j=1}^{N} \langle T(v), T(e_j)\rangle T(e_j) = \sum_{j=1}^{N} \langle v, e_j\rangle Ae_j = Av,$$

as desired. □

**Theorem D.7.** *A mapping $T\colon \mathbb{R}^N \to \mathbb{R}^N$ is an isometry if and only if there exists $A \in \mathrm{O}(N)$ such that*

$$T(v) = Av + T(0) \tag{D.3}$$

*for each $v \in \mathbb{R}^N$. The matrix $A$ is uniquely determined by $T$.*

*Proof.* Proposition D.4 provides one direction. For the other one note that if $T$ is an isometry $\mathbb{R}^N \to \mathbb{R}^N$ then $T - T(0)$ is also an isometry $\mathbb{R}^N \to \mathbb{R}^N$ that additionally fixes the origin. Thus by Proposition D.6 there is an orthogonal matrix $A$ such that $T(v) = Av + T(0)$ for each $v \in \mathbb{R}^N$. The matrix $A$ is unique since its columns are the images of the standard basis vectors for $\mathbb{R}^N$ under the action of $T - T(0)$. □

Let $\mathbb{B}^N$ denote the closed unit ball of $\mathbb{R}^N$.

**Corollary D.8.** *$T\colon \mathbb{B}^N \to \mathbb{B}^N$ is an isometry if and only if there exists $A \in \mathrm{O}(N)$ such that $Tv = Av$ for every $v \in \mathbb{B}^N$.*

*Proof.* The proof of Theorem D.7 actually showed that:

*An isometry* $T\colon \mathbb{B}^N \to \mathbb{R}^N$ *must have the form* (D.3) *for each* $v \in \mathbb{B}^N$.

Suppose, in addition, that $T$ maps $\mathbb{B}^N$ into itself. It seems obvious that the translation vector $T(0)$ must then equal 0; for a picture-free argument let's suppose this is not the case. Let $x_0 = T(0)$ and $u = A^{-1}(x_0/\|x_0\|)$. Since $A^{-1} = A^t$ is also an orthogonal matrix, $u$ is a unit vector (Proposition D.4), so belongs to $\mathbb{B}^N$. However

$$Tu = \frac{x_0}{\|x_0\|} + x_0 = (1 + \|x_0\|)\frac{x_0}{\|x_0\|},$$

so $\|Tu\| > 1$, contradicting our assumption that $T(\mathbb{B}^N) \subset \mathbb{B}^N$. Thus $x_0 = 0$. □

**Corollary D.9.** *Isometries* $\mathbb{R}^N \to \mathbb{R}^N$ *and* $\mathbb{B}^N \to \mathbb{B}^N$ *must be surjective.*

**Corollary D.10.** *For* $N \geq 2$ *the isometry groups of* $\mathbb{R}^N$ *and* $\mathbb{B}^N$ *are not commutative.*

*Proof.* In view of Theorem D.7 and Corollary D.8 it's enough to note that:

*If* $N \geq 2$ *then the matrix group* $\mathrm{O}(N)$ *is not commutative.*

Indeed, here are two matrices in $\mathrm{O}(2)$ that do not commute:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the first of which induces rotation through an angle of 45 degrees, while the second induces reflection about the horizontal axis. To get an example in $\mathrm{O}(N)$ for $N > 2$ just put each of the above matrices in the upper left-hand corner of an $N \times N$ matrix, and fill in the remaining entries with ones on the main diagonal and zeros off it. □

## D.2 Rotations of $\mathbb{R}^2$ and $\mathbb{R}^3$

Every orthogonal matrix has determinant $\pm 1$; those with determinant 1 are called *special-orthogonal*. The special-orthogonal matrices of a given size form a subgroup of all the invertible matrices of that size. Here we'll be concerned with $\mathrm{SO}(2)$ and $\mathrm{SO}(3)$, the special-orthogonal matrices respectively of sizes $2 \times 2$ and $3 \times 3$.

**Proposition D.11.** *Each matrix in* $\mathrm{SO}(2)$ *induces on* $\mathbb{R}^2$ *a rotation about the origin. If a matrix in* $\mathrm{O}(2)$ *has determinant* $-1$*, then it induces on* $\mathbb{R}^2$ *a reflection in a line through the origin.*

*Proof.* Each $A \in \mathrm{O}(2)$ takes the pair of unit vectors $(e_1, e_2)$ (respectively along the horizontal and vertical axes) to an orthogonal pair $(u, v)$ of unit vectors, where $u$ is the rotate of $e_1$ through some angle $\theta$, and $v$ is either the rotate of $e_2$ through that angle—in which case the determinant of $A$ is 1 and $A$ is the mapping of "rotation by $\theta$"—or $v$ is the negative of that vector. In this latter case $\det A = -1$, and $A$ effects the mapping of reflection in the line through the origin parallel to $u$. □

**Proposition D.12.** *If $A \in \mathrm{SO}(3)$ then the map $x \to Ax$ is a rotation of $\mathbb{R}^3$, with center at the origin.*

We're saying that for each $A \in \mathrm{SO}(3)$ the associated linear transformation fixes a line through the origin, and acts as a rotation about this line (the so-called axis of rotation).

*Proof.* Suppose $A \in \mathrm{SO}(3)$. To find the axis of rotation we need to show that $Av = v$ for some unit vector $v \in \mathbb{R}^3$, i.e., that 1 is an eigenvalue of $A$, or equivalently that $\det(A - I) = 0$. For this, note that since $AA^t = I$ we have

$$(A-I)A^t = AA^t - A^t = I - A^t = -(A-I)^t$$

hence, since $\det A = \det A^t = 1$:

$$\begin{aligned}\det(A-I) &= \det(A-I)\det(A^t) = \det[(A-I)A^t] \\ &= \det[-(A-I)^t] = (-1)^3\det(A-I)^t \\ &= -\det(A-I)\end{aligned}$$

so $\det(A-I) = 0$, as desired.

Let $v_1 \in \mathbb{R}^3$ be the unit vector promised by the last paragraph: $Av_1 = v_1$. Let $(v_2, v_3)$ be an orthonormal basis for the subspace $E$ of $\mathbb{R}^3$ orthogonal to $v_1$. Then $(v_1, v_2, v_3)$ is an orthonormal basis for $\mathbb{R}^3$, relative to which the matrix of the transformation $x \to Ax$ has block diagonal form

$$\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \tag{D.4}$$

where $B$ is a $2 \times 2$ orthogonal matrix. Thus $A$ and $B$ have the same determinant, so $\det B = 1$, i.e., $B \in \mathrm{SO}(2)$, so by the previous proposition $B$ induces on $E$ either the identity map or a rotation about the origin. □

**Corollary D.13.** *A map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation about the origin if and only if there exists a matrix $A \in \mathrm{SO}(3)$ such that $T(v) = Av$ for every $v \in \mathbb{R}^3$.*

*Proof.* We already know (Proposition D.12) that maps of $\mathbb{R}^3$ represented by matrices in $\mathrm{SO}(3)$ are rotations about the origin. For the converse, suppose $T$ is a rotation of $\mathbb{R}^3$ about the origin, i.e., an isometry of $\mathbb{R}^3$ that fixes a line through the origin about which it acts as a two dimensional rotation. By Corollary D.8 we know that $T$ is represented as left multiplication by an orthogonal matrix $A$. Thus $A$ must have the block-diagonal form (D.4) with $B \in \mathrm{SO}(2)$, i.e., $A \in \mathrm{SO}(3)$. □

It's easy to see that the rotation group of the ball does *not* share the commutativity of that of the disc; take, for an example, a pair of 45° rotations about different orthogonal axes. Thus, while the matrix group $\mathrm{SO}(2)$ is commutative, $\mathrm{SO}(3)$ is not.

*Exercise* D.2. $\mathrm{SO}(N)$ is not commutative for every $N \geq 3$.

**The matrix of a rotation in space.** Rotations about the origin in three-space are linear transformations, and linear transformations have matrix representations. Let $R_u(\rho)$ denote the matrix (with respect to the standard basis of $\mathbb{R}^3$) of the transformation of rotation about the origin through angle $\rho$ with axis the unit vector $u \in \mathbb{R}^3$ (the "right-hand rule" determining the positive direction of $\rho$). Although somewhat complicated, this matrix factors readily as a product of simpler matrices. We start with the three "elementary" rotation matrices; the ones that represent rotations about the coordinate axes:

1. Rotation through angle $\rho$ about the $z$-axis

$$R_z := \begin{bmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Rotation through angle $\rho$ about the $x$-axis

$$R_x := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\rho & -\sin\rho \\ 0 & \sin\rho & \cos\rho \end{bmatrix}$$

3. Rotation through angle $\rho$ about the $y$-axis

$$R_y := \begin{bmatrix} \cos\rho & 0 & \sin\rho \\ 0 & 1 & 0 \\ -\sin\rho & 0 & \cos\rho \end{bmatrix}$$

Now fix the unit vector $u \in \mathbb{R}^3$ and the angle $\rho \in [-\pi, \pi)$, and let $L_u$ denote the oriented line through the origin in the direction of $u$. We're going to understand the transformation $R_u(\rho)$ of rotation about $L_u$ through angle $\rho$ by factoring it into a product of several elementary ones. For this let $(\varphi, \theta)$ be the spherical coordinates of $u$, i.e.,

$$u = [\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi]^t$$

where $\varphi \in [0, \pi]$ is the angle between $u$ and the $z$-axis, and $\theta \in [-\pi, \pi]$ is the angle between the $x$-axis and the projection of the $u$ into the $x, y$-plane.

Let $T = R_y(-\varphi)R_z(-\theta)$, so that $T$ rotates $u$ through angle $-\theta$ about the $z$-axis, depositing it into the $x,z$-plane, then in that plane (i.e., about the $y$-axis) rotates the resulting vector through angle $\rho$ so that it ends up at the "north pole" $e_3 := [0,0,1]^t$. Thus $T^{-1}R_z(\rho)T$ fixes $u$ and, since $T$ belongs to SO(3), and therefore preserves both distances and angles, it rotates points of $\mathbb{R}^3$ about $L_u$ through angle $\rho$, i.e., it's none other than $R_u(\rho)$. Explicitly:

$$R_u(\rho) = R_z(\theta)R_y(\varphi)R_z(\rho)R_y(-\varphi)R_z(-\theta). \tag{D.5}$$

To find the matrix of $R_u(\rho)$ (with respect to the standard unit vector basis of $\mathbb{R}^3$) one "need only" multiply the elementary matrices for the five transformations on the right-hand side of (D.5). This is best done with your favorite computer-algebra program; the result is nevertheless quite a mess. To bring it into some kind of reasonable form it helps to invert the spherical coordinate representation of $u$, noting that $\cos\varphi = z$, $\sin\varphi = \sqrt{x^2+y^2} = \sqrt{1-z^2}$ (non-negative square root because $0 \le \varphi \le \pi$), $\cos\theta = x/\sqrt{1-z^2}$, and $\sin\theta = y/\sqrt{1-z^2}$.

Again with the help of your computer-algebra program, most likely aided by some paper and pencil algebraic simplifications, there will result the following matrix representation of $R_u(\rho)$:

$$\begin{bmatrix} x^2+(1-x^2)\cos\rho & xy(1-\cos\rho)-z\sin\rho & xz(1-\cos\rho)+y\sin\rho \\ xy(1-\cos\rho)+z\sin\rho & y^2+(1-y^2)\cos\rho & yz(1-\cos\rho)-x\sin\rho \\ xz(1-\cos\rho)-y\sin\rho & yz(1-\cos\rho)+x\sin\rho & z^2+(1-z^2)\cos\rho \end{bmatrix}.$$

## Notes

*Rotations in $\mathbb{R}^3$ and beyond.* Proposition D.12 (or more accurately, the statement that each rotation in $\mathbb{R}^3$ about the origin has a fixed axis), was first proved by Euler in 1775–1776; it's called "Euler's Rotation Theorem." For a lively article that gives much more detail about this result, see [90]. The results above on SO(2) and SO(3) generalize to higher dimensions, but now reflections can be present. For O($N$) the full story is this (see, e.g., [14, Theorem 10.12, p. 152]):

> *For $A \in O(N)$ there exists an orthonormal basis for $\mathbb{R}^N$ relative to which the transformation $x \to Ax$ has block diagonal matrix $(I_p, -I_q, B_1, \ldots, B_r)$ where the I's are identity matrices of orders $p$ and $q$ respectively, the B's are $2\times 2$ orthogonal matrices, and $p+q+2r = n$.*

# Appendix E
# A Little Group Theory, a Little Set Theory

We'll write groups multiplicatively, denoting the identity element by "$e$". For subsets $A$ and $B$ of a group $G$ we'll write $AB$ for the collection of all products $ab$ with $a \in A$ and $b \in B$, using the abbreviation $aB$ for the product $\{a\}B$. If $H$ is a subgroup of $G$ (i.e., a group in the operation inherited from $G$, whose identity is the identity element of $G$), and $g \in G$, then $gH$ is the *left coset of $G$ modulo $H$,* and $Hg$ is the corresponding *right coset.*

## E.1 Normal Subgroups

**Definition E.1** (Normal Subgroup)**.** Suppose $G$ is a group and $H$ a subgroup (notation: $H < G$). To say $H$ is a *normal* subgroup of $G$ (notation: $H \triangleleft G$) means that $gH = Hg$ for any $g \in G$, i.e., there is no distinction between left and right cosets. In this case we'll use $G/H$ to denote the collection of all cosets of $G$ modulo $H$.

**Proposition E.2.** *Suppose $H$ is a subgroup of $G$. Then $H \triangleleft G$ if and only if $G/H$ forms a group under the multiplication*

$$(g_1H)(g_2H) = g_1g_2H \qquad (g_1, g_2 \in G).$$

*Proof.* If $H \triangleleft G$ and $g_1, g_2 \in G$, then

$$(g_1H)(g_2H) = g_1(Hg_2)H = g_1g_2HH = g_1g_2H$$

and from this we have for each $g \in G$:

$$(gH)(g^{-1}H) = gg^{-1}H = eH = H.$$

Thus $G/H$ is a group under the inherited multiplication, with $H$ being the identity and $(gH)^{-1} = g^{-1}H$.



J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7

Conversely, if $G/H$ is a group under the multiplication $(g_1H)(g_2H) = g_1g_2H$ (where it's being assumed that the multiplication is well defined), then $H$ is the identity, and for any $g \in G$:

$$gHg^{-1} \subset gHg^{-1}H = gg^{-1}H = eH = H$$

so $gH \subset Hg$. For the opposite inclusion just replace $g$ by its inverse in this one and take inverses of both sides of the resulting inclusion. □

## E.2 Solvable Groups

Let $G$ be a group. For each pair $a,b$ of elements of $G$ let $[a,b]$ denote the commutator $a^{-1}b^{-1}ab$ (so named because if $c = [a,b]$ then $ab = bac$). To think about solvability for $G$ let's consider *chains of subgroups*

$$\{e\} = G_0 < G_1 < \ldots < G_n = G. \tag{E.1}$$

Here's a restatement of Definition 12.2(c) of "solvable group."

**Definition E.3.** To say that $G$ is a *solvable group* means that there is a chain of subgroups (E.1) with each subgroup $G_k$ containing all the commutators of $G_{k+1}$ $(0 \le k < n)$.

Good things happen whenever a subgroup contains all the commutators of its parent group.

**Proposition E.4.** *Suppose G is a group and H a subgroup of G. Then the following are equivalent:*

*(a) H contains all the commutators of G.*
*(b) H is a normal subgroup of G and the quotient group G/H is abelian.*

*Proof.* (a) → (b): Suppose $H$ contains all the commutators of $G$, i.e., for every pair $a,b$ of elements of $G$ there is an element $h \in H$ (namely $h = [a,b] = a^{-1}b^{-1}ab$) such that $ab = bah$. In particular, for any $a \in G$ and $h_1 \in H$ there exists $h \in H$ such that

$$a^{-1}(h_1a) = a^{-1}(ah_1h) = h_1h \in H.$$

Thus for each $a \in G$ we have $a^{-1}Ha \subset H$, so $Ha \subset aH$, hence—as we've seen above (last part of proof of Proposition E.2)—this implies $Ha = aH$, i.e., $H$ is a normal subgroup of $G$. As for the commutativity of the quotient group $G/H$, note that if $a,b \in G$ then, as noted above, $ab = bah$ for $h = [a,b] \in H$ hence $abH \subset baH$ and so $abH = baH$.

(b) → (a): Consider the statement

(*) $H < G$ and $abH = baH$ for each pair $a,b$ of elements of $G$.

Then for all $a, b \in G$ and $h_1 \in H$ there exists $h_2 \in H$ such that $abh_1 = bah_2$, i.e., $[a,b] = h_2 h_1^{-1} \in H$. Thus statement (*) implies that each commutator of $G$ belongs to $H$, which by the first part of our proof is enough to guarantee normality for $H$, and hence—again by (*)—commutativity for $G/H$. □

## E.3 The Axiom of Choice and Zorn's Lemma

If we are given a family of sets, the Axiom of Choice allows us to choose one element from each member of the family. More precisely:

**The Axiom of Choice.** *Suppose $X$ is a set and $\mathscr{E}$ is a family of nonempty subsets of $X$. Then there is a "choice function" $f : \mathscr{E} \to X$ such that $f(E) \in E$ for each $E \in \mathscr{E}$.*

**Definition.** A *partial order* on a set $X$ is a binary relation "$\leq$" such that for all $x, y, z \in X$:

$x \leq x$ (reflexivity),
$x \leq y$ and $y \leq x \implies x = y$ (antisymmetry), and
$x \leq y$ and $y \leq z \implies x \leq z$ (transitivity).

Suppose "$\leq$" is a partial order on $X$ and $S \subset X$. An element $b \in X$ for which each $s \in S$ is $\leq b$ is called an *upper bound* for $S$. An element $m \in X$ is called *maximal* if no other element of $X$ "exceeds" $m$, i.e., if $x \in X$ and $m \leq x$ then $x = m$. $S \subset X$ is said to be *totally ordered* if for every pair of elements $s, t \in S$ either $s \leq t$ or $t \leq s$.

**Zorn's Lemma.** *If $X$ is a partially ordered set in which every totally ordered subset has an upper bound, then $X$ has a maximal element.*

Zorn's Lemma and the Axiom of Choice are *equivalent* in the sense that each one can be derived from the other. Halmos [46, Sects. 15 and 16] gives an delightful introduction to these two principles of mathematics, with a proof of their equivalence.

## Notes

*Non-normal pathology.* The following exercise, taken from taken from Milne's freely downloadable introduction to group theory [77], shows that we can't expect to extend the definition used to multiply cosets modulo normal subgroups to cosets modulo arbitrary subgroups.

*Exercise* E.1. Let $G$ denote the collection of all $2 \times 2$ invertible matrices with rational entries, and let $H$ denote the set of $2 \times 2$ matrices of the form $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$ where $n$ runs through the integers. Then $G$ is a group under matrix multiplication and $H$ is a subgroup (isomorphic to the group of integers under addition).

(a) Show that $H$ is not a normal subgroup of $G$. In fact, if $g = \left[\begin{smallmatrix} 5 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ then $g \in G$ and $gH$ is a proper subset of $Hg$. Moreover, $g^{-1}H$ is not even contained in $Hg^{-1}$.

(b) Show that for $g$ as in part (a), $g^{-1}HgH \neq H$.

*The usual definition of "solvable."* Proposition E.4 shows that our definition of "solvable group" can be restated

*G is a solvable group if and only if there exists a chain* (E.1) *of normal subgroups such that each group $G_k/G_{k-1}$ is commutative.*

This is the usual definition of "solvable" for groups.

*The Axiom of Choice.* Consequences such as the existence of non-measurable subsets of the real line, and more spectacularly the Banach-Tarski Paradox, initially gave the Axiom of Choice something of a bad reputation. To quote Halmos [46, Sect. 15, p. 60]:

> It used to be considered important to examine, for each consequence of the axiom of choice, the extent to which the axiom is needed in the proof of the consequence. An alternative proof without the axiom of choice spelled victory; a converse proof, showing that the consequence is equivalent to the axiom ... meant honorable defeat. Anything in between was considered exasperating.

The Axiom of Choice is now much better understood. In Chap. 13 of Wagon's book [121] there is a detailed discussion of the role it plays in set theory. On page 214 (Fig. 13.1) of that chapter there is a useful diagram showing the logical connections between the Axiom of Choice and various well-known theorems of mathematics that follow from it (e.g., the Tychonoff Product Theorem, the Hahn-Banach Theorem). Jech's monograph [55] provides an in-depth treatment of such matters.

# References

1. Agnew, R.P., Morse, A.P.: Extensions of linear functionals, with applications to limits, integrals, measures, and densities. Ann. of Math (2), **39**, 20–30 (1938)
2. Apostol, T.M.: Mathematical Analysis, A Modern Approach to Advanced Calculus. Addison-Wesley, Reading (1957)
3. Argyros, S.A., Haydon, R.G.: A hereditarily indecomposable $\mathscr{L}_\infty$-space that solves the scalar-plus-compact problem. Acta Math. **206**, 1–54 (2011)
4. Aronszajn, N., Smith, K.T.: Invariant subspaces of completely continuous operators. Ann. of Math (2), **60**, 345–350 (1954)
5. Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. **3**, 133–181 (1922)
6. Banach, S.: Sur le probleme de la mesure. Fund. Math. **4**, 7–33 (1923)
7. Banach, S.: Un théorème sur les transformations biunivoques. Fund. Math. **6**, 236–239 (1924)
8. Banach, S.: On Haar's measure. In: Theory of the Integral, by Stanisław Saks, Monografie Matematyczne, Warsaw, vol. 7, pp. 314–319 (1937) Reprinted by Dover Pub. 1964 & 2005
9. Banach, S.: Théorie des Opérations Linéaires, 2nd edn. Chelsea (1978). First ed. published in 1932, Monografje Matematyczne, vol. I, Warsaw
10. Banach, S., Tarski, A.: Sur la décomposition des ensembles de points en parties respectivement congruentes. Fund. Math. **6**, 244–277 (1924)
11. Bernstein, A.R., Robinson, A.: Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos. Pacific J. Math. **16**, 421–431 (1966)
12. Bilton, N., Sandhaus, E.: # of links on the homepage of 98 top websites. Available at http://www.nickbilton.com/98/
13. Birkhoff, G.: A note on topological groups. Compos. Math. **3**, 427–430 (1936)
14. Blyth, T., Robertson, E.F.: Further Linear Algebra. Springer, New York (2006). Second printing
15. Boyce, W.M.: Commuting functions with no common fixed point. Trans. Amer. Math. Soc. **137**, 77–92 (1969)
16. Bressan, A.: Noncooperative Differential Games. A Tutorial. Lecture Notes available online at http://www.math.psu.edu/bressan/PSPDF/game-lnew.pdf (2010)
17. Brin, S., Page, L.: The anatomy of a large-scale hypertextual web search engine. In: Seventh International World-Wide Web Conference (WWW 1998), Brisbane, 14–18 April 1998. Available online at http://infolab.stanford. edu/pub/papers/google.pdf (1998)
18. Brouwer, L.E.J.: Über Abbildung von Mannigfaltigkeiten. Math. Ann. **71**, 97–115 (1912)
19. Brouwer, L.E.J.: An intuitionist correction of the fixed-point theorem on the sphere. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **213**, 1–2 (1952)



J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7

20. Bryan, K., Leise, T.: The $25,000,000,000 eigenvector: the linear algebra behind Google. SIAM Rev. **48**, 569–581 (2006)
21. Casti, J.L.: Five Golden Rules: Great Theories of 20th Century Mathematics, and Why They Matter. Wiley, New York (1965)
22. Cauty, R.: Solution du problème de point fixe de Schauder. Fund. Math. **170**, 231–246 (2001)
23. Cellina, A.: Approximation of set valued functions and fixed point theorems. Ann. Mat. Pura Appl. **82**(4), 17–24 (1969)
24. Chalendar, I., Partington, J.R.: Modern Approaches to the Invariant-Subspace Problem. Cambridge University Press, Cambridge (2011)
25. Chernoff, P.R.: A simple proof of Tychonoff's theorem via nets. Amer. Math. Monthly **99**, 932–934 (1992)
26. Cohen, D.I.A.: On the Sperner Lemma. J. Comben. Theory **2**, 585–587 (1967)
27. Day, M.M.: Means for the bounded functions and ergodicity of the bounded representations of semi-groups. Trans. Amer. Math. Soc. **69**, 276–291 (1950)
28. Day, M.M.: Amenable semigroups. Illinois J. Math. **1**, 509–544 (1957)
29. Diestel, J., Spalsbury, A.: The Joys of Haar Measure. Graduate Studies in Mathematics, vol. 150. American Mathematical Society, Providence (2014)
30. Ding, J., Zhou, A.: Non-Negative Matrices, Positive Operators, and Applications. World Scientific, Singapore (2009)
31. Dobrowolski, T.: Revisiting Cauty's proof of the Schauder conjecture. Abstr. Appl. Anal. **7**, 407–433 (2003)
32. Drinfel'd, V.G.: Finitely additive measures on $S^2$ and $S^3$, invariant with respect to rotations. Funct. Anal. Appl. **18**, 245–246 (2014)
33. Driver, R.D.: Torricelli's law-an ideal example of an elementary ODE. Amer. Math. Monthly **105**, 453–455 (1998)
34. Dugundji, J.: An extension of Tietze's theorem. Pacific. J. Math. **1**, 353–367 (1951)
35. Dugundji, J., Granas, A.: A proof of the Ryll-Nardzewski fixed point theorem. J. Math. Anal. Appl. **97**, 301–305 (1983)
36. Dym, H., McKean, H.P.: Fourier Series and Integrals. Academic, New York (1972)
37. Edwards, R.E.: Functional Analysis: Theory and Applications. Holt, Rinehart, Winston, New York (1965)
38. Enflo, P.: On the invariant subspace problem for Banach spaces. Acta Math. **158**, 213–313 (1987)
39. Folland, G.B.: A Course in Abstract Harmonic Analysis. CRC, Boca Raton (1995)
40. Franklin, J.: Methods of Mathematical Economics. Springer, Heidelberg (1980)
41. Galilei, G.: Dialogues Concerning Two New Sciences. Dover, New York (1914). Translation by Crew and de Salvio of: Discorsi e Dimonstrazione Matematiche Intorno à Due Nuove Scienze. Elzevir, Leiden (1638). This translation originally published by Macmillan, New York (1914)
42. Haar, A.: Der Masbegriff in der Theorie der Kontinuierlichen Gruppen. Ann. of Math (2), **34**, 147–169 (1933)
43. Hadlock, C.R.: Field Theory and its Classical Problems. Carus Monograph, vol. 19. Mathematical Association of America, Washington (1978)
44. Hadwin, D.W., Nordgren, E.A., Radjavi, H., Rosenthal, P.: An operator not satisfying Lomonosov's hypotheses. J. Funct. Anal. **38**, 410–415 (1980)
45. Hahn, F.: A fixed-point theorem. Math. Syst. Theory **1**, 55–57 (1968)
46. Halmos, P.R.: Naive Set Theory. van Nostrand, Princeton (1960); Undergraduate Texts in Mathematics. Reprinted by Springer, New York (1974)
47. Hardin, G.: The tragedy of the commons. Science **162**, 1243–1248 (1968)
48. Hausdorff, F.: Grundzüge der Mengenlehre. Veig, Leipzig (1914)
49. Hawkins, T.: Weyl and the topology of continuous groups. In: James, I.M. (ed.) History of Topology. North Holland, Amsterdam (1999)
50. Hiriart-Urrutz, J.B., Lemaréchal, C.: Convex Analysis and Minimization Algorithms I. Springer, Heidelberg (1996)

51. Horn, R.A., Johnson, C.: Matrix Analysis, 2nd edn. Cambridge University Press, New York (2013)
52. Huneke, J.P.: On common fixed points of commuting continuous functions on an interval. Proc. Amer. Math. Soc. **139**, 371–381 (1969)
53. Ivanov, N.V.: A differential forms perspective on the Lax proof of the change of variable formula. Amer. Math. Monthly **112**, 799–806 (2005)
54. Izzo, A.J.: A functional analysis proof of the existence of Haar measure on locally compact abelian groups. Proc. Amer. Math. Soc. **115**, 581–583 (1992)
55. Jech, T.J.: The Axiom of Choice. North Holland, Amsterdam (1973). Republished by Dover, New York (2008)
56. Kakutani, S.: Two fixed-point theorems concerning bicompact convex sets. Proc. Imp. Acad. (Jap.) **14**, 242–245 (1938)
57. Kakutani, S.: Uber die Metrisation der topologischen Gruppen. Proc. Imp. Acad. (Jap.) **14**, 82–84 (1938)
58. Kakutani, S.: A generalization of Brouwer's fixed point theorem. Duke Math. J. **8**, 457–459 (1941)
59. Kakutani, S.: Topological properties of the unit sphere of a Hilbert space. Proc. Imp. Acad. Tokyo **19**, 269–271 (1943)
60. Kelley, J.L., Namioka, I.: Linear Topological Spaces. van Nostrand, New York (1963)
61. Klarreich, E.: The Mathematics of Strategy. Article available online from http://www.pnas.org/site/classics/classics5.xhtml
62. Klee, V.L.: Some topological properties of convex sets. Trans. Amer. Math. Soc. **78**, 30–45 (1955)
63. Knaster, B.: Un théorème sur les fonctions d'ensembles. Ann. Soc. Polon. Math. **6**, 133–134 (1928)
64. Kuhn, H.W.: Simplicial approximation of fixed points. Proc. Natl. Acad. Sci. U.S.A. **61**, 1238–1242 (1968)
65. Kuhn, H.W., Tucker, A.W.: John von Neumann's work in the theory of games and mathematical economics. Bull. Am. Math. Soc. **64**, 100–122 (1958)
66. Lalley, S.: Markov Chains: Basic Theory. Lecture Notes available from http://galton.uchicago.edu/ lalley/Courses/312/MarkovChains.pdf (2009)
67. Lax, P.D.: Change of variables in multiple integrals. Amer. Math. Monthly **106**, 497–501 (1999)
68. Lemke, C.E.: Bimatrix equilibrium points in mathematical programming. Managment Sci. **11**, 681–689 (1965)
69. Li, M.C.: An elementary proof of a generalization of Banach's Mapping Theorem. Amer. Math. Monthly **121**, 445–446 (2014)
70. Lindelöf, M.E.: Sur l'application de la méthode des approximations successives aux équations différentiales ordinaires du premier ordre. C. R. Math. Acad. Sci. Paris **114**, 454–457 (1894)
71. Lomonosov, V.I.: Invariant subspaces for the family of operators which commute with a completely continuous operator. Funct. Anal. Appl. **7**, 213–214 (1973)
72. MacCluer, C.R.: The many proofs and applications of Perron's Theorem. SIAM Rev. **42**, 487–498 (2000)
73. Magnus, W., Karrass, A., Solitar, D.: Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, 2nd edn. Dover, New York (1976). Originally published by Interscience Publishers, New York (1966)
74. Margulis, G.: Some remarks on invariant means. Monatsh. Math. **90**, 233–235 (1980)
75. Markov, A.A.: Quelques théoremes sur les ensembles abéliens. C.R. URSS **2**, p. 311 (1936)
76. Mason, O.: NonNegative Matrices and Related Topics. Lecture notes available at www.hamilton.ie/ollie/Downloads/Pos1.pdf
77. Milne, J.S.: Group Theory (v3.13). Available at www.jmilne.org/math/ (2013)
78. Milnor, J.: Analytic proofs of the "hairy ball theorem" and the Brouwer fixed point theorem. Amer. Math. Monthly **85**, 521–524 (1978)

79. Milnor, J.: A Nobel Prize for John Nash. Math. Intelligence **17**, 11–17 (1995)
80. Montgomery, D., Zippin, L.: Topological Transformation Groups. Interscience, New York (1955)
81. Myerson, R.B.: Nash equilibrium and the history of economic theory. J. Econ. Lit. **37**, 1067–1082 (1999)
82. Namioka, I., Asplund, E.: A geometric proof of Ryll-Nardzewski's fixed point theorem. Bull. Amer. Math. Soc. **73**, 443–445 (1967)
83. Nasar, S.: The lost years of a Nobel laureate. The N. Y. Times, Nov 13, 1994
84. Nasar, S.: A Beautiful Mind. Simon & Schuster, New York (1998)
85. Nash Jr., J.F.: Equilibrium points in N-person games. Proc. Natl. Acad. Sci. U.S.A. **36**, 48–49 (1950)
86. Nash Jr., J.F.: Non-cooperative games, pp. 1–32. Dissertation, Princeton (1950)
87. Nash Jr., J.F.: Non-cooperative games. Ann. of Math (2), **54**, 286–295 (1951)
88. von Neumann, J.L.: Zur algemeinnen theorie des masses. Fund. Math. **13**, 73–116 (1929)
89. von Neumann, J.L.: A model of general economic equilibrium. Rev. Econ. Studies **13**, 1–9 (1945)
90. Palais, B., Palais, R., Rodi, S.: A disorienting look at Euler's theorem on the axis of a rotation. Amer. Math. Monthly **116**, 892–909 (2009)
91. Park, S.: Ninety years of the Brouwer fixed point theorem. Vietnam J. Math. **27**, 187–222 (1999)
92. Pearcy, C., Shields, A.L.: A survey of the Lomonosov technique. Topics in Operator Theory, Mathematical Surveys, vol. 13, pp. 221–229. American Mathematical Society, Providence (1974)
93. Perko, L.: Differential Equations and Dynamical Systems, 3rd edn. Springer, New York (2000)
94. Radjavi, H., Rosenthal, P.: Invariant Subspaces, 2nd edn. Dover, New York (2003). First edition published by Springer, New York (1973)
95. Read, C.J.: A short proof concerning the invariant subspace problem. J. Lond. Math. Soc. (2) **34**, 335–348 (1986)
96. Read, C.J.: The invariant subspace problem for a class of Banach spaces 2: hypercyclic operators. Isreal J. Math. **63**, 1–40 (1988)
97. Riesz, F.: Sur les opérations fonctionelles linéaires. C. R. Math. Acad. Sci. Paris **149**, 974–977 (1909)
98. Riesz, F., Nagy, B.S.: Functional Analysis, 2nd edn. Frederick Ungar, New York (1955). Reprinted by Dover Publications, New York (1990)
99. Rogers, C.A.: A less strange version of Milnor's proof of Brouwer's fixed-point theorem. Amer. Math. Monthly **67**, 525–527 (2010)
100. Rudin, W.: Fourier Analysis on Groups. Interscience, vol. 12 Wiley, New York (1962)
101. Rudin, W.: Principles of Mathematical Analysis, 3rd edn. McGraw-Hill, New York (1976)
102. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)
103. Rudin, W.: Functional Analysis, 2nd edn. McGraw-Hill, New York (1991)
104. Runde, V.: Lectures on Amenability. Springer Lecture Notes, vol. 1774. Springer, Berlin (2002)
105. Ryll-Nardzewski, C.: On fixed points of semigroups of endomorphisms of linear spaces. In: Proceeding of the Fifth Berkeley Sympos Math. Statistics & Probability, pp. 55–61 (1967)
106. Scarf, H.E.: Fixed-point theorems and economic analysis: mathematical theorems can be used to predict the probable effects of changes in economic policy. Am. Sci. **71**, 289–296 (1983)
107. Schauder, J.: Der Fixpunktsatz in Funktionalräumen. Studia. Math. **2**, 171–180 (1930)
108. Schilling, F., Piper, W.S.: Basic Abstract Algebra. Allyn & Bacon, Boston (1975)
109. Schur, I.: Über die charakteristischen Wurzeln einerlinearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen. Math. Ann. **66**, 488–510 (1909)
110. Shurman, J.: Multivariable Calculus. Lecture Notes for a Course at Reed College. Available online at http://people.reed.edu/ jerry/211/vcalc.pdf

111. Sperner, E.: Neur Beweis für die Invarianz der Dimensionzal und des Gebietes. Abh. Math. Sem. Univ. Hamburg **6**, 265–272 (1928)
112. Stuckless, T.: Brouwer's Fixed Point Theorem: Methods of Proof and Generalizations. Available from http://oldweb.cecm.sfu.ca/Preprints03/ preprints03.html (2003)
113. Su, F.E.: Rental harmony: Sperner's lemma in fair division. Amer. Math. Monthly **106**, 930–942 (1999)
114. Sullivan, D.: For $n > 3$ there is only one finitely additive rotationally invariant measure on the $n$-sphere defined on all Lebesgue measurable subsets. Bull. Amer. Math. Soc. **4**, 121–123 (1981)
115. Tao, T.: The Banach-Tarski Paradox. Available online from www.math.ucla.edu/~tao/preprints/Expository/banach-tarski.pdf
116. Tarski, A.: A lattice-theoretical fixpoint theorem and its applications. Pacific J. Math. **5**, 255–309 (1955)
117. Trefethen, L.N., Bau III, D.: Numerical Linear Algebra. SIAM, Philadelphia (1997)
118. Trench, W.F.: Introduction to Real Analysis. Freely available online at http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS .PDF (2013)
119. Troitsky, V.G.: Lomonosov's theorem cannot be extended to chains of four operators. Proc. Amer. Math. Soc. **128**, 521–525 (2000)
120. Tychonoff, A.: Ein Fixpunktsatz. Math. Ann. **111**, 767–776 (1935)
121. Wagon, S.: The Banach-Tarski Paradox, paperback edn. Cambridge University Press, Cambridge (1993)
122. Wapner, L.: The Pea and the Sun: A Mathematical Paradox. A. K. Peters Ltd., Wellesley (2005)
123. Willard, S.: General Topology. Addison Wesley, Reading (1970). Reprinted by Dover Publications, Mineola (2004)
124. Williamson, J.H.: Compact linear operators in linear topological spaces. J. Lond. Math. Soc. (2) **29**, 149–156 (1954)
125. Young, N.: An Introduction to Hilbert Space. Cambridge University Press, Cambridge (1998)

# Index




J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7

# List of symbols





J.H. Shapiro, *A Fixed-Point Farrago*, Universitext,
DOI 10.1007/978-3-319-27978-7