Appendix A
List of Symbols

Roman Letter

A	action
$\mathcal{A}, \mathcal{A}_s, \mathcal{A}_f$	Helmholtz free energy, and for solid and fluid phase
$\mathscr{A}, \mathscr{A}_s, \mathscr{A}_f$	Helmholtz free energy density, and for solid and fluid phase
$\overline{\mathscr{A}}, \overline{\mathscr{A}}_s, \overline{\mathscr{A}}_f$	REV averaged Helmholtz free energy density, and strain energy density function, and for solid and fluid phase
a^f, a^f_{ij}	fluid apparent mass constitutive constant scalar and tensor
a_k	chemical activity of species k
a_L, a_T	longitudinal and transverse dispersivity
$B, \vec{B}, B_{ij}$	Skempton pore pressure coefficient B scalar, vector (6×1), and tensor (3×3)
B_c	Skempton pore pressure coefficient under constant solute concentration
B_s, B_f, B_ψ	constitutive constants for porothermoelastic quadratic form
B^b_k, B^p_k, B^ψ_k	constitutive constants for porochemoelastic quadratic form for species k
B^b_s, B^p_s, B^ψ_s	constitutive constants for porochemoelastic quadratic form for solute
B^b_w, B^p_w, B^ψ_w	constitutive constants for porochemoelastic quadratic form for solvent (water)
$\mathfrak{B}, \mathfrak{B}^*$	generalized boundary normal derivative operator and its adjoint operator
$b, \mathbf{b}, b_{ij}$	$(= \mathbf{k}^{-1})$ resistivity scalar and tensor
b_{ij}	$\{i,j\} = \{T,p\}$, cross resistivity coefficient
C	constant stress storage coefficient
C_c	constant stress storage coefficient under constant solute concentration
C_{ijkl}	drained compliance tensor
$\boldsymbol{C}, C_{ij}$	drained compliance tenor engineering notation (6×6)

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C^s_{ijkl}	compliance tensor of the solid constituent
C^u_{ijkl}	undrained compliance tensor
C_{bp}, C_{pp}	compliance tensor coefficient
C_{bc}	$(= 1/K = C_K)$ drained frame compressibility
C_{pc}	$(= 1/K_p)$ pore volume compressibility
C_K	$(= 1/K)$ drained frame compressibility
C_u	$(= 1/K_u)$ undrained frame compressibility
C'_s	$(= 1/K'_s)$ unjacketed frame compressibility
C''_s	$(= 1/K''_s)$ unjacketed pore volume compressibility
C_s	$(= 1/K_s)$ solid compressibility
C_φ	$(= 1/K_\varphi)$ porosity compressibility
C_ϕ	$(= 1/K_\phi)$ porosity compressibility
C_f	$(= 1/K_f)$ fluid compressibility
C^*_K	$(= 1/K^* = C_{iijj})$ generalized drained frame compressibility
C_a, C^a_{ij}	added mass coefficient scalar and tensor
c_{ij}	$(= C^s_{ijkk})$ contracted solid compliance tensor
c^s_{ij}	micromechanical solid compliance tensor
c	consolidation coefficient $[\mathrm{L}^2/\mathrm{T}]$
c_c	adiabatic undrained consolidation coefficient
c_α	consolidation coefficient under constant concentration
c_β	concentration diffusivity coefficient under constant pressure
c_a	entropy diffusivity coefficient
c_b	entropy diffusivity coefficient
c_v	specific heat at constant strain
c_d	drained specific heat at constant strain
c_u	undrained specific heat at constant strain
c_f	specific heat of fluid phase at constant pressure
c_s	specific heat of solid phase at constant stress
c_p	specific heat at constant stress
c'	cohesion
$\bar{c}$	average crack half length
c	total molar concentration
c_k	molar concentration (molarity) of species k
c_s, c_w	molar concentration of solute and solvent (water)
c^o_s, c^o_w	initial molar concentration of solute and solvent (water)
c^o	$(= c^o_s)$ initial molar concentration of solute
$\bar{c}^p_k, \bar{c}^b_k$	volume averaged molar concentration of species k in pore fluid and bound fluid
$\bar{c}^b_s, \bar{c}^b_w$	volume averaged molar concentration of solute and solvent (water) in bound fluid
$\bar{c}^p_s, \bar{c}^p_w$	volume averaged molar concentration of solute and solvent (water) in pore fluid
c_s	$(= \Delta\bar{c}^p_s)$ molar concentration of solute in pore fluid

c_s^{po}, c_w^{po}	reference molar concentration of the solute and solvent of the pore fluid
$\mathcal{D}$	dissipation function (rate of energy dissipation)
$\mathcal{D}$, $\mathcal{D}_s$, $\mathcal{D}_f$	dissipation density function, and for solid and fluid phase
$\hat{D}$	relative fluid to solid displacement discontinuity
$\mathbb{D}, D_{ij}$	dispersion coefficient tensor
D_m	mass diffusion coefficient
D_c, D_c'	apparent mass diffusion coefficient
d	mean grain size
$\hat{d}, \hat{d}_i, \hat{d}_{ij}$	displacement discontinuity scalar, vector and tensor
$\hat{d}_n, \hat{d}_s$	normal and shear displacement discontinuity
E, E_i	Young's modulus scalar and vector (anisotropic)
E_u	undrained Young's modulus
E_d	drained dynamic modulus
E_{du}	undrained dynamic modulus
E_{ijkl}	displacement discontinuity forcing function
E_{ij}	eigenstrain tensor
$\mathbb{E}_1, \mathbb{E}_2$	evolutional function equivalent to Young's modulus in plate bending problem
$\mathcal{E}$, $\mathcal{E}_s$, $\mathcal{E}_f$	internal energy, and for solid and fluid phase
$\mathcal{E}$	McNamee-Gibson displacement function
e	$(= e_{ii})$ volumetric strain, positive for dilatation
e_{ij}, $\vec{e}$	total strain tensor (3×3) and vector (engineering notation) (6×1)
e_{ij}^p	pore strain tensor
e_{ij}^s	solid strain tensor
e^f	fluid strain
$\overline{e}_{ij}^s$	solid external strain tensor
$\overline{e}^f$	fluid external strain
$\langle e_{ij} \rangle$	deviatoric strain tensor
$\langle \overline{e}_{ij}^s \rangle$	solid external deviatoric strain tensor
e_{ij}^o	elastic strain tensor of Biot decomposition
e_{ij}^*	total strain tensor (including eigenstrain)
$\vec{F}, F_i$	total body force vector
$F(\kappa_\mu)$	viscosity correction factor
$\mathcal{F}$	dissipation energy function (dissipated energy)
$\mathcal{F}$	dissipation energy density function
$\mathfrak{F}$	Faraday constant
f	$(= \omega/2\pi)$ frequency (in Hz)
$\vec{f}, f_i$	fluid body force vector
G, G_{ij}	shear modulus scalar and tensor
G_s	shear modulus of solid constituent
$\mathcal{G}$	Gibbs free energy
g	gravitational acceleration

$\vec{H}, H_i$	entropy displacement vector
h	piezometric head [L]
h^*	Hubbert's potential
$\hbar$	heat density (per unit volume)
$\vec{J}_T$	heat flux
$\vec{J}_p$	fluid volume flux
$\vec{J}_c$	solute mass flux
$\vec{J}_e$	electrical current
$\vec{J}_s, \vec{J}_w$	flux of solvent and solute (water) per cross-sectional area of porous medium
$\vec{J}_k^s, \vec{J}_k^d$	diffusive and dispersive mass flux of species k per unit area of porous medium per unit time
$\mathsf{K}, \mathsf{K}_{ij}$	hydraulic conductivity tensor [L/T]
K	drained bulk modulus
K_u	undrained bulk modulus
K_p	pore volume bulk modulus
K_s	solid bulk modulus
K_s'	unjacketed frame bulk modulus
K_s''	unjacketed pore volume bulk modulus
K_f	fluid bulk modulus
K_α	solid bulk modulus for porosity preserving deformation
K_φ	micromechanical porosity bulk modulus
K_ϕ	intrinsic porosity bulk modulus
K_ψ	micro inhomogeneity and anisotropy bulk modulus
K^*	generalized drained bulk modulus
K_a	adiabatic drained bulk modulus
K_b	adiabatic undrained bulk modulus
K_d	adiabatic bulk modulus
K_c	constant solute concentration drained bulk modulus
K_ν	constant solute concentration undrained bulk modulus
$\mathcal{K}$	kinetic energy
$\mathcal{K}, \mathcal{K}_s, \mathcal{K}_f$	kinetic energy density, and for solid and fluid phase
k, k	intrinsic permeability scalar and tensor [L^2]
k^*	dimensionless intrinsic permeability tensor
k_T	thermal conductivity
k_{pT}	thermo-osmosis coefficient
k_{Tp}	mechano-caloric coefficient
k_f	fluid kinetic energy per unit mass due to local velocity fluctuation
k_s	solid kinetic energy per unit mass due to local velocity fluctuation
L	wavelength
L_c	characteristic length
L_{ij}	$\{i,j\} = \{T,p,c,e\}$, Onsager law coefficient

L_{ij}	$\{i,j\} = \{s,w\}$, diffusion coefficient for flux law between solute and solvent (water)
L_{ij}	$\{i,j\} = \{p,c\}$, diffusion coefficient for flux law between pore pressure and solute concentration
$\mathcal{L}$	Lagrangian
$\mathcal{L}$	Lagrangian density
$\mathscr{L}$	Laplace transform
$\mathfrak{L}$, $\mathfrak{L}^*$	linear partial differential operator and its adjoint operator
$\ell, \boldsymbol{\ell}$	wave number scaler and vector, real and complex
ℓ_r, ℓ_i	real and imaginary part of complex wave number
M	Biot modulus
M_{sq}	complex Biot modulus for squirt flow
M_c	constant concentration Biot modulus
M_{ijkl}	drained elastic modulus tensor
$\boldsymbol{M}, M_{ij}$	drained elastic modulus tenor engineering notation (6×6)
M^u_{ijkl}	undrained elastic modulus tensor
$\mathcal{M}$	moment
m_α, m_β	constitutive constants for porothermoelastic quadratic form
m_d	$(= c_d/\mathcal{T}_o)$ drained thermoelastic constitutive constant
m_u	$(= c_u/\mathcal{T}_o)$ undrained thermoelastic constitutive constant
m_f	$(= c_f/\mathcal{T}^o_f)$ thermoelastic constitutive constant for fluid phase
m_s	$(= c_s/\mathcal{T}^o_s)$ thermoelastic constitutive constant for solid phase
m	$(= c_p/\mathcal{T}_o)$ thermoelastic constitutive constant at constant stress
$\mathcal{m}_f$	fluid mass
N	total number of particles or moles of atoms or molecules
N_k	number of particles or moles of *k*th species
N_s, N_w	number of particles of solute and solvent (water)
$\vec{n}$, n_i	unit outward normal vector
P	$(= -\sigma_{ii}/3)$ mean total compressive stress
P_o	far-field mean total compressive stress, or loading magnitude
P'	$(= P - p)$ Terzaghi effective compressive stress
P''	$(= P - \alpha p)$ Biot effective compressive stress
p	pore pressure
$\overline{p}^f$	fluid external pressure
$\overline{p}^s$	$(= -\overline{\sigma}^s_{ii}/3)$ solid external pressure
p_b	borehole breakdown pressure
Q	volume of injected fluid per unit porous medium volume (dimensionless)
$\mathcal{Q}$	heat energy
$\mathcal{Q}$	McNamee-Gibson displacement function
Q^{-1}	inverse of quality factor
$\vec{q}$, q_i	fluid specific discharge vector [L/T]
q	$(= q_i n_i)$ normal specific discharge

$\vec{q}$, q_i	heat flux vector
$\vec{q}_r$	solute to solvent relative flux
R	$(= \|\vec{\chi} - \vec{x}\|)$ radial distance, 3D
$\mathcal{R}$	gas constant
$\mathscr{R}$	osmotic reflection coefficient
$\mathscr{R}_a$	chemo-mechanical reflection coefficient
$\mathscr{R}_b$	chemo-hydraulic reflection coefficient
R_q	$(= \sqrt{c/\kappa_T})$ square root of ratio of consolidation coefficient and the heat diffusivity coefficient
R_{p1}, R_{p2}, R_s	Amplitude ratio of reflected or refracted wave to incident wave
r	$(= \|\vec{\chi} - \vec{x}\|)$ radial distance, 2D or 3D
S	storage coefficient (constant stress uniaxial strain)
S_a	adiabatic drained thermal storage coefficient
S_b	adiabatic storage coefficient for entropy diffusion equation
S_c	adiabatic storage coefficient for variation in fluid content diffusion equation
S_e	$(= 1/M)$ constant strain storage coefficient
S_σ	$(= C)$ constant stress storage coefficient
S_α	constant concentration storage coefficient
S_β	constant pressure storage coefficient for solute diffusion
S_s	specific storativity [1/L]
S_o	far-field deviatoric stress
S_r	degree of saturation
$\mathcal{S}$	McNamee-Gibson displacement function
$\mathscr{S}$	entropy
$\mathcal{S}_s$	specific surface (per porous medium volume)
$\mathcal{S}_v$	$(= (1-\phi)\mathcal{S}_s)$ specific surface (per solid volume)
s	Laplace transform parameter
s^*	dimensionless Laplace transform parameter
$\hat{s}, \hat{s}_i, \hat{s}_{ij}$	stress discontinuity scalar, vector, and tensor
$\bar{s}^f$	fluid internal stress
$\bar{s}^s_{ij}$	solid internal stress tensor
$\bar{s}^s$	$(= \bar{s}^s_{ii}/3)$ solid internal average stress
$\mathscr{s}$	entropy density
$\bar{\mathscr{s}}_s, \bar{\mathscr{s}}_f$	solid and fluid volume averaged entropy density
T	incremental temperature
$\overline{T}, \overline{T}_s, \overline{T}_f$	volume averaged incremental temperature, and for solid and fluid phase
$\mathcal{T}$	absolute temperature
$\overline{\mathcal{T}}, \overline{\mathcal{T}}_s, \overline{\mathcal{T}}_f$	volume averaged absolute temperature, and for solid and fluid phase
$\mathcal{T}_o, \mathcal{T}^o_s, \mathcal{T}^o_f$	reference absolute temperature, and for solid and fluid phase
$\mathscr{T}$	wave period
t	time

t^*	dimensionless time
t_c	characteristic time
t_i	surface traction vector
$\mathcal{U}$	internal energy
$\mathcal{U}, \mathcal{U}_s, \mathcal{U}_f$	strain energy density function, and for solid and fluid phase
$\vec{U}, U_i$	fluid displacement vector
$\vec{u}, u_i$	solid frame displacement vector
u_i^s, u_i^f	solid and fluid displacement vector (microscopic)
u_i^{fe}	fluid displacement vector extended to solid space
u_i^{se}	solid displacement vector extended to fluid space
u_i^o	elastic displacement vector of Biot decomposition
V	$(= V_s + V_p)$ total volume of solid frame, or total volume of solution
V_f	$(= V_p)$ fluid volume
V_s	solid volume
V_p	pore volume
ΔV	change of volume of solid frame
ΔV_f	change of fluid volume of the original group of fluid mass
$\Delta V_f'$	change of fluid volume with respect to porous frame
ΔV_s	change of solid volume
ΔV_p	change of pore volume
$\vec{V}$	$(= \vec{q}/\phi)$ seepage velocity
$\mathcal{V}$	potential energy
$\mathcal{V}$	potential energy density
v_i^s, v_i^f	solid and fluid velocity vector
$\overline{v}_i$	relative fluid to solid velocity vector
v_{pe}, v_{se}	elastic compressional and shear wave velocity
v_{p1}, v_{p2}, v_s	first and second compressional wave and shear wave velocity
v_w	wave velocity in water
v_{ph}	phase velocity
v_{gr}	group velocity
v	average molar volume
v_k	partial molar volume of species k
v_s, v_w	partial molar volume of solute and solvent (water)
$\mathcal{W}, \mathcal{W}_s, \mathcal{W}_f$	total work, and of the solid and fluid constituent
$\mathcal{W}_m, \mathcal{W}_c$	mechanical and chemical work
$\mathcal{W}_{\text{irr}}$	irreversible work
$\mathcal{W}, \mathcal{W}_s, \mathcal{W}_f$	work density (per unit volume), and of solid and fluid constituent
$\vec{w}, w_i$	$(= \phi(U_i - u_i))$ specific relative fluid to solid displacement vector
w	$(= w_i n_i)$ normal specific relative fluid to solid displacement
$\vec{X}_T$	temperature gradient driving force
$\vec{X}_p$	pore pressure gradient driving force
$\vec{X}_c$	solute mass concentration (chemical potential) gradient driving force

$\vec{X}_e$	electrical potential gradient driving force
x_k	molar fraction of species k
x_s, x_w	molar fraction of solute and solvent (water)
z^+, z^-	valence of cations and anions

Greek Letter

$\alpha, \vec{\alpha}, \alpha_{ij}$	Biot effective stress coefficient scalar, vector, and tensor
α_a	adiabatic drained Biot effective stress coefficient
α_b	adiabatic undrained Biot effective stress coefficient
α_T	$(= 3K\beta_l)$ thermoelastic effective stress coefficient
α_d	$(= K\beta_d)$ drained thermoelastic effective stress coefficient
α_u	$(= K_u\beta_u)$ undrained thermoelastic effective stress coefficient
α_e	thermoelastic effective stress coefficient in pore pressure diffusion equation
α_p, α_t	poroelastic effective stress coefficient in thermal diffusion equation
α_c, α_c'	constant solute concentration effective stress coefficient
α_μ	chemical effective stress coefficient
$\alpha_\alpha, \alpha_\beta$	chemical effective stress coefficients in pressure and solution diffusion equation
α_l	wave attenuation coefficient $[\mathrm{L}^{-1}]$
$\alpha_{p1}', \alpha_{p2}', \alpha_s'$	complex wave reflection or refraction angle for P1, P2 and S wave
β	pore volume effective stress coefficient
β_l	coefficient of linear thermal expansion
β_d	drained coefficient of volumetric thermal expansion for porous medium frame
β_u	undrained coefficient of volumetric thermal expansion for porous medium frame
β_v	coefficient of volumetric thermal expansion for variation in fluid content
β_e	coefficient of volumetric thermal expansion for variation in fluid content at constant frame volume
β_f	coefficient of volumetric thermal expansion for fluid
β_s	coefficient of volumetric thermal expansion for solid
β_ψ	coefficient of volumetric thermal expansion for porosity
β_b, β_c	thermal expansion coefficient
β_g, β_h	thermal expansion coefficient
β_k^b, β_k^ψ	solid and porosity swelling coefficient of species k in bound water
β_k^p	fluid swelling coefficient of species k in pore fluid
β_μ, β_μ'	chemical swelling coefficients under free stress and pore pressure condition

β_ν	chemical swelling coefficients under free stress and undrained condition
β_ε	chemical swelling coefficients under constant frame and drained condition
$\beta_\alpha, \beta_\beta$	chemical swelling coefficients
β_{ij}	$(= \phi^2 \mu_f b_{ij})$ resistant coefficient tensor
$\Gamma, \Gamma_s, \Gamma_f$	total, solid, and fluid external surface
Γ_k	production of mass of species k per unit volume of porous medium
γ	fluid source intensity, volume of injected fluid per unit porous medium volume per unit time [1/T]
γ_T	heat source
γ_b	coefficient of fluid content
γ_s, γ_f	specific weight of solid and fluid
γ_{be}	barometric efficiency
γ_{te}	tidal efficiency
γ_{ij}	$(= 2e_{ij})$ shear strain (engineering notation)
γ_k	chemical activity coefficient of species k
γ_μ, γ'_μ	constitutive constant for chemical potential under constant stress and constant strain condition
γ_π	constitutive constant for chemical potential associated with osmotic pressure
δ_b	$(= 1/K'_s)$ unjacketed compressibility
δ_ℓ	boundary layer thickness
δ_p	pore geometry factor
δ', δ''	Coulomb shear and volumetric damping coefficient
ϵ	crack density parameter
$\overline{\epsilon}^s, \overline{\epsilon}^f$	solid fluid internal volumetric strain
$\overline{\epsilon}^s_{ij}$	solid internal strain tensor
$\langle \overline{\epsilon}^s_{ij} \rangle$	solid internal deviatoric strain tensor
η	poroelastic stress coefficient
η_d	thermoelastic stress coefficient
η_b	adiabatic undrained poroelastic stress coefficient
η_u	adiabatic undrained thermoelastic stress coefficient
$\theta_{p1}, \theta_{p2}, \theta_s$	incident wave angle for P1, P2 and S wave
$\theta'_{p1}, \theta'_{p2}, \theta'_s$	reflected or refracted wave phase angle for P1, P2 and S wave
$\overline{\theta}$	Cauchy principal value integration coefficient
ζ	variation in fluid content, positive for fluid entering the frame
$\kappa, \boldsymbol{\kappa}, \kappa_{ij}$	$(= k/\mu_f)$ permeability (mobility) coefficient scalar and tensor [L^4/FT]
κ_T	heat diffusivity coefficient
κ_{Tp}	mechano-caloric diffusivity coefficient
κ_D	permeability coefficient of osmotic pressure
κ_{pc}, κ_{cp}	osmotic-hydraulic permeability coefficient
κ_p	constant concentration permeability coefficient for pressure

κ_b	$(= 1/K)$ jacketed compressibility
λ	Lame constant
$\hat{\lambda}_s$	Lame constant of solid phase
λ_u	undrained Lame constant
λ_b	adiabatic undrained Lame constant
λ_D	osmotic transmission coefficient
λ_f	fluid volumetric viscosity
μ	$(= G)$ Lame constant
$\hat{\mu}_s$	$(= G_s)$ Lame constant of solid phase
μ_k	chemical potential of the kth species
μ_s, μ_w	chemical potential for solute and solvent (water)
μ_s	$(= \Delta\overline{\mu}_s)$ chemical potential for solute in pore fluid
$\overline{\mu}_k^p, \overline{\mu}_k^b$	volume averaged chemical potential of species k in pore water and bound water
$\overline{\mu}_k$	$(= \overline{\mu}_k^p = \overline{\mu}_k^b)$ volume averaged chemical potential of species k
$\overline{\mu}_s, \overline{\mu}_w$	volume averaged chemical potential of solute and solvent (water)
$\hat{\mu}_k$	electrochemical potential of the kth species
μ_k^c	integration constant for species k dependent on $\mathcal{T}$ and N_i, $i \neq k$
μ_k^o	integration constant for species k dependent on $\mathcal{T}$ only
μ_s^o, μ_w^o	integration constant for solute and solvent (water) dependent on $\mathcal{T}$ only
$\mu(\chi)$	double layer potential distribution density
$\mu_i(\chi)$	displacement discontinuity distribution density
$\upmu_f$	fluid dynamic viscosity
$\upmu_a$	fluid apparent viscosity
$\upmu$	viscoelastic viscosity
$\upmu_K, \upmu_G$	viscosity for the porous frame and shear modulus
$\upmu_s, \upmu_\phi$	viscosity for solid phase and porosity
ν, ν_{ij}	drained Poisson ratio scalar and tensor
ν_u	undrained Poisson ratio
ν_b	adiabatic undrained Poisson ratio
ν_s	Poisson ratio of solid phase
ν_c	constant solute concentration drained Poisson ratio
ν^+, ν^-	number of cations and anions
Π	osmotic pressure
ρ, ρ_s, ρ_f	total, solid, and fluid density
ρ_a, ρ_{ij}^a	added mass density scalar and tensor
Σ_s, Σ_f	solid and fluid surface
Σ_i	solid/fluid interface
σ	entropy production
$\sigma_{ij}, \vec{\sigma}$	total stress tensor (3×3) and vector (engineering notation) (6×1)
σ'_{ij}	Terzaghi effective stress tensor
σ''_{ij}	Biot effective stress tensor

σ_n	normal stress
σ_{ij}^s	solid stress tensor
$\overline{\sigma}_{ij}^s$	solid external stress tensor
$\langle\sigma_{ij}\rangle$	deviatoric stress tensor
σ_{ij}^o	elastic stress tensor of Biot decomposition
$\sigma(\chi)$	single layer potential distribution density
$\sigma_i(\chi)$	stress discontinuity distribution density
τ	shear stress
τ_t	tortuosity
Φ	porous medium geometric factor
ϕ	porosity
ϕ_a	areal porosity
ϕ_v	volumetric porosity
ϕ'	friction angle
Ψ_i	Boussinesq-Papkovitch function
ψ	Boussinesq-Papkovitch function
$\Omega, \Omega_s, \Omega_f$	total, solid, and fluid volume
$\vec{\Omega}$	rotation vector of displacement
$\boldsymbol{\Omega}_s, \boldsymbol{\Omega}_f$	rotation vector of displacement of solid and fluid
$\boldsymbol{\Omega}_r$	relative fluid to solid displacement rotation vector
ω	angular frequency (in radian)
ω^*	dimensionless frequency
ω_c	characteristic frequency
ω_e	membrane efficiency
ω_k^b, ω_k^p	constitutive constants for porochemoelastic quadratic form

Mathematical Function

ber	real part of Kelvin function of the first kind
bei	imaginary part of Kelvin function of the first kind
ber′	derivative of ber
bei′	derivative of bei
E	complete elliptic integral of the second kind
E_1	elliptic integral
erf	error function
erfc	$(= 1 - \mathrm{erf})$ complimentary error function
$\mathrm{H}(t - t')$	Heaviside unit step function
$\mathrm{H}_0^{(1)}$	Hankel function of the first kind of order zero
I_i	modified Bessel function of the first kind of order i
J_i	Bessel function of the first kind of order i
K_i	modified Bessel function of the second kind of order i

K	complete elliptic integral of the first kind
Y_i	Bessel function of the second kind of order i
$\delta(x, x')$	Dirac delta function
δ_{ij}	Kronecker delta
ε_{ij}	Levi-Cevita permutation symbol
ϑ_{ijk}	permutation symbol

Appendix B
Poroelastic Constants

B.1 Relations Among Bulk Poroelastic Constants

For isotropic materials, there exist only four independent poroelastic constants (see Sect. 2.3). For different purposes, they can be expressed into different forms. In the following, we present the conversion of constants from one set to the other.

$$\lambda = \frac{2G\nu}{1-2\nu} = K - \frac{2G}{3} \tag{B.1}$$

$$\lambda_u = \frac{2G\nu_u}{1-2\nu_u} = K_u - \frac{2G}{3} \tag{B.2}$$

$$E = 2G(1+\nu) = \frac{9KG}{3K+G} \tag{B.3}$$

$$E_u = 2G(1+\nu_u) = \frac{9K_uG}{3K_u+G} \tag{B.4}$$

$$K = \frac{2G(1+\nu)}{3(1-2\nu)} = K_u - \alpha^2 M = K_u(1-\alpha B) = \frac{\alpha}{BC} = \frac{\alpha K_p}{\phi} \tag{B.5}$$

$$K_u = \frac{2G(1+\nu_u)}{3(1-2\nu_u)} = K + \alpha^2 M = \frac{\alpha M}{B} = \frac{K}{1-\alpha B} \tag{B.6}$$

$$K_p = \frac{\phi K}{\alpha} = \frac{\phi}{CB} \tag{B.7}$$

$$\begin{aligned} \nu &= \frac{3K-2G}{2(3K+G)} = \frac{2G\nu_u - \alpha^2 M(1-2\nu_u)}{2[G-\alpha^2 M(1-2\nu_u)]} \\ &= \frac{9\nu_u - 2GCB^2(1+\nu_u)}{9+2GCB^2(1+\nu_u)} = \frac{GS\nu_u - 2\eta^2(1-\nu_u)}{GS-2\eta^2(1-\nu_u)} \end{aligned} \tag{B.8}$$

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$$\nu_u = \frac{3K_u - 2G}{2(3K_u + G)} = \frac{2G\nu + \alpha^2 M(1-2\nu)}{2\left[G + \alpha^2 M(1-2\nu)\right]}$$
$$= \frac{9\nu + 2GCB^2(1+\nu)}{9 - 2GCB^2(1+\nu)} = \nu + \frac{9\alpha^2 GM}{2(G+3K)\left[G + 3\left(K + \alpha^2 M\right)\right]} \tag{B.9}$$

$$\alpha = KCB = \frac{K_u CB}{1 + K_u CB^2} = \frac{K_u B}{M} = \frac{K_u - K}{K_u B} = \frac{2\eta(1-\nu)}{1-2\nu}$$
$$= \frac{\eta}{3G}(3K + 4G) = \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)} = \frac{\phi K}{K_p} \tag{B.10}$$

$$\eta = \frac{3\alpha G}{3K + 4G} = \frac{\alpha(1-2\nu)}{2(1-\nu)} \tag{B.11}$$

$$B = \frac{\alpha M}{K_u} = \frac{\alpha M}{K + \alpha^2 M} = \frac{\alpha}{CK} = \frac{K_u - K}{\alpha K_u} = \frac{3(\nu_u - \nu)}{\alpha(1-2\nu)(1+\nu_u)}$$
$$= \frac{3(\nu_u - \nu)}{2\eta(1-\nu)(1+\nu_u)} = \frac{\phi}{K_p C} \tag{B.12}$$

$$M = \frac{1}{C} + K_u B^2 = \frac{1}{C(1 - KCB^2)} = \frac{K_u B}{\alpha} = \frac{K_u}{KC} = \frac{K_u - K}{\alpha^2}$$
$$= \frac{K}{KC - \alpha^2} = \frac{BK}{\alpha(1 - \alpha B)} = \frac{(1+\nu_u)(1-2\nu)}{C(1+\nu)(1-2\nu_u)} = \frac{2GB(1+\nu_u)}{3\alpha(1-2\nu_u)}$$
$$= \frac{2G(\nu_u - \nu)}{\alpha^2(1-2\nu_u)(1-2\nu)} = \frac{G(1-2\nu)(\nu_u - \nu)}{2\eta^2(1-\nu)^2(1-2\nu_u)} = \frac{G}{GS - \alpha\eta}$$
$$= \frac{(1-\nu_u)(1-2\nu)}{S(1-\nu)(1-2\nu_u)} \tag{B.13}$$

$$S = \frac{3K_u + 4G}{M(3K + 4G)} = \frac{1}{M} + \frac{3\alpha^2}{3K + 4G} = \frac{1}{M} + \frac{\alpha^2(1-2\nu)}{2G(1-\nu)}$$
$$= \frac{1}{M} + \frac{\alpha\eta}{G} = \frac{\alpha^2(1-2\nu)^2(1-\nu_u)}{2G(1-\nu)(\nu_u - \nu)} = \frac{2\eta^2(1-\nu)(1-\nu_u)}{G(\nu_u - \nu)}$$
$$= \frac{(1-\nu_u)(1-2\nu)}{M(1-\nu)(1-2\nu_u)} = C - \frac{4\alpha^2 G}{K(3K + 4G)}$$
$$= \frac{C(1-\nu_u)(1+\nu)}{(1-\nu)(1+\nu_u)} \tag{B.14}$$

$$C = \frac{K + \alpha^2 M}{MK} = \frac{K_u}{M(K_u - \alpha^2 M)} = \frac{K_u}{MK} = \frac{\alpha}{KB} = \frac{\phi}{K_p B}$$
$$= \frac{(1+\nu_u)(1-2\nu)}{M(1+\nu)(1-2\nu_u)} = \frac{\alpha^2(1-2\nu)^2(1+\nu_u)}{2G(1+\nu)(\nu_u - \nu)} \tag{B.15}$$

$$c = \frac{\kappa}{S} = \frac{2\kappa G(1-\nu)(\nu_u-\nu)}{\alpha^2(1-2\nu)^2(1-\nu_u)} = \frac{\kappa G(\nu_u-\nu)}{2\eta^2(1-\nu)(1-\nu_u)}$$
$$= \frac{\kappa M(3K+4G)}{3K_u+4G} \tag{B.16}$$

Limiting Case For the soil mechanics case, with incompressible solid and incompressible fluid, we find

$$K_u \to \infty \tag{B.17}$$
$$\nu_u = \frac{1}{2} \tag{B.18}$$
$$\alpha = 1 \tag{B.19}$$
$$B = 1 \tag{B.20}$$
$$K_p = \phi K \tag{B.21}$$
$$C = \frac{1}{K} \tag{B.22}$$
$$M \to \infty \tag{B.23}$$
$$S = \frac{3}{3K+4G} = \frac{1-2\nu}{2G(1-\nu)} = \frac{\eta}{G} \tag{B.24}$$
$$\eta = \frac{3G}{3K+4G} = \frac{1-2\nu}{2(1-\nu)} \tag{B.25}$$
$$c = \frac{2\kappa G(1-\nu)}{1-2\nu} = \kappa\left(K+\frac{4G}{3}\right) = \frac{\kappa G}{\eta} \tag{B.26}$$

B.2 Relations Among Bulk and Micromechanical Constants

Based on the analysis in Sect. 3.1, there exist four independent Rice-Cleary micromechanical constant [6], K, K_s', K_s'', and K_f, together with porosity ϕ; and related constants are K_φ and β. Their association with the bulk material constants is presented in the following.

$$\frac{1}{K} = \frac{1}{K_\varphi} + \frac{1}{(1-\phi)K_s'} = \frac{1}{K_s'} + \frac{\phi}{K_p} \tag{B.27}$$
$$\frac{1}{K_p} = \frac{1}{\phi}\left(\frac{1}{K} - \frac{1}{K_s'}\right) = \frac{1}{\phi K_\varphi} + \frac{1}{(1-\phi)K_s'} \tag{B.28}$$

$$K_u = K + \frac{K_f K_s''(K_s' - K)^2}{K_f K_s''(K_s' - K) + \phi K_s'^2 (K_s'' - K_f)} \tag{B.29}$$

$$\alpha = 1 - \frac{K}{K_s'} \tag{B.30}$$

$$\beta = 1 - \frac{K_p}{K_s''} = 1 - \frac{\phi K K_s'}{K_s''(K_s' - K)} \tag{B.31}$$

$$B = 1 - \frac{\phi K K_s'(K_s' - K_f)}{K_f K_s''(K_s' - K) + \phi K K_s'(K_s' - K_f)} \tag{B.32}$$

$$C = \frac{\phi}{K_f} + \frac{1}{K} - \frac{1}{K_s'} - \frac{\phi}{K_s''} \tag{B.33}$$

$$M = \frac{K_f K_s'^2 K_s''}{K_f K_s''(K_s' - K) + \phi K_s'^2 (K_s'' - K_f)} \tag{B.34}$$

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{1}{K_s'} - \frac{\phi}{K_s''} - \frac{K}{K_s'^2} \tag{B.35}$$

Ideal Porous Medium For ideal porous medium, $K_s' = K_s'' = K_s$, the above relations simplify to:

$$\frac{1}{K} = \frac{1}{K_\varphi} + \frac{1}{(1-\phi)K_s} = \frac{1}{K_s} + \frac{\phi}{K_p} \tag{B.36}$$

$$\frac{1}{K_p} = \frac{1}{\phi}\left(\frac{1}{K} - \frac{1}{K_s}\right) = \frac{1}{\phi K_\varphi} + \frac{1}{(1-\phi)K_s} \tag{B.37}$$

$$K_u = K + \frac{K_f (K_s - K)^2}{K_f (K_s - K) + \phi K_s (K_s - K_f)} \tag{B.38}$$

$$\alpha = 1 - \frac{K}{K_s} \tag{B.39}$$

$$\beta = 1 - \frac{K_p}{K_s} = 1 - \frac{\phi K}{K_s - K} \tag{B.40}$$

$$B = 1 - \frac{\phi K (K_s - K_f)}{K_f (K_s - K) + \phi K (K_s - K_f)} \tag{B.41}$$

$$C = \frac{\phi}{K_f} + \frac{1}{K} - \frac{1 + \phi}{K_s} \tag{B.42}$$

$$M = \frac{K_f {K_s}^2}{K_f (K_s - K) + \phi K_s (K_s - K_f)} \tag{B.43}$$

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{1-\phi}{K_s} - \frac{K}{{K_s}^2} \tag{B.44}$$

Limiting Cases For the limiting case $(K_s', K_s'') \to \infty$, we obtain

$$K = \frac{K_p}{\phi} = K_\varphi \tag{B.45}$$

$$K_u = K + \frac{K_f}{\phi} \tag{B.46}$$

$$\alpha = 1 \tag{B.47}$$

$$\beta = 1 \tag{B.48}$$

$$B = 1 - \frac{\phi K}{K_f + \phi K} \tag{B.49}$$

$$C = \frac{\phi}{K_f} + \frac{1}{K} \tag{B.50}$$

$$M = \frac{K_f}{\phi} \tag{B.51}$$

Furthermore, if $(K_s', K_s'', K_f) \to \infty$, we obtain

$$K_u \to \infty \tag{B.52}$$

$$B = 1 \tag{B.53}$$

$$C = \frac{1}{K} \tag{B.54}$$

$$M \to \infty \tag{B.55}$$

For the limiting case $K_f \to \infty$, we obtain

$$K_u = K + \frac{K_s''(K_s' - K)^2}{K_s'(K_s'' - \phi K_s') - K_s'' K} \tag{B.56}$$

$$B = 1 + \frac{\phi K K_s'}{K_s'' K_s' - K_s'' K - \phi K K_s'} \tag{B.57}$$

$$C = \frac{1}{K} - \frac{1}{K_s'} - \frac{\phi}{K_s''} \tag{B.58}$$

$$M = \frac{K_s'^2 K_s''}{K_s'' K_s' - K_s'' K - \phi K_s'^2} \tag{B.59}$$

For the limiting case $K_f \to \infty$, together with ideal porous medium assumption, $K_s' = K_s'' = K_s$:

$$K_u = K + \frac{(K_s - K)^2}{(1-\phi)K_s - K} \tag{B.60}$$

$$B = 1 + \frac{\phi K}{K_s - (1+\phi)K} \tag{B.61}$$

$$C = \frac{1}{K} - \frac{1+\phi}{K_s} \tag{B.62}$$

$$M = \frac{{K_s}^2}{(1-\phi)K_s - K} \tag{B.63}$$

For the limiting case $K_f \to 0$, and ideal porous medium ($K_s' = K_s'' = K_s$),

$$K_u \approx K + \frac{(K_s - K)^2 K_f}{\phi {K_s}^2} \tag{B.64}$$

$$B \approx \frac{(K_s - K)K_f}{\phi K K_s} \tag{B.65}$$

$$C \approx \frac{\phi}{K_f} \tag{B.66}$$

$$M \approx \frac{K_f}{\phi} \tag{B.67}$$

B.3 Relations Among Bulk and Intrinsic Constants

As demonstrated in Sects. 4.4 and 4.5, for macroscopically isotropic material, there exist four intrinsic micromechanical material constants, K_s, K_ϕ, K_ψ, and K_f, associated with volumetric deformation. Alternatively, K_s can be replaced by K_α. The bulk material constants can be expressed in terms of the intrinsic constants as:

$$K_\alpha = K_s + \frac{K_\psi^2}{K_\phi} \tag{B.68}$$

$$K = \frac{(1-\phi)^3 K_s K_\phi^2}{K_s K_\phi + [(1-\phi)K_\phi - K_\psi]^2} = \frac{(1-\phi)^3 (K_\alpha K_\phi - K_\psi^2)}{K_\alpha + (1-\phi)^2 K_\phi - 2(1-\phi)K_\psi} \tag{B.69}$$

$$K_p = \frac{\phi(1-\phi)^3 K_s K_\phi^2}{K_s K_\phi + [(1-\phi)K_\phi - K_\psi][\phi(1-\phi)K_\phi - K_\psi]}$$
$$= \frac{\phi(1-\phi)^3 (K_\alpha K_\phi - K_\psi^2)}{K_\alpha + \phi(1-\phi)^2 K_\phi - (1-\phi^2)K_\psi} \tag{B.70}$$

$$K_u = \frac{K_f\{K_s K_\phi + [\phi(1-\phi)K_\phi - K_\psi]^2\} + \phi(1-\phi)^3 K_s K_\phi^2}{(1-\phi)K_f K_\phi + \phi\{K_s K_\phi + [(1-\phi)K_\phi - K_\psi]^2\}}$$
$$= \frac{K_f[K_\alpha + \phi^2(1-\phi)^2 K_\phi - 2\phi(1-\phi)K_\psi] + \phi(1-\phi)^3(K_\alpha K_\phi - K_\psi^2)}{(1-\phi)K_f + \phi[K_\alpha + (1-\phi)^2 K_\phi - 2(1-\phi)K_\psi]} \tag{B.71}$$

$$\alpha = 1 - \frac{(1-\phi)^2 K_\phi[(1-\phi)K_\phi - K_\psi]}{K_s K_\phi + [(1-\phi)K_\phi - K_\psi]^2}$$
$$= 1 - \frac{(1-\phi)^2[(1-\phi)K_\phi - K_\psi]}{K_\alpha + (1-\phi)^2 K_\phi - 2(1-\phi)K_\psi} \tag{B.72}$$

$$B = \frac{K_f\{K_s K_\phi + [(1-\phi)K_\phi - K_\psi][\phi(1-\phi)K_\phi - K_\psi]\}}{K_f\{K_s K_\phi + [\phi(1-\phi)K_\phi - K_\psi]^2\} + \phi(1-\phi)^3 K_s K_\phi^2}$$
$$= \frac{K_f[K_\alpha + \phi(1-\phi)^2 K_\phi - (1-\phi^2)K_\psi]}{K_f[K_\alpha + \phi^2(1-\phi)^2 K_\phi - 2\phi(1-\phi)K_\psi] + \phi(1-\phi)^3(K_\alpha K_\phi - K_\psi^2)} \tag{B.73}$$

$$C = \frac{\phi}{K_f} + \frac{K_s + [\phi(1-\phi)K_\phi - K_\psi]^2}{(1-\phi)^3 K_s K_\phi^2}$$
$$= \frac{\phi}{K_f} + \frac{K_\alpha + \phi^2(1-\phi)^2 K_\phi - 2\phi(1-\phi)K_\psi}{(1-\phi)^3(K_\alpha K_\phi - K_\psi^2)} \tag{B.74}$$

$$M = \frac{K_f\{K_s K_\phi + [(1-\phi)K_\phi - K_\psi]^2\}}{(1-\phi)K_f K_\phi + \phi\{K_s K_\phi + [(1-\phi)K_\phi - K_\psi]^2\}}$$
$$= \frac{K_f[K_\alpha + (1-\phi)^2 K_\phi - 2(1-\phi)K_\psi]}{(1-\phi)K_f + \phi[K_\alpha + (1-\phi)^2 K_\phi - 2(1-\phi)K_\psi]} \tag{B.75}$$

Ideal Porous Medium For ideal porous medium, $K_\psi = 0$, the above relations simplify to:

$$K_\alpha = K_s \tag{B.76}$$

$$K = \frac{(1-\phi)^3 K_s K_\phi}{K_s + (1-\phi)^2 K_\phi} \tag{B.77}$$

$$\frac{1}{K} = \frac{1}{(1-\phi)K_s} + \frac{1}{(1-\phi)^3 K_\phi} \tag{B.78}$$

$$K_p = \frac{\phi(1-\phi)^3 K_s K_\phi}{K_s + \phi(1-\phi)^2 K_\phi} \tag{B.79}$$

$$\frac{1}{K_p} = \frac{1}{(1-\phi)K_s} + \frac{1}{\phi(1-\phi)^3 K_\phi} \tag{B.80}$$

$$K_u = \frac{K_f[K_s + \phi^2(1-\phi)^2 K_\phi] + \phi(1-\phi)^3 K_s K_\phi}{(1-\phi)K_f + \phi[K_s + (1-\phi)^2 K_\phi]} \tag{B.81}$$

$$\alpha = 1 - \frac{(1-\phi)^3 K_\phi}{K_s + (1-\phi)^2 K_\phi} \tag{B.82}$$

$$B = 1 - \frac{\phi(1-\phi)^3 K_\phi (K_s - K_f)}{K_f[K_s + \phi^2(1-\phi)^2 K_\phi] + \phi(1-\phi)^3 K_s K_\phi} \tag{B.83}$$

$$C = \frac{\phi}{K_f} + \frac{1}{(1-\phi)^3 K_\phi} + \frac{\phi^2}{(1-\phi)K_s} \tag{B.84}$$

$$M = \frac{K_f[K_s + (1-\phi)^2 K_\phi]}{(1-\phi)K_f + \phi[K_s + (1-\phi)^2 K_\phi]} \tag{B.85}$$

Limiting Cases For the limiting case $K_s \to \infty$, we obtain

$$K = (1-\phi)^3 K_\phi \tag{B.86}$$

$$K_p = \phi(1-\phi)^3 K_\phi \tag{B.87}$$

$$K_u = \frac{K_f}{\phi} + (1-\phi)^3 K_\phi = \frac{K_f}{\phi} + K \tag{B.88}$$

$$\alpha = 1 \tag{B.89}$$

$$B = 1 - \frac{\phi(1-\phi)^3 K_\phi}{K_f + \phi(1-\phi)^3 K_\phi} = 1 - \frac{\phi K}{K_f + \phi K} \tag{B.90}$$

$$C = \frac{\phi}{K_f} + \frac{1}{(1-\phi)^3 K_\phi} = \frac{\phi}{K_f} + \frac{1}{K} \tag{B.91}$$

$$M = \frac{K_f}{\phi} \tag{B.92}$$

Furthermore, if both $(K_s, K_f) \to \infty$, we obtain

$$K_u \to \infty \tag{B.93}$$

$$B = 1 \tag{B.94}$$

$$C = \frac{1}{(1-\phi)^3 K_\phi} = \frac{1}{K} \tag{B.95}$$

$$M \to \infty \tag{B.96}$$

For the limiting case $K_f \to \infty$, we obtain

$$K_u = \frac{K_s}{1-\phi} + \phi^2(1-\phi)K_\phi - 2\phi K_\psi \tag{B.97}$$

$$B = 1 + \frac{(1-\phi)^2[\phi(1-\phi)K_\phi - K_\psi]}{K_s + \phi(1-\phi)[\phi(1-\phi)K_\phi - 2K_\psi]} \tag{B.98}$$

$$C = \frac{K_s + \phi^2(1-\phi)^2 K_\phi - 2\phi(1-\phi)K_\psi}{(1-\phi)^3(K_s K_\phi - K_\psi^2)} \tag{B.99}$$

$$M = \frac{K_s}{1-\phi} + (1-\phi)K_\phi - 2K_\psi \tag{B.100}$$

For the limiting case $K_f \to \infty$, and ideal porous medium, $K_\psi = 0$:

$$K_u = \frac{K_s}{1-\phi} + \phi^2(1-\phi)K_\phi \tag{B.101}$$

$$B = 1 + \frac{\phi(1-\phi)^3 K_\phi}{K_s + \phi^2(1-\phi)^2 K_\phi} \tag{B.102}$$

$$C = \frac{\phi^2(1-\phi)}{K_s} + \frac{1}{(1-\phi)K_\phi} \tag{B.103}$$

$$M = \frac{K_s}{1-\phi} + (1-\phi)K_\phi \tag{B.104}$$

For the limiting case $K_f \to 0$, and ideal porous medium ($K_\psi = 0$)

$$K_u \approx \frac{(1-\phi)^3 K_s K_\phi}{K_s + (1-\phi)^2 K_\phi} + \frac{[K_s + \phi(1-\phi)^2 K_\phi]^2}{\phi[K_s + (1-\phi)^2 K_\phi]^2} K_f \tag{B.105}$$

$$B \approx \frac{K_f[K_s + \phi(1-\phi)^2 K_\phi]}{\phi(1-\phi)^3 K_s K_\phi} \tag{B.106}$$

$$C \approx \frac{\phi}{K_f} \tag{B.107}$$

$$M \approx \frac{K_f}{\phi} \tag{B.108}$$

B.4 Relations Among Micromechanical and Intrinsic Constants

Relations among the micromechanical constants K_s', K_s'', K_φ, β, and the intrinsic constants K_s (or K_α), K_ϕ, K_ψ, are presented below. Relations for K are contained in Sect. B.3.

$$K_s' = \frac{(1-\phi)K_s K_\phi}{(1-\phi)K_\phi - K_\psi} = \frac{(1-\phi)(K_\alpha K_\phi - K_\psi^2)}{(1-\phi)K_\phi - K_\psi} \tag{B.109}$$

$$K_s'' = \frac{\phi(1-\phi)K_s K_\phi}{\phi(1-\phi)K_\phi - K_\psi} = \frac{\phi(1-\phi)(K_\alpha K_\phi - K_\psi^2)}{\phi(1-\phi)K_\phi - K_\psi} \tag{B.110}$$

$$K_\varphi = \frac{(1-\phi)^3 K_s K_\phi^2}{K_s K_\phi - (1-\phi)K_\phi K_\psi + K_\psi^2} = \frac{(1-\phi)^3(K_\alpha K_\phi - K_\psi^2)}{K_\alpha - (1-\phi)K_\psi} \tag{B.111}$$

$$\begin{aligned}\beta &= 1 - \frac{(1-\phi)^2 K_\phi[\phi(1-\phi)K_\phi - K_\psi]}{K_s K_\phi + [(1-\phi)K_\phi - K_\psi][\phi(1-\phi)K_\phi - K_\psi]} \\ &= 1 - \frac{(1-\phi)^2[\phi(1-\phi)K_\phi - K_\psi]}{K_\alpha + \phi(1-\phi)^2 K_\phi - (1-\phi^2)K_\psi}\end{aligned} \tag{B.112}$$

$$K_s = \frac{(1-\phi)K_s' K_s''}{K_s'' - \phi K_s'} \tag{B.113}$$

$$K_\alpha = \frac{K_s'[(1-\phi)^2 K_s' K_s'' - \phi K K_s' - (1-2\phi)K K_s'']}{(1-\phi)[K_s'(K_s'' - \phi K_s') - K K_s'']} \tag{B.114}$$

$$K_\phi = \frac{K K_s'(K_s'' - \phi K_s')}{(1-\phi)^3[K_s'(K_s'' - \phi K_s') - K K_s'']} \tag{B.115}$$

$$K_\psi = \frac{\phi K K_s'(K_s'' - K_s')}{(1-\phi)^2[K_s'(K_s'' - \phi K_s') - K K_s'']} \tag{B.116}$$

Ideal Porous Medium For ideal porous medium, $K_\psi = 0$ and $K_s' = K_s'' = K_s$, and

$$K_\varphi = (1-\phi)^3 K_\phi \tag{B.117}$$

$$\beta = 1 - \frac{\phi(1-\phi)^3 K_\phi}{K_s + \phi(1-\phi)^2 K_\phi} \tag{B.118}$$

$$K_\phi = \frac{K K_s}{(1-\phi)^2[(1-\phi)K_s - K]} \tag{B.119}$$

Limiting Case For the limiting case $K_s \to \infty$, we obtain

$$\beta = 1 \tag{B.120}$$

$$K_\phi = \frac{K}{(1-\phi)^3} \tag{B.121}$$

B.5 Relations Among Biot-Willis and Micromechanical Constants

Here we give a summary of relations already established in Sect. 3.5, between the Biot-Willis laboratory constants and the Rice-Cleary micromechanical constants.

$$\kappa_b = \frac{1}{K} \tag{B.122}$$

$$\delta_b = \frac{1}{K_s'} \tag{B.123}$$

$$\gamma_b = \phi\left(\frac{1}{K_f} - \frac{1}{K_s''}\right) \tag{B.124}$$

B.6 Biot Notations

To allow easy comparison between the constants and notations used by Biot in his publications and those introduced in this book, we shall establish their relations below.

Biot 1941 The constitutive constants H, R and Q (two of them are independent) introduced by Biot in 1941 [1] are here denoted as H', R' and Q' to avoid confusion, as these same symbols were redefined by Biot in 1955 [2], as well as in subsequent papers [3–5]. The equivalence between $\{H', R', Q'\}$ and bulk material constants can be expressed as

$$Q' = M \tag{B.125}$$

$$H' = \frac{K}{\alpha} \tag{B.126}$$

$$R' = \frac{BK}{\alpha} \tag{B.127}$$

Biot 1955 The Biot [2] four independent constants A, N, Q, R, were discussed in detail by Biot and Willis [5]. They are related to the current bulk material constants as

$$A = \lambda + M(\alpha - \phi)^2 \tag{B.128}$$

$$N = G \tag{B.129}$$

$$Q = \phi M(\alpha - \phi) \tag{B.130}$$

$$R = \phi^2 M \tag{B.131}$$

Biot further defined

$$P = A + 2N = \lambda + 2G + M(\alpha - \phi)^2 \tag{B.132}$$

$$S = A - \frac{Q^2}{R} = \lambda \tag{B.133}$$

A useful formula is

$$\alpha = \frac{\phi(Q + R)}{R} \tag{B.134}$$

Biot 1956 For poroelastodynamics, Biot [4] introduced

$$H = P + R + 2Q = \lambda_u + 2G \tag{B.135}$$

B.7 Porothermoelasticity Constants

Relations Among Bulk Constants

$$\beta_b = \beta_e - \frac{\alpha_u \eta}{MGS} = \beta_e - \frac{K_u \eta}{MGS}\beta_u \tag{B.136}$$

$$\begin{aligned}\beta_c &= \beta_e - \frac{3\alpha K}{3K + 4G}\beta_d = \beta_e - \frac{K\eta}{G}\beta_d = \beta_e - \frac{\alpha_d\,\eta}{G} \\ &= \beta_e - \frac{\alpha \eta_d}{G} = \beta_v + \frac{4}{3}\eta\beta_d\end{aligned} \tag{B.137}$$

$$\beta_e = \alpha\beta_d + \beta_v \tag{B.138}$$

$$\beta_g = \beta_e - \frac{3\alpha_b(m_d\alpha_u - M m_u \alpha_b \beta_e)}{m_d(3K_d + 4G)} \tag{B.139}$$

$$\beta_h = \beta_e + \frac{3\alpha_u(\alpha_u\beta_e - \alpha_b m_u)}{m_u(3K_u + 4G)} \tag{B.140}$$

$$\beta_u = \beta_d + B\beta_v \tag{B.141}$$

$$\alpha_a = \alpha + \frac{\alpha_d \beta_e}{m_d} \tag{B.142}$$

$$\alpha_b = \alpha + \frac{\alpha_u \beta_v}{m_u} \tag{B.143}$$

$$\alpha_d = K\beta_d = \frac{2\eta_d(1-\nu)}{1-2\nu} = \frac{\alpha\,\eta_d}{\eta} \tag{B.144}$$

$$\alpha_u = K_u\beta_u = \frac{M\beta_u}{B}\alpha = K\beta_d + \alpha M\beta_e = K_u\beta_d + \alpha M\beta_v \tag{B.145}$$

$$\alpha_e = \frac{\beta_c}{S} \tag{B.146}$$

$$\alpha_t = \frac{\beta_b}{S_b} \tag{B.147}$$

$$\alpha_p = \frac{\beta_c}{S_a} \tag{B.148}$$

$$m_u = m_d - M\beta_e^2 \tag{B.149}$$

$$K_a = K + \frac{\alpha_d^2}{m_d} \tag{B.150}$$

$$K_b = K_u + \frac{\alpha_u^2}{m_u} \tag{B.151}$$

$$K_d = K_b - \frac{Mm_u\alpha_b^2}{m_d} \tag{B.152}$$

$$\eta_d = \frac{3\alpha_d\,G}{3K+4G} = \frac{\eta\,\alpha_d}{\alpha} = \frac{\alpha_d(1-2\nu)}{2(1-\nu)} \tag{B.153}$$

$$\eta_b = \frac{3\alpha_b G}{3K_b+4G} \tag{B.154}$$

$$\eta_u = \frac{3\alpha_u G}{3K_b+4G} \tag{B.155}$$

$$S_a = \frac{m_d(3K_a+4G)}{3K+4G} = m_d + \frac{3\alpha_d^2}{3K+4G} = m_d + \frac{\alpha_d\,\eta_d}{G} \tag{B.156}$$

$$S_b = \frac{m_u(3K_b+4G)}{3K_u+4G} = m_u + \frac{3\alpha_u^2}{3K_u+4G} = m_u + \frac{\alpha_u\eta\beta_u}{GSB} \tag{B.157}$$

$$S_c = \frac{m_u(3K_b+4G)}{m_d M(3K_d+4G)} \tag{B.158}$$

$$c_a = \frac{m_d\kappa_T}{S_a} \tag{B.159}$$

$$c_b = \frac{m_d\kappa_T}{S_b} \tag{B.160}$$

$$c_c = \frac{\kappa}{S_c} \tag{B.161}$$

$$\kappa_T = \frac{k_T}{c_d} = \frac{k_T}{m_d \mathcal{T}_o} \tag{B.162}$$

Limits for Thermally Uncoupled Model

$$\left(\frac{\alpha_d}{m_d},\ \frac{\beta_e}{m_d},\ \alpha_p\right) \to 0 \tag{B.163}$$

$$(\alpha_a,\ \alpha_b) \to \alpha \tag{B.164}$$

$$(m_u,\ S_a) \to m_d \tag{B.165}$$

$$K_a \to K \tag{B.166}$$

$$K_b \to K_u \tag{B.167}$$

$$c_a \to \kappa_T \tag{B.168}$$

Relation among Bulk and Intrinsic Constants

$$\beta_d = \beta_s + \frac{\beta_\psi}{1-\phi} \tag{B.169}$$

$$\beta_v = \phi\,(\beta_f - \beta_s) - \frac{\beta_\psi}{1-\phi} \tag{B.170}$$

$$\beta_u = (1-\phi B)\beta_s + \phi B \beta_f + \frac{1-B}{1-\phi}\beta_\psi \tag{B.171}$$

$$\beta_e = (\alpha-\phi)\beta_s + \phi\beta_f - \frac{1-\alpha}{1-\phi}\beta_\psi \tag{B.172}$$

$$m_d = m - K\left(\beta_s + \frac{\beta_\psi}{1-\phi}\right)^2 \tag{B.173}$$

$$m_u = m - M\left[(\alpha-\phi)\beta_s + \phi\beta_f - \frac{1-\alpha}{1-\phi}\beta_\psi\right]^2 \tag{B.174}$$

with K, α, B, and M defined in (B.69), (B.72), (B.73), and (B.75).

For ideal porous medium

$$\beta_d = \beta_s \tag{B.175}$$

$$\beta_v = \phi\,(\beta_f - \beta_s) \tag{B.176}$$

$$\beta_u = (1-\phi B)\beta_s + \phi B\beta_f \tag{B.177}$$

$$\beta_e = (\alpha-\phi)\beta_s + \phi\beta_f \tag{B.178}$$

$$m_d = m - K\beta_s^2 \tag{B.179}$$

$$m_u = m - M\left[(\alpha - \phi)\beta_s + \phi\beta_f\right]^2 \tag{B.180}$$

with K, α, B, and M defined in (B.77), (B.82), (B.83), and (B.85).

B.8 Porochemoelasticity Constants

Relations Among Bulk Constants

$$K_\nu = \frac{K_c}{1 - \alpha_c B_c} = K_c + M_c \alpha_c \alpha_c' \tag{B.181}$$

$$M_c = \frac{K_c B_c}{\alpha_c'(1 - \alpha_c B_c)} \tag{B.182}$$

$$\nu_c = \frac{3K_c - 2G}{2(3K_c + G)} \tag{B.183}$$

$$C_c = \frac{\alpha_c'}{K_c B_c} \tag{B.184}$$

$$B_c = \frac{\alpha_c' M_c}{K_\nu} = \frac{\alpha_c' M_c}{K_c + \alpha_c \alpha_c' M_c} = \frac{\alpha_c'}{C_c K_c} = \frac{K_\nu - K_c}{\alpha_c K_\nu} \tag{B.185}$$

$$\alpha_a = \alpha + \frac{\alpha_d \beta_e}{m_d} \tag{B.186}$$

$$\alpha_b = \alpha + \frac{\alpha_u \beta_e}{m_u} \tag{B.187}$$

$$\alpha_d = K\beta_d \tag{B.188}$$

$$\alpha_e = \frac{\beta_c}{S} \tag{B.189}$$

$$\alpha_p = \frac{\beta_c}{S_a} \tag{B.190}$$

$$\alpha_u = K_u \beta_u \tag{B.191}$$

$$\alpha_\alpha = \frac{M_c[\beta_\alpha(4G + 3K_c) - 3\alpha_c' \alpha_\mu]}{4G + 3K_\nu} = \frac{M_c(G\beta_\alpha - \alpha_c' \eta_\mu)}{G + M_c\, \alpha_c' \eta_c} \tag{B.192}$$

$$\alpha_\beta = \frac{M_c[\beta_\beta(4G + 3K_c) - 3\alpha_c' \alpha_\mu]}{4G + 3K_\nu} = \frac{M_c(G\beta_\beta - \alpha_c' \eta_\mu)}{G + M_c\, \alpha_c' \eta_c} \tag{B.193}$$

$$\eta_c = \frac{3\alpha_c G}{3K_c + 4G} \tag{B.194}$$

$$\eta_\mu = \frac{3\alpha_\mu G}{3K_c + 4G} = \frac{\eta_c \alpha_\mu}{\alpha_c} \tag{B.195}$$

$$\beta_e = \alpha\beta_d + \beta_v \tag{B.196}$$

$$\beta_\nu = \frac{\alpha_\mu}{K_c} - \frac{B_c \alpha_c \beta_\mu}{\alpha'_c} \tag{B.197}$$

$$\beta_\varepsilon = \frac{\alpha'_c \alpha_\mu}{K_c} - \beta_\mu \tag{B.198}$$

$$\beta_\beta = \frac{\phi}{(1-\mathcal{R})c_o} - \beta_\varepsilon \tag{B.199}$$

$$\beta_\alpha = \beta_\varepsilon + \frac{\mathcal{R}\kappa\mathcal{R}\mathcal{T}_o\phi}{(1-\mathcal{R})D_c - \mathcal{R}\,c_o\kappa\mathcal{R}\mathcal{T}_o} \tag{B.200}$$

$$\gamma'_\mu = \gamma_\mu - \frac{B_c \beta'_\mu (K_c \beta_\mu - \alpha'_c \alpha_\mu)}{\alpha'_c (1 - \alpha_c B_c)} \tag{B.201}$$

$$\gamma_\pi = \frac{\gamma_\mu}{\mathcal{R}\mathcal{T}_o} \tag{B.202}$$

$$\kappa_p = \left[1 + \frac{\mathcal{R}^2 c_o \kappa \mathcal{R}\mathcal{T}_o}{(1-\mathcal{R})D_c - \mathcal{R}\,c_o\kappa\mathcal{R}\mathcal{T}_o}\right]\kappa \tag{B.203}$$

$$D'_c = \frac{\phi\,(1-\mathcal{R})(D_c - \mathcal{R}\,c_o\kappa\mathcal{R}\mathcal{T}_o)}{\phi - \mathcal{R}\,c_o\beta_\varepsilon} \tag{B.204}$$

$$c_\alpha = \frac{\kappa_p}{S_\alpha} \tag{B.205}$$

$$c_\beta = \frac{\beta_\beta D'_c}{S_\beta} \tag{B.206}$$

$$S_\alpha = \frac{4G + 3K_\nu}{M_c(4G + 3K_c)} = \frac{1}{M_c} + \frac{\alpha'_c \eta_c}{G} \tag{B.207}$$

$$S_\beta = \beta_\beta - \frac{3\alpha'_c \alpha_\mu}{4G + 3K_c} = \beta_\beta - \frac{\alpha'_c\,\eta_\mu}{G} \tag{B.208}$$

$$\mathcal{R}_a = \frac{\alpha_\mu}{\alpha_c \mathcal{R}\mathcal{T}_o} \tag{B.209}$$

$$\mathcal{R}_b = \frac{\beta_\mu}{C_c \mathcal{R}\mathcal{T}_o} \tag{B.210}$$

Relation Among Bulk and Intrinsic Constants Full expressions too lengthy to list.

For ideal porous medium, $K_\psi = \beta_s^\phi = \beta_w^\phi = 0$,

$$K_c = \frac{(1-\phi)^3 K_s K_\phi}{D_1}\left[\omega_s^b\,\omega_w^b - K_s(\beta_w^b \beta_w^b\,\omega_s^b + \beta_s^b \beta_s^b\,\omega_w^b)\right] \tag{B.211}$$

$$\alpha_c = \alpha_c' - \frac{K_s K_\phi (1-\phi)^3}{D_1 D_2} \left\{ [(1 + c_s^{po} \beta_s^p) \beta_w^b \, \omega_s^b - c_w^{po} \beta_s^b \beta_s^p \, \omega_w^b] \, \omega_w^p \right.$$
$$\left. - K_f \beta_w^p (\beta_w^b \beta_w^p \, \omega_s^b + \beta_s^b \beta_s^p \, \omega_w^b) \right\} \tag{B.212}$$

$$\alpha_c' = 1 - \frac{(1-\phi)^3 K_\phi \, \omega_s^b \, \omega_w^b}{D_1} \tag{B.213}$$

$$\alpha_\mu = \frac{(1-\phi)^3 K_s K_\phi}{D_1 D_2} \left[\omega_s^p \, \omega_w^p - K_f (\beta_w^p \, \beta_w^p \, \omega_s^p + \beta_s^p \, \beta_s^p \, \omega_w^p) \right]$$
$$\times (c_w^{po} \beta_s^b \, \omega_w^b - c_s^{po} \beta_w^b \, \omega_s^b) \tag{B.214}$$

$$B_c = \frac{K_f}{D_3} [c_w^{po} \omega_w^p - K_f \beta_w^p (c_s^{po} \beta_s^p + c_w^{po} \beta_w^p)]$$
$$\times \left\{ [K_s + \phi(1-\phi)^2 K_\phi] \, \omega_s^b \, \omega_w^b - K_s^2 (\beta_w^b \beta_w^b \, \omega_s^b + \beta_s^b \beta_s^b \, \omega_w^b) \right\} \tag{B.215}$$

$$\beta_\mu = \frac{\phi}{D_4} \left\{ -(c_w^{po} \beta_s^p \, \omega_w^p - c_s^{po} \beta_w^p \, \omega_s^p) [\, \omega_s^b \, \omega_w^b - K_s (\beta_w^b \, \beta_w^b \, \omega_s^b + \beta_s^b \, \beta_s^b \, \omega_w^b)] \right.$$
$$\left. + (c_w^{po} \beta_s^b \, \omega_w^b - c_s^{po} \beta_w^b \, \omega_s^b) [\, \omega_s^p \, \omega_w^p - K_f (\beta_w^p \, \beta_w^p \, \omega_s^p + \beta_s^p \, \beta_s^p \, \omega_w^p)] \right\} \tag{B.216}$$

$$\beta_\mu' = \frac{\beta_s^p}{D_2} (c_w^{po} \omega_w^p + K_f \beta_w^p) \tag{B.217}$$

$$\gamma_\mu = \frac{c_w^{po}}{D_2} \left[\omega_s^p \, \omega_w^p - K_f (\beta_w^p \beta_w^p \, \omega_s^p + \beta_s^p \beta_s^p \, \omega_w^p) \right] \tag{B.218}$$

where

$$D_1 = [K_s + (1-\phi)^2 K_\phi] \, \omega_s^b \, \omega_w^b - K_s^2 (\beta_w^b \beta_w^b \, \omega_s^b + \beta_s^b \beta_s^b \, \omega_w^b) \tag{B.219}$$

$$D_2 = c_w^{po} \, \omega_w^p - K_f \beta_w^p (c_s^{po} \beta_s^p + c_w^{po} \beta_w^p) \tag{B.220}$$

$$D_3 = K_f^2 \beta_w^p (c_s^{po} \beta_s^p + c_w^{po} \beta_w^p) \left\{ K_s^2 (\beta_w^b \beta_w^b \, \omega_s^b + \beta_s^b \beta_s^b \, \omega_w^b) \right.$$
$$\left. - [K_s + \phi^2 (1-\phi)^2 K_\phi] \, \omega_s^b \, \omega_w^b \right\}$$
$$- \phi (1-\phi)^3 K_s K_\phi K_f \left[K_s \beta_w^p (\beta_w^b \beta_w^b \, \omega_s^b + \beta_s^b \beta_s^b \, \omega_w^b) \right.$$
$$- K_f \beta_w^p (\beta_w^b \beta_w^p \, \omega_s^b + \beta_s^b \beta_s^p \, \omega_w^b)$$
$$\left. + \beta_s^p (c_s^{po} \beta_w^b \, \omega_s^b - c_w^{po} \beta_s^b \, \omega_w^b) \, \omega_w^p - \omega_s^b (\beta_w^p \, \omega_w^b - \beta_w^b \, \omega_w^p) \right]$$
$$+ c_w^{po} \left\{ \phi (1-\phi)^3 K_s K_\phi + K_f [K_s + \phi^2 (1-\phi)^2 K_\phi] \omega_w^p \right\} \omega_s^b \, \omega_w^b$$
$$- c_w^{po} K_s^2 [K_f + \phi (1-\phi)^3 K_\phi] (\beta_w^b \beta_w^b \, \omega_s^b + \beta_s^b \beta_s^b \, \omega_w^b) \, \omega_w^p \tag{B.221}$$

$$D_4 = D_2 [\, \omega_s^b \, \omega_w^b - K_s (\beta_w^b \beta_w^b \, \omega_s^b + \beta_s^b \beta_s^b \, \omega_w^b)] \tag{B.222}$$

For dilute solution, $c_s^{po} = \beta_s^p = 0$,

$$K_c = \frac{(1-\phi)^3 K_s K_\phi}{D_1}\left[\omega_s^b \omega_w^b - K_s(\beta_w^b \beta_w^b \omega_s^b + \beta_s^b \beta_s^b \omega_w^b)\right] \tag{B.223}$$

$$\alpha_c = \alpha_c' - \frac{(1-\phi)^3 K_s K_\phi \beta_w^b \omega_s^b}{c_w^{po} D_1} \tag{B.224}$$

$$\alpha_c' = 1 - \frac{(1-\phi)^3 K_\phi \omega_s^b \omega_w^b}{D_1} \tag{B.225}$$

$$\alpha_\mu = \frac{(1-\phi)^3 K_s K_\phi \beta_s^b \omega_s^p \omega_w^b}{D_1} \tag{B.226}$$

$$\begin{aligned} B_c = \frac{K_f c_w^{po}}{D_3}(\omega_w^p - K_f \beta_w^p \beta_w^p)\left\{[K_s + \phi(1-\phi)^2 K_\phi]\,\omega_s^b \omega_w^b \right.\\ \left. - K_s^2(\beta_w^b \beta_w^b \omega_s^b + \beta_s^b \beta_s^b \omega_w^b)\right\} \end{aligned} \tag{B.227}$$

$$\beta_\mu = \frac{\phi \beta_s^b \omega_w^b \omega_s^p}{\omega_s^b \omega_w^b - K_s(\beta_w^b \beta_w^b \omega_s^b + \beta_s^b \beta_s^b \omega_w^b)} \tag{B.228}$$

$$\beta_\mu' = 0 \tag{B.229}$$

$$\gamma_\mu = \omega_s^p \tag{B.230}$$

where

$$D_1 = [K_s + (1-\phi)^2 K_\phi]\,\omega_s^b \omega_w^b - K_s^2(\beta_w^b \beta_w^b \omega_s^b + \beta_s^b \beta_s^b \omega_w^b) \tag{B.231}$$

$$\begin{aligned} D_3 = {} & K_f^2 c_w^{po} \beta_w^p \beta_w^p \left\{K_s^2(\beta_w^b \beta_w^b \omega_s^b + \beta_s^b \beta_s^b \omega_w^b)\right.\\ & \left. - [K_s + \phi^2(1-\phi)^2 K_\phi]\,\omega_s^b \omega_w^b\right\}\\ & - \phi(1-\phi)^3 K_s K_\phi K_f \left[K_s \beta_w^p(\beta_w^b \beta_w^b \omega_s^b + \beta_s^b \beta_s^b \omega_w^b)\right.\\ & \left. - K_f \beta_w^b \beta_w^p \beta_w^p \omega_s^b - \omega_s^b(\beta_w^p \omega_w^b - \beta_w^b \omega_w^p)\right]\\ & + c_w^{po}\left\{\phi(1-\phi)^3 K_s K_\phi + K_f[K_s + \phi^2(1-\phi)^2 K_\phi]\omega_w^p\right\}\omega_s^b \omega_w^b\\ & - c_w^{po} K_s^2[K_f + \phi(1-\phi)^3 K_\phi](\beta_w^b \beta_w^b \omega_s^b + \beta_s^b \beta_s^b \omega_w^b)\,\omega_w^p \end{aligned} \tag{B.232}$$

References

1. Biot MA (1941) General theory of three-dimensional consolidation. J Appl Phys 12(2):155–164
2. Biot MA (1955) Theory of elasticity and consolidation for a porous anisotropic solid. J Appl Phys 26(2):182–185
3. Biot MA (1956) General solutions of the equations of elasticity and consolidation for a porous material. J Appl Mech ASME 78:91–96

4. Biot MA (1956) Theory of propagation of elastic waves in a fluid-saturated porous solid.1. Low-frequency range. J Acoust Soc Am 28(2):168–178
5. Biot MA, Willis DG (1957) The elastic coefficients of the theory of consolidation. J Appl Mech ASME 24:594–601
6. Rice JR, Cleary MP (1976) Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents. Rev Geophys 14(2):227–241

Appendix C
Cartesian and Curvilinear Coordinate System

For convenience of reference, the stress, strain, constitutive equations, and governing equations are expressed in Cartesian, spherical, and cylindrical coordinate systems in the following.

C.1 Cartesian Coordinate System

In Cartesian coordinates (x, y, z), the gradient operator is [1]

$$\nabla p = \frac{\partial p}{\partial x}\,\hat{i} + \frac{\partial p}{\partial y}\,\hat{j} + \frac{\partial p}{\partial z}\,\hat{k} \tag{C.1}$$

the divergence operator is

$$\nabla \cdot \vec{u} = e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{C.2}$$

the curl operator is

$$\nabla \times \vec{u} = 2\,\vec{\Omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_x & u_y & u_z \end{vmatrix} \tag{C.3}$$

and the Laplacian operator is given by

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \tag{C.4}$$

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In the above, the volumetric strain e is defined as

$$e = e_{xx} + e_{yy} + e_{zz} \tag{C.5}$$

and the strains are

$$e_{xx} = \frac{\partial u_x}{\partial x} \tag{C.6}$$

$$e_{yy} = \frac{\partial u_y}{\partial y} \tag{C.7}$$

$$e_{zz} = \frac{\partial u_z}{\partial z} \tag{C.8}$$

$$e_{xy} = e_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{C.9}$$

$$e_{yz} = e_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{C.10}$$

$$e_{zx} = e_{xz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \tag{C.11}$$

The rotation vector is

$$\vec{\Omega} = \Omega_x\,\hat{i} + \Omega_y\,\hat{j} + \Omega_z\,\hat{k} \tag{C.12}$$

with

$$\Omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \tag{C.13}$$

$$\Omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \tag{C.14}$$

$$\Omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{C.15}$$

Stress strain relations are

$$\sigma_{ij} = 2Ge_{ij} + \frac{2G\nu}{1-2\nu}\delta_{ij}e - \alpha\delta_{ij}p; \quad i,j = (x,y,z) \tag{C.16}$$

The equilibrium equations are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{C.17}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \tag{C.18}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{C.19}$$

Navier equations are obtained by substituting the constitutive equations into the equilibrium equations.

C.2 Cylindrical Coordinate System

In cylindrical coordinates (r, θ, z), the gradient operator is [1]

$$\nabla p = \frac{\partial p}{\partial r}\,\hat{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\,\hat{e}_\theta + \frac{\partial p}{\partial z}\,\hat{e}_z \tag{C.20}$$

the divergence operator is

$$\nabla \cdot \vec{u} = e = \frac{1}{r}\frac{\partial\, r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{C.21}$$

the curl operator is

$$\nabla \times \vec{u} = 2\,\vec{\Omega} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ u_r & r u_\theta & u_z \end{vmatrix} \tag{C.22}$$

and the Laplacian operator is given by

$$\nabla^2 p = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,\frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\,\frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\,\frac{\partial p}{\partial z}\right)\right] \tag{C.23}$$

In the above, the volumetric strain e is defined as

$$e = e_{rr} + e_{\theta\theta} + e_{zz} \tag{C.24}$$

and the strains are

$$e_{rr} = \frac{\partial u_r}{\partial r} \tag{C.25}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \tag{C.26}$$

$$e_{zz} = \frac{\partial u_z}{\partial z} \tag{C.27}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \tag{C.28}$$

$$e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{C.29}$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) \tag{C.30}$$

The rotation vector is

$$\vec{\Omega} = \Omega_r\, \hat{e}_r + \Omega_\theta\, \hat{e}_\theta + \Omega_z\, \hat{e}_z \tag{C.31}$$

with

$$\Omega_r = \frac{1}{2r}\left(\frac{\partial u_z}{\partial \theta} - \frac{\partial r u_\theta}{\partial z}\right) \tag{C.32}$$

$$\Omega_\theta = \frac{1}{2r}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \tag{C.33}$$

$$\Omega_z = \frac{1}{2r}\left(\frac{\partial\, r u_\theta}{\partial r} - \frac{\partial u_r}{\partial \theta}\right) \tag{C.34}$$

Stress strain relations are

$$\sigma_{ij} = 2Ge_{ij} + \frac{2G\nu}{1-2\nu}\delta_{ij}e - \alpha\delta_{ij}p; \quad i,j = (r,\theta,z) \tag{C.35}$$

The equilibrium equations are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{C.36}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0 \tag{C.37}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{C.38}$$

Navier equations are obtained by substituting the constitutive equations into the equilibrium equations.

C.3 Spherical Coordinate System

In spherical coordinates (R, θ, φ), the gradient operator is [1]

$$\nabla p = \frac{\partial p}{\partial R}\,\hat{e}_R + \frac{1}{R}\frac{\partial p}{\partial \theta}\,\hat{e}_\theta + \frac{1}{R\sin\theta}\frac{\partial p}{\partial \varphi}\,\hat{e}_\varphi \tag{C.39}$$

the divergence operator is

$$\nabla \cdot \vec{u} = e = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial R}(R^2 u_R \sin\theta) + \frac{\partial}{\partial \theta}(R\,u_\theta \sin\theta) + \frac{\partial}{\partial \varphi}(R\,u_\varphi)\right] \tag{C.40}$$

the curl operator is

$$\nabla \times \vec{u} = 2\,\vec{\Omega} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{e}_R & R\,\hat{e}_\theta & R\sin\theta\,\hat{e}_\varphi \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial\varphi \\ u_R & R\,u_\theta & R\sin\theta\,u_\varphi \end{vmatrix} \tag{C.41}$$

and the Laplacian operator is given by

$$\nabla^2 p = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial R}\left(R^2\sin\theta\,\frac{\partial p}{\partial R}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\theta}\,\frac{\partial p}{\partial \varphi}\right)\right] \tag{C.42}$$

In the above, the volumetric strain e is defined as

$$e = e_{RR} + e_{\theta\theta} + e_{\varphi\varphi} \tag{C.43}$$

and the strains are

$$e_{RR} = \frac{\partial u_R}{\partial R} \tag{C.44}$$

$$e_{\theta\theta} = \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R} \tag{C.45}$$

$$e_{\varphi\varphi} = \frac{1}{R\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_R}{R} + \frac{u_\theta\cot\theta}{R} \tag{C.46}$$

$$e_{R\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R} + \frac{1}{R}\frac{\partial u_R}{\partial \theta}\right) \tag{C.47}$$

$$e_{R\varphi} = \frac{1}{2}\left(\frac{1}{R\sin\theta}\frac{\partial u_R}{\partial \varphi} + \frac{\partial u_\varphi}{\partial R} - \frac{u_\varphi}{R}\right) \tag{C.48}$$

$$e_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{R\sin\theta}\frac{\partial u_\theta}{\partial\varphi} + \frac{1}{R}\frac{\partial u_\varphi}{\partial\theta} - \frac{u_\varphi\cot\theta}{R}\right) \tag{C.49}$$

The rotation vector is

$$\vec{\Omega} = \Omega_R\,\hat{e}_R + \Omega_\theta\,\hat{e}_\theta + \Omega_\varphi\,\hat{e}_\varphi \tag{C.50}$$

with

$$\Omega_R = \frac{1}{2R^2\sin\theta}\left(\frac{\partial R\sin\theta\,u_\varphi}{\partial\theta} - \frac{\partial R\,u_\theta}{\partial\varphi}\right) \tag{C.51}$$

$$\Omega_\theta = \frac{1}{2R\sin\theta}\left(\frac{\partial u_R}{\partial\varphi} - \frac{\partial R\sin\theta\,u_\varphi}{\partial R}\right) \tag{C.52}$$

$$\Omega_\varphi = \frac{1}{2R}\left(\frac{\partial\,R\,u_\theta}{\partial R} - \frac{\partial u_R}{\partial\theta}\right) \tag{C.53}$$

Stress strain relations are

$$\sigma_{ij} = 2Ge_{ij} + \frac{2G\nu}{1-2\nu}\delta_{ij}e - \alpha\delta_{ij}p; \quad i,j = (R,\theta,\varphi) \tag{C.54}$$

The equilibrium equations are

$$\frac{\partial\sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_{R\theta}}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial\sigma_{R\varphi}}{\partial\varphi} + \frac{1}{R}\left(2\,\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{R\theta}\cot\theta\right) = 0 \tag{C.55}$$

$$\frac{\partial\sigma_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial\sigma_{\theta\varphi}}{\partial\varphi} + \frac{1}{R}\left[(\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cot\theta + 3\,\sigma_{R\theta}\right] = 0 \tag{C.56}$$

$$\frac{\partial\sigma_{R\varphi}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_{\theta\varphi}}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} + \frac{1}{R}\left(2\,\sigma_{\theta\varphi}\cot\theta + 3\,\sigma_{R\varphi}\right) = 0 \tag{C.57}$$

Navier equations are obtained by substituting the constitutive equations into the equilibrium equations.

Reference

1. Aris R (1990) Vectors, tensors and the basic equations of fluid mechanics. Dover, New York, 320pp

Appendix D
Laplace Transform and Inversion

D.1 Laplace Transform

Laplace transform is a powerful tool that can reduce a differential operator into an algebraic operator, for the solution of partial or ordinary differential equations. It is normally applied to the time variable to resolve the time derivative. Given a function of time, $f(t)$, its Laplace transform $\tilde{f}(s)$ is defined as

$$\mathscr{L}\{f(t)\} = \tilde{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{D.1}$$

where s is the *Laplace transform parameter* and is a real variable. The inverse transformation is given by the *Bromwich integral* in the complex s plane as

$$\mathscr{L}^{-1}\{\tilde{f}(s)\} = f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{f}(s)\, e^{st}\, ds \tag{D.2}$$

where the integration is performed along the vertical line $\Re(s) = \gamma$ such that γ is greater than the real part of all singularities (poles) of $\tilde{f}(s)$.

As textbooks on Laplace transform [3, 6] are readily available, there is no intention to give an introduction to Laplace transform in this appendix. We shall, however, summarize a few essential formulae that are used in this book to obtain solutions of poroelastic boundary values problems.

The Laplace transform can remove a time derivative and turn it into an algebraic form as

$$\mathscr{L}\left\{\frac{\partial f(t)}{\partial t}\right\} = s\tilde{f}(s) - f(0) \tag{D.3}$$

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where $f(0)$ is the initial value. The order of Laplace transform and differentiation is interchangeable, such that

$$\mathscr{L}\left\{\frac{\partial f(x,t)}{\partial x}\right\} = \frac{\partial \mathscr{L}\{f(x,t)\}}{\partial x} = \frac{\partial \tilde{f}(x,s)}{\partial x} \tag{D.4}$$

With (D.3) and (D.4), we can transform the diffusion equation (7.372) from a partial differential equation to an ordinary differential equation (7.374), as shown in Sect. 7.10.

To simplify the presentation of the solution, it is often desirable to normalize the variables and the equations to *dimensionless* forms. For example, the time variable t is typically combined with a diffusion coefficient c (dim. $[L^2/T]$) and a length scale L to form a dimensionless time

$$t^* = \frac{ct}{L^2} \tag{D.5}$$

We can either perform such nondimensionalization on the original differential equation, and then perform Laplace transform regarding t^* as t, or we can work on the Laplace transformed equation and define a dimensionless Laplace transform parameter

$$s^* = \frac{L^2 s}{c} \tag{D.6}$$

to rearrange the equation into dimensionless form. As the inverse Laplace transform is performed on the variable s, the following formula is handy to transform the equation in terms of s^*, such as (7.379), into an equation in terms of t^*, such as (7.395):

$$\mathscr{L}^{-1}\left\{\tilde{f}(as)\right\} = \frac{1}{a} f\left(\frac{t}{a}\right) \tag{D.7}$$

There is a powerful property of the Laplace transform, known as the *convolutional theorem*. The theorem takes the form

$$\mathscr{L}^{-1}\left\{\tilde{f}_1(s)\tilde{f}_2(s)\right\} = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau \tag{D.8}$$

The integral on the right hand side is a *convolutional integral*. Equation (D.8) can be used to find the Laplace inverse of the product of $\tilde{f}_1(s)$ and $\tilde{f}_2(s)$, if the individual inverse transforms, $f_1(t)$ and $f_2(t)$, are known. Or, this theorem in its inverse form can be used to find the Laplace transform of the convolutional integral, which can result from the application of *Duhamel principle of superposition*, in the following form

$$\mathscr{L}\left\{\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right\} = \tilde{f}_1(s)\tilde{f}_2(s) \tag{D.9}$$

Finally, we present a short table of Laplace transform and inversion [8] for formulae used in this book, as Table D.1.

Table D.1 A short table of Laplace transform and inversion [3]

$f(t)$	$\tilde{f}(s)$
1	s^{-1}
$2(t/\pi)^{1/2}$	$s^{-3/2}$
t	s^{-2}
$\mathrm{H}(t-t_o)$	$s^{-1}\exp(-st_o)$
$\mathrm{H}(t-0)$	s^{-1}
$\delta(t-t_o)$	$\exp(-st_o)$
$\delta(t-0)$	1
$\exp(at)$	$1/(s-a)$
$t\exp(-t)$	$1/(s+1)^2$
$\gamma-\ln t$	$(\ln s)/s$
$\mathrm{erfc}(a/2\sqrt{t})$	$\exp(-a\sqrt{s})/s$
$2\sqrt{t/\pi}\,\exp(-a^2/4t)-a\,\mathrm{erfc}(a/2\sqrt{t})$	$\exp(-a\sqrt{s})/s^{3/2}$
$-a\sqrt{t/\pi}\,\exp(-a^2/4t)$ $+(t+a^2/2)\,\mathrm{erfc}(a/2\sqrt{t})$	$\exp(-a\sqrt{s})/s^2$
$[2\sqrt{t/\pi}\,(a^2+4t)\,\exp(-a^2/4t)$ $-(a^3+6at)\,\mathrm{erfc}(a/2\sqrt{t})]/6$	$\exp(-a\sqrt{s})/s^{5/2}$
$-b\exp(b^2t+ab)\,\mathrm{erfc}[b\sqrt{t}+(a/2\sqrt{t})]$ $+\exp(-a^2/4t)$	$\exp(-a\sqrt{s})/(b+\sqrt{s})$
$\{-\exp(b^2t+ab)\,\mathrm{erfc}[b\sqrt{t}+(a/2\sqrt{t})]$ $+\mathrm{erfc}(a/2\sqrt{t})\}/b$	$\exp(-a\sqrt{s})/[s(b+\sqrt{s})]$
$\exp(-a^2/4t)/2t$	$\mathrm{K}_0(a\sqrt{s})$
$\mathrm{E}_1(a^2/4t)/2$	$\mathrm{K}_0(a\sqrt{s})/s$
$[-4t\exp(-a^2/4t)$ $+(a^2+4t)\mathrm{E}_1(a^2/4t)]/8$	$\mathrm{K}_0(a\sqrt{s})/s^2$
$1/\sqrt{\pi t}-a\exp(a^2t)\,\mathrm{erfc}(a\sqrt{t})$	$1/(\sqrt{s}+a)$
$\exp(-b^2/4t)/\sqrt{\pi t}-a\exp(ab+a^2t)\times$ $\mathrm{erfc}[(b/2\sqrt{t})+a\sqrt{t}]$	$\exp(-b\sqrt{s})/(\sqrt{s}+a)$
$\sum_{n=1}^{m}[p(a_n)/q'(a_n)]\exp(a_nt)$	$p(s)/q(s);\ q(s)=(s-a_1)(s-a_2)$ $\cdots(s-a_m)$

D.2 Inverse Laplace Transform

To obtain the solution in the time domain, we can carry out the inverse Laplace transform by performing the line integral on the complex s plane as given by (D.2). The integration can in fact be performed on the closed semi-circular contour as shown in Fig. D.1, in which the semi-circle is to be pushed to infinity. For the problems investigated here, if the integrand decays fast enough as $s \to \infty$ for the integration on the open semi-circle to vanish, the closed contour integral is equivalent to the line integral in (D.2).

According to the *Cauchy integral formula*, if $f(z)$ is *analytic* (meaning it is continuous and without singularities) in the area enclosed by a closed contour C in the complex plane $z = x + iy$, then

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz = f(a) \tag{D.10}$$

if the point a is contained in C.

On the other hand, if $f(z)$ is analytic in C, except at a number of singularity points a_i, $i = 1, 2, \ldots, n$, known as *poles*, around which $f(z)$ can be expanded into a Laurent series, then the *Cauchy residue theorem* states

$$\frac{1}{2\pi i} \oint_C f(z)\, dz = \sum_{i=1}^{n} b_i \tag{D.11}$$

where b_i are the coefficients associated with the $1/(z - a_i)$ term of the Laurent series around a_i.

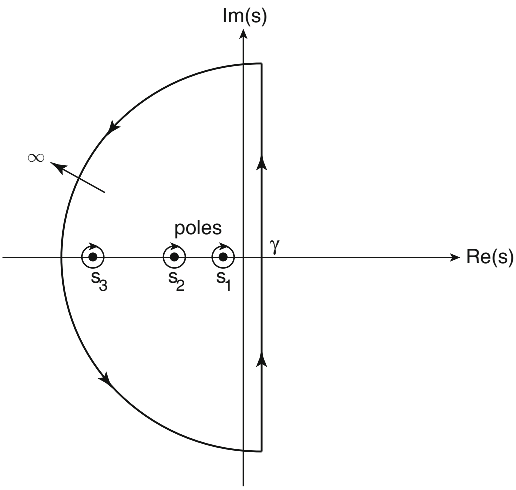

Fig. D.1 Integration contour for inverse Laplace transform

If the integrand can be expressed as the quotient of two analytic functions, $f(z)$ and $g(z)$, and $g(z)$ contains zeros at a_i, that is, the complex roots of the equation $g(z) = 0$, then the following formula exists [7]

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{g(z)}\,dz = \sum_{i=1}^{n}\frac{f(a_i)}{g'(a_i)} \tag{D.12}$$

where $g'(z)$ is the derivative of $g(z)$, and $g'(a_i)$ is the coefficient of the Taylor series expansion term $(z-a_i)$ around the point a_i. In the above, we assume that $g'(a) \neq 0$. If $g(a) = g'(a) = 0$, but $g''(a) \neq 0$, then the residual should come from the term associated with $(z-a_i)^2$, so on and so forth.

Finally, we come to the Laplace inverse transform of the function $\tilde{f}(s)$, which can be expressed as the ratio of two analytic functions $\tilde{f}(s) = h(s)/g(s)$. If $g(s)$ contains zeros at s_i, $i = 1, 2, \ldots, n$, then the Laplace inversion becomes

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\tilde{f}(s)\,e^{st}\,ds = \frac{1}{2\pi i}\oint_C \frac{h(s)}{g(s)}\,e^{st}\,ds = \sum_{i=1}^{n}\frac{h(s_i)}{g'(s_i)}\,e^{s_i t} \tag{D.13}$$

where C is the closed semi-circle contour in Fig. D.1.

D.3 Approximate Inverse Laplace Transform

As not all Laplace transform expressions can be analytically inverted, approximate inversion based on numerical algorithms is useful. Here we shall give a brief discussion and present one particular method for ease of use.

There exist a large number of approximate inverse Laplace transform algorithms [9, 10]. With issues of computational efficiency and accuracy in mind, a handful of those techniques have been tested and compared side by side [2, 4, 5]. One of the methods widely in use for engineering purposes is the *Stehfest method* [11, 12]. All approximate inverse Laplace transform results found in this book are obtained using this algorithm; hence a brief presentation is given below.

We denote $F(s)$ the Laplace transform of the function $f(t)$. The Stehfest inversion seeks to find the approximate value of $f(t)$ at a given t by sampling $F(s)$ at a set of discrete s values. The resultant series approximation is as follows:

$$f(t) \approx \frac{\ln 2}{t}\sum_{i=1}^{n} c_i\,F\left(\frac{i\ln 2}{t}\right) \tag{D.14}$$

where the coefficients c_i are given by

$$c_i = (-1)^{i+\frac{n}{2}} \sum_{k=\lfloor \frac{i+1}{2} \rfloor}^{\min(i,\frac{n}{2})} \frac{k^{\frac{n}{2}}\,(2k)!}{(\frac{n}{2}-k)!\,k!\,(k-1)!\,(i-k)!\,(2k-i)!} \tag{D.15}$$

In the above, n is the number of terms in the series, which must be even, and the brackets $\lfloor\ \rfloor$ is the *floor function* that gives the greatest integer not larger than its argument (e.g., $\lfloor \frac{3}{2} \rfloor = 1$).

Some caution needs to be exercised in using the above, and in fact in all approximate Laplace inverse algorithms. It is important to recognize that the inverse Laplace transform is inherently unstable, that is, a small perturbation of the data in the inversion space (values of $F(s)$) can lead to large error in $f(t)$ [1]. For example, one should not attempt to numerically invert a Laplace transform of a periodic function in time (such as $\sin t$), or functions that have spike-like behavior, such as a (Dirac) delta convergence sequence. However, for functions that have a steady state limit as $t \to \infty$, most approximate inversion methods should perform well.

For the Stehfest method, we should note that the series presented in (D.14) is an asymptotic series, and not a convergent series. The series is made of alternating (positive and negative) terms in pairs that get ever larger as the number of terms increases. With finite precision computation, the result becomes unstable when the number of terms becomes too large. So there exist an optimal number of terms for a stable result, which can be obtained only empirically. Generally speaking, with double precision computation, 6–10 terms in the series can give good "engineering" results, with error less than 1%. For better results, up to 20 terms may be tolerated, but one should watch for instability of the result in that range. For even higher accuracy inversion, high precision computation, such as quadruple precision, is recommended.

References

1. Bellman R, Kalaba RE, Lockett JA (1966) Numerical in version of the Laplace transform: applications to biology, economics, engineering, and physics. American Elsevier, New York, 211pp
2. Cheng AHD, Sidauruk P, Abousleiman Y (1994) Approximate inversion of the Laplace transform. Math J 4(2):76–82
3. Churchill RV (1972) Operational mathematics. McGraw-Hill, New York, 3rd edn, 481pp
4. Cost TL (1964) Approximate Laplace transform inversions in viscoelastic stress analysis. AIAA J 2(12):2157–2166
5. Davies B, Martin B (1979) Numerical inversion of the Laplace transform–survey and comparison of methods. J Comput Phys 33(1):1–32
6. LaPage WR (2010) Complex variables and the Laplace transform for engineers. Dover, New York, 512pp
7. Mei CC (1997) Mathematical analysis in engineering. Cambridge University Press, Cambridge/New York, 461pp

8. Oberhettinger F, Badii L (1973) Tables of Laplace transforms. Springer, Berlin/New York, 428pp
9. Piessens R (1975) A bibliography on numerical inversion of the Laplace transform and applications. J Comput Appl Math 1(2):115–128
10. Piessens R, Dang NDP (1976) A bibliography on numerical inversion of the Laplace transform and applications: a supplement. J Comput Appl Math 2(3):225–228
11. Stehfest H (1970) Numerical inversion of Laplace transforms. Commun ACM 13(1):47–49
12. Stehfest H (1970) Correction. Commun ACM 13(10):624

Appendix E
Poroelasticity Fundamental Solution

A comprehensive list of Green's functions for quasi-static isotropic poroelasticity is given below. The following abbreviated notations are utilized:

$$r = \left|\vec{x} - \vec{\chi}\right| \tag{E.1}$$

$$r_{,i} = \frac{\partial r}{\partial x_i} = \frac{x_i - \chi_i}{r} \tag{E.2}$$

$$\xi = \frac{r}{\sqrt{4c\,(t-\tau)}} \tag{E.3}$$

E.1 Continuous Fluid Source

$\gamma = \delta(\vec{x} - \vec{\chi})\,\mathrm{H}(t-\tau)$

2D

$$u_i^{sc} = \frac{\eta}{8\pi G\kappa} r\, r_{,i} \left[\mathrm{E}_1(\xi^2) + \frac{1 - e^{-\xi^2}}{\xi^2} \right] \tag{E.4}$$

$$\sigma_{ij}^{sc} = \frac{\eta}{4\pi\kappa} \left[(\delta_{ij} - 2r_{,i}r_{,j}) \frac{1 - e^{-\xi^2}}{\xi^2} - \delta_{ij}\mathrm{E}_1(\xi^2) \right] \tag{E.5}$$

$$p^{sc} = \frac{1}{4\pi\kappa}\mathrm{E}_1(\xi^2) = \frac{1}{S}\zeta^{sc} \tag{E.6}$$

$$w_i^{sc} = \frac{1}{8\pi c} r\, r_{,i} \left[\frac{e^{-\xi^2}}{\xi^2} - \mathrm{E}_1(\xi^2) \right] \tag{E.7}$$

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$$q_i^{sc} = \frac{1}{2\pi}\frac{r_{,i}}{r}e^{-\xi^2} = w_i^{si} = \kappa\, p_i^{pc} = c\,\zeta_i^{pc} = w_i^{lc} = p_i^{fc}$$
$$= \frac{1}{S}\zeta_i^{fc} = -\frac{G}{\eta}\zeta_i^{Fc} \tag{E.8}$$
$$\zeta^{sc} = Sp^{sc} \tag{E.9}$$

3D

$$u_i^{sc} = \frac{\eta}{16\pi G\kappa}r_{,i}\left[2\,\mathrm{erfc}(\xi) + \frac{\mathrm{erf}(\xi)}{\xi^2} - \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right] \tag{E.10}$$
$$\sigma_{ij}^{sc} = \frac{\eta}{8\pi\kappa}\frac{1}{r}\left\{(\delta_{ij} - 3r_{,i}r_{,j})\left[\frac{\mathrm{erf}(\xi)}{\xi^2} - \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right]\right.$$
$$\left. - 2(\delta_{ij} + r_{,i}r_{,j})\,\mathrm{erfc}(\xi)\right\} \tag{E.11}$$
$$p^{sc} = \frac{1}{4\pi\kappa}\frac{\mathrm{erfc}(\xi)}{r} = \frac{1}{S}\zeta^{sc} \tag{E.12}$$
$$w_i^{sc} = \frac{1}{16\pi c}r_{,i}\left[\frac{\mathrm{erfc}(\xi)}{\xi^2} - 2\,\mathrm{erfc}(\xi) + \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right] \tag{E.13}$$
$$q_i^{sc} = \frac{1}{4\pi}\frac{r_{,i}}{r^2}\left[\frac{2}{\sqrt{\pi}}\xi e^{-\xi^2} + \mathrm{erfc}(\xi)\right] = w_i^{si} = \kappa\, p_i^{pc} = c\,\zeta_i^{pc}$$
$$= w_i^{lc} = p_i^{fc} = \frac{1}{S}\zeta_i^{fc} = -\frac{G}{\eta}\zeta_i^{Fc} \tag{E.14}$$
$$\zeta^{sc} = Sp^{sc} \tag{E.15}$$

E.2 Instantaneous Fluid Source

$\gamma = \delta(\vec{x} - \vec{\chi})\,\delta(t-\tau)$

2D

$$u_i^{si} = \frac{\eta}{2\pi GS}\frac{r_{,i}}{r}(1 - e^{-\xi^2}) = p_i^{Fc} = u_i^{lc} \tag{E.16}$$
$$\sigma_{ij}^{si} = \frac{\eta}{\pi S}\frac{1}{r^2}\left[(\delta_{ij} - 2r_{,i}r_{,j})(1 - e^{-\xi^2}) - 2(\delta_{ij} - r_{,i}r_{,j})\xi^2 e^{-\xi^2}\right]$$
$$= \sigma_{ij}^{lc} = p_{ij}^{dc} \tag{E.17}$$
$$p^{si} = \frac{1}{\pi S}\frac{1}{r^2}\xi^2 e^{-\xi^2} = \frac{1}{S}\zeta^{si} = p^{lc} = \frac{1}{S}\zeta^{lc} \tag{E.18}$$

$$w_i^{si} = q_i^{sc} \;=\; \kappa p_i^{pc} \;=\; c\,\zeta_i^{pc} \;=\; w_i^{lc} \;=\; p_i^{fc} \;=\; \frac{1}{S}\zeta_i^{fc} \tag{E.19}$$

$$q_i^{si} = \frac{2c}{\pi}\frac{r_{,i}}{r^3}\xi^4 e^{-\xi^2} \;=\; \kappa p_i^{pi} \;=\; c\,\zeta_i^{pi} \;=\; q_i^{lc} \;=\; w_i^{li} \;=\; p_i^{fi}$$
$$= \frac{1}{S}\zeta_i^{fi} \;=\; -\frac{G}{\eta}\zeta_i^{Fi} \tag{E.20}$$

$$\zeta^{si} = Sp^{si} \;=\; Sp^{lc} \;=\; \zeta^{lc} \tag{E.21}$$

3D

$$u_i^{si} = \frac{\eta}{4\pi GS}\frac{r_{,i}}{r^2}\left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}}\,\xi e^{-\xi^2}\right] \;=\; p_i^{Fc} \;=\; u_i^{lc} \tag{E.22}$$

$$\sigma_{ij}^{si} = \frac{\eta}{2\pi S}\frac{1}{r^3}\left\{(\delta_{ij} - 3r_{,i}r_{,j})\left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}}\,\xi e^{-\xi^2}\right]\right.$$
$$\left. - \frac{4}{\sqrt{\pi}}(\delta_{ij} - r_{,i}r_{,j})\xi^3 e^{-\xi^2}\right\}$$
$$= \sigma_{ij}^{lc} \;=\; p_{ij}^{dc} \tag{E.23}$$

$$p^{si} = \frac{1}{\pi^{3/2}S}\frac{1}{r^3}\xi^3 e^{-\xi^2} \;=\; \frac{1}{S}\zeta^{si} \;=\; p^{lc} \;=\; \frac{1}{S}\zeta^{lc} \tag{E.24}$$

$$w_i^{si} = q_i^{sc} \;=\; \kappa p_i^{pc} \;=\; c\,\zeta_i^{pc} \;=\; w_i^{lc} \;=\; p_i^{fc} \;=\; \frac{1}{S}\zeta_i^{fc} \tag{E.25}$$

$$q_i^{si} = \frac{2c}{\pi^{3/2}}\frac{r_{,i}}{r^4}\xi^5 e^{-\xi^2} \;=\; \kappa p_i^{pi} \;=\; c\,\zeta_i^{pi} \;=\; q_i^{lc} \;=\; w_i^{li}$$
$$= p_i^{fi} \;=\; \frac{1}{S}\zeta_i^{fi} \;=\; -\frac{G}{\eta}\zeta_i^{Fi} \tag{E.26}$$

$$\zeta^{si} = Sp^{si} \;=\; Sp^{lc} \;=\; \zeta^{lc} \tag{E.27}$$

E.3 Continuous Fluid Dipole

$\gamma_k = -\delta(\vec{x} - \vec{\chi})_{,k}\,\mathrm{H}(t-\tau)$
2D

$$u_{ik}^{pc} = \frac{\eta}{8\pi G\kappa}\left[(2r_{,i}r_{,k} - \delta_{ik})\frac{1 - e^{-\xi^2}}{\xi^2} - \delta_{ik}\mathrm{E}_1(\xi^2)\right]$$
$$= \frac{1}{\kappa}u_{ik}^{fc} \;=\; -\frac{\eta}{\kappa GS}w_{ik}^{fc} \;=\; \frac{1}{\kappa}w_{ik}^{Fc} \tag{E.28}$$

$$\sigma_{ijk}^{pc} = \frac{\eta}{2\pi\kappa}\frac{1}{r}\left[(\delta_{ik}r_{,j} + \delta_{jk}r_{,i} + \delta_{ij}r_{,k} - 4r_{,i}r_{,j}r_{,k})\frac{1-e^{-\xi^2}}{\xi^2} + 2(r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k})e^{-\xi^2}\right]$$

$$= \frac{1}{\kappa}\sigma_{ijk}^{fc} = \frac{1}{\kappa}w_{kji}^{dc} \tag{E.29}$$

$$p_k^{pc} = \frac{1}{\kappa}q_k^{sc} = \frac{1}{\kappa}w_k^{si} = \frac{c}{k}\zeta_k^{pc} = \frac{1}{\kappa}w_k^{lc} = \frac{1}{\kappa}p_k^{fc}$$

$$= \frac{1}{\kappa S}\zeta_k^{fc} = -\frac{G}{\eta\kappa}\zeta_k^{Fc} \tag{E.30}$$

$$w_{ik}^{pc} = \frac{1}{8\pi c}\left[(2r_{,i}r_{,k} - \delta_{ik})\frac{e^{-\xi^2}}{\xi^2} + \delta_{ik}\mathrm{E}_1(\xi^2)\right] \tag{E.31}$$

$$q_{ik}^{pc} = \frac{1}{2\pi}\frac{1}{r^2}\left[2r_{,i}r_{,k}(1+\xi^2) - \delta_{ik}\right]e^{-\xi^2} = w_{ik}^{pi} \tag{E.32}$$

$$\zeta_k^{pc} = \frac{1}{c}q_k^{sc} = \frac{1}{c}w_k^{si} = Sp_k^{pc} = \frac{1}{c}w_k^{lc} = \frac{1}{c}p_k^{fc}$$

$$= \frac{1}{\kappa}\zeta_k^{fc} = -\frac{G}{\eta c}\zeta_k^{Fc} \tag{E.33}$$

3D

$$u_{ik}^{pc} = \frac{\eta}{16\pi G\kappa}\frac{1}{r}\left\{(3r_{,i}r_{,k} - \delta_{ik})\left[\frac{\mathrm{erf}(\xi)}{\xi^2} - \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right] + 2(r_{,i}r_{,k} - \delta_{ik})\mathrm{erfc}(\xi)\right\}$$

$$= \frac{1}{\kappa}u_{ik}^{fc} = -\frac{\eta}{\kappa GS}w_{ik}^{fc} = \frac{1}{\kappa}w_{ik}^{Fc} \tag{E.34}$$

$$\sigma_{ijk}^{pc} = \frac{\eta}{8\pi\kappa}\frac{1}{r^2}\left\{(\delta_{ij}r_{,k} + \delta_{ik}r_{,j} + \delta_{jk}r_{,i} - 5r_{,i}r_{,j}r_{,k}) \times \left[\frac{3\,\mathrm{erf}(\xi)}{\xi^2} - \frac{6}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi} + 2\,\mathrm{erfc}(\xi)\right] + 4(r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k})\left[\mathrm{erfc}(\xi) + \frac{2}{\sqrt{\pi}}\xi e^{-\xi^2}\right]\right\}$$

$$= \frac{1}{\kappa}\sigma_{ijk}^{fc} = \frac{1}{\kappa}w_{kji}^{dc} \tag{E.35}$$

$$p_k^{pc} = \frac{1}{\kappa} q_k^{sc} = \frac{1}{\kappa} w_k^{si} = \frac{1}{S} \zeta_k^{pc} = \frac{1}{\kappa} w_k^{lc} = \frac{1}{\kappa} p_k^{fc}$$
$$= \frac{1}{\kappa S} \zeta_k^{fc} = -\frac{G}{\eta\kappa} \zeta_k^{Fc} \tag{E.36}$$

$$w_{ik}^{pc} = \frac{1}{16\pi c} \frac{1}{r} \left\{ (3r_{,i}r_{,k} - \delta_{ik}) \left[\frac{\text{erfc}(\xi)}{\xi^2} + \frac{2}{\sqrt{\pi}} \frac{e^{-\xi^2}}{\xi} \right] \right.$$
$$\left. + 2(\delta_{ik} - r_{,i}r_{,k})\text{erfc}(\xi) \right\} \tag{E.37}$$

$$q_{ik}^{pc} = \frac{1}{4\pi} \frac{1}{r^3} \left\{ (3r_{,i}r_{,k} - \delta_{ik}) \left[\text{erfc}(\xi) + \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right] \right.$$
$$\left. + \frac{4}{\sqrt{\pi}} r_{,i}r_{,k}\xi^3 e^{-\xi^2} \right\}$$
$$= w_{ik}^{pi} \tag{E.38}$$

$$\zeta_k^{pc} = \frac{1}{c} q_k^{sc} = \frac{1}{c} w_k^{si} = S p_k^{pc} = \frac{1}{c} w_k^{lc} = \frac{1}{c} p_k^{fc} = \frac{1}{\kappa} \zeta_k^{fc} = -\frac{G}{\eta c} \zeta_k^{Fc} \tag{E.39}$$

E.4 Instantaneous Fluid Dipole

$\gamma_k = \delta(\vec{x} - \vec{\chi})_{,k}\, \delta(t - \tau)$
2D

$$u_{ik}^{pi} = \frac{\eta}{2\pi GS} \frac{1}{r^2} \left[(2r_{,i}r_{,k} - \delta_{ik})(1 - e^{-\xi^2}) - 2r_{,i}r_{,k}\xi^2 e^{-\xi^2} \right]$$
$$= -\frac{\eta}{\kappa GS} q_{ik}^{fc} = \frac{1}{\kappa} u_{ik}^{fi} = -\frac{\eta}{\kappa GS} w_{ik}^{fi} = \frac{1}{\kappa} q_{ik}^{Fc} = \frac{1}{\kappa} w_{ik}^{Fi} \tag{E.40}$$

$$\sigma_{ijk}^{pi} = \frac{2\eta}{\pi S} \frac{1}{r^3} \left[(\delta_{ik}r_{,j} + \delta_{jk}r_{,i} + \delta_{ij}r_{,k} - 4r_{,i}r_{,j}r_{,k})(1 - e^{-\xi^2} - \xi^2 e^{-\xi^2}) \right.$$
$$\left. + 2(r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k})\xi^4 e^{-\xi^2} \right]$$
$$= \frac{1}{\kappa} \sigma_{ijk}^{fi} = \frac{1}{\kappa} q_{kji}^{dc} = \frac{1}{\kappa} w_{kji}^{di} \tag{E.41}$$

$$p_k^{pi} = \frac{1}{\kappa} q_k^{si} = \frac{1}{S} \zeta_k^{pi} = \frac{1}{\kappa} q_k^{lc} = \frac{1}{\kappa} w_k^{li} = \frac{1}{\kappa} p_k^{fi} = \frac{1}{\kappa S} \zeta_k^{fi} = -\frac{G}{\eta\kappa} \zeta_k^{Fi} \tag{E.42}$$

$$w_{ik}^{pi} = q_{ik}^{pc} \tag{E.43}$$

$$q_{ik}^{pi} = \frac{2c}{\pi} \frac{1}{r^4} (2r_{,i} r_{,k} \xi^2 - \delta_{ik})\, \xi^4 e^{-\xi^2} \tag{E.44}$$

$$\zeta_k^{pi} = \frac{1}{c} q_k^{si} = S p_k^{pi} = \frac{1}{c} q_k^{lc} = \frac{1}{c} w_k^{li} = \frac{1}{c} p_k^{fi} = \frac{1}{\kappa} \zeta_k^{fi} = -\frac{G}{\eta c} \zeta_k^{Fi} \tag{E.45}$$

3D

$$\begin{aligned} u_{ik}^{pi} &= \frac{\eta}{4\pi GS} \frac{1}{r^3} \left\{ (3r_{,i} r_{,k} - \delta_{ik}) \left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right] - \frac{4}{\sqrt{\pi}} r_{,i} r_{,k} \xi^3 e^{-\xi^2} \right\} \\ &= -\frac{\eta}{\kappa GS} q_{ik}^{fc} \;=\; \frac{1}{\kappa} u_{ik}^{fi} \;=\; -\frac{\eta}{\kappa GS} w_{ik}^{fi} \;=\; \frac{1}{\kappa} q_{ik}^{Fc} \;=\; \frac{1}{\kappa} w_{ik}^{Fi} \end{aligned} \tag{E.46}$$

$$\begin{aligned} \sigma_{ijk}^{pi} &= \frac{\eta}{2\pi S} \frac{1}{r^4} \left\{ \frac{8}{\sqrt{\pi}} (r_{,i} r_{,j} r_{,k} - \delta_{ij} r_{,k}) \xi^5 e^{-\xi^2} \right. \\ &\qquad + (\delta_{ij} r_{,k} + \delta_{ik} r_{,j} + \delta_{jk} r_{,i} - 5 r_{,i} r_{,j} r_{,k}) \\ &\qquad \left. \times \left[3 \operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}} (3 + 2\xi^2) \xi e^{-\xi^2} \right] \right\} \\ &= \frac{1}{\kappa} \sigma_{ijk}^{fi} \end{aligned} \tag{E.47}$$

$$p_k^{pi} = \frac{1}{\kappa} q_k^{si} = \frac{1}{S} \zeta_k^{pi} = \frac{1}{\kappa} q_k^{lc} = \frac{1}{\kappa} w_k^{li} = \frac{1}{\kappa} p_k^{fi} = \frac{1}{\kappa S} \zeta_k^{fi} = -\frac{G}{\eta\kappa} \zeta_k^{Fi} \tag{E.48}$$

$$w_{ik}^{pi} = q_{ik}^{pc} \tag{E.49}$$

$$q_{ik}^{pi} = \frac{2c}{\pi^{3/2}} \frac{1}{r^5} (2r_{,i} r_{,k} \xi^2 - \delta_{ik}) \xi^5 e^{-\xi^2} \tag{E.50}$$

$$\zeta_k^{pi} = \frac{1}{c} q_k^{si} = S p_k^{pi} = \frac{1}{c} q_k^{lc} = \frac{1}{c} w_k^{li} = \frac{1}{c} p_k^{fi} = \frac{1}{\kappa} \zeta_k^{fi} = -\frac{G}{\eta c} \zeta_k^{Fi} \tag{E.51}$$

E.5 Continuous Fluid Dilatation

$Q = \delta(\vec{x} - \vec{\chi})\,\mathrm{H}(t - \tau)$
2D, 3D

$$\left(u_i^{lc}, \sigma_{ij}^{lc}, p^{lc}, w_i^{lc}, q_i^{lc}, \zeta^{lc} \right) = \left(u_i^{si}, \sigma_{ij}^{si}, p^{si}, w_i^{si}, q_i^{si}, \zeta^{si} \right) \tag{E.52}$$

E.6 Instantaneous Fluid Dilatation

$Q = \delta(\vec{x} - \vec{\chi})\,\delta(t - \tau)$

2D

$$
\begin{aligned}
u_i^{li} &= \frac{\eta}{2\pi GS}\frac{r_{,i}}{r}\left[\delta(t-\tau) - 4c\frac{1}{r^2}\xi^4 e^{-\xi^2}\right] \\
&= p_i^{Fi}
\end{aligned}
\tag{E.53}
$$

$$
\begin{aligned}
\sigma_{ij}^{li} &= \frac{\eta}{\pi S}\,\frac{1}{r^2}\Big\{(\delta_{ij} - 2r_{,i}r_{,j})\delta(t-\tau) \\
&\qquad + 4c\frac{1}{r^2}\left[\delta_{ij}(1-2\xi^2) + 2r_{,i}r_{,j}\xi^2\right]\xi^4 e^{-\xi^2}\Big\} \\
&= p_{ij}^{di}
\end{aligned}
\tag{E.54}
$$

$$
p^{li} = \frac{4c}{\pi S}\frac{1}{r^4}(\xi^2 - 1)\xi^4 e^{-\xi^2} \;=\; \frac{1}{S}\zeta^{li}
\tag{E.55}
$$

$$
w_i^{li} = q_i^{si} \;=\; \kappa p_i^{pi} \;=\; c\,\zeta_i^{pi} \;=\; q_i^{lc} \;=\; p_i^{fi} \;=\; \frac{1}{S}\zeta_i^{fi} \;=\; -\frac{G}{\eta}\zeta_i^{Fi}
\tag{E.56}
$$

$$
q_i^{li} = \frac{8c^2}{\pi}\frac{r_{,i}}{r^5}(\xi^2 - 2)\,\xi^6 e^{-\xi^2}
\tag{E.57}
$$

$$
\zeta^{li} = Sp^{li}
\tag{E.58}
$$

3D

$$
\begin{aligned}
u_i^{li} &= \frac{\eta}{4\pi GS}\frac{r_{,i}}{r^2}\left[\delta(t-\tau) - \frac{8c}{\sqrt{\pi}}\frac{1}{r^2}\xi^5 e^{-\xi^2}\right] \\
&= p_i^{Fi}
\end{aligned}
\tag{E.59}
$$

$$
\begin{aligned}
\sigma_{ij}^{li} &= \frac{\eta}{2\pi S}\,\frac{1}{r^3}\Big\{(\delta_{ij} - 3r_{,i}r_{,j})\delta(t-\tau) \\
&\qquad + \frac{16c}{\sqrt{\pi}}\frac{1}{r^2}\left[\delta_{ij}(1-\xi^2) + r_{,i}r_{,j}\xi^2\right]\xi^5 e^{-\xi^2}\Big\} \\
&= p_{ij}^{di}
\end{aligned}
\tag{E.60}
$$

$$
p^{li} = \frac{2c}{\pi^{3/2}S}\frac{1}{r^5}(2\xi^2 - 3)\xi^5 e^{-\xi^2} \;=\; \frac{1}{S}\zeta^{li}
\tag{E.61}
$$

$$
w_i^{li} = q_i^{si} \;=\; \kappa p_i^{pi} \;=\; c\,\zeta_i^{pi} \;=\; q_i^{lc} \;=\; p_i^{fi} \;=\; \frac{1}{S}\zeta_i^{fi} \;=\; -\frac{G}{\eta}\zeta_i^{Fi}
\tag{E.62}
$$

$$q_i^{li} = \frac{4c^2}{\pi^{3/2}} \frac{r_{,i}}{r^6}(2\xi^2 - 5)\xi^7 e^{-\xi^2} \tag{E.63}$$

$$\zeta^{li} = Sp^{li} \tag{E.64}$$

E.7 Continuous Fluid Force

$f_{ik} = \delta_{ik}\delta(\vec{x} - \vec{\chi})\mathrm{H}(t - \tau)$
2D, 3D

$$u_{ik}^{fc} = \kappa u_{ik}^{pc} = -\frac{\eta}{GS} w_{ik}^{fc} = w_{ik}^{Fc} \tag{E.65}$$

$$\sigma_{ijk}^{fc} = \kappa \sigma_{ijk}^{pc} = w_{kji}^{dc} \tag{E.66}$$

$$p_k^{fc} = q_k^{sc} = w_k^{si} = \kappa p_k^{pc} = c\,\zeta_k^{pc} = w_k^{lc} = \frac{1}{S}\zeta_k^{fc} = -\frac{G}{\eta}\zeta_k^{Fc} \tag{E.67}$$

$$w_{ik}^{fc} = -\frac{\kappa GS}{\eta} u_{ik}^{pc} = -\frac{GS}{\eta} u_{ik}^{fc} = -\frac{GS}{\eta} w_{ik}^{Fc} \tag{E.68}$$

$$q_{ik}^{fc} = -\frac{\kappa GS}{\eta} u_{ik}^{pi} = -\frac{GS}{\eta} u_{ik}^{fi} = w_{ik}^{fi} = -\frac{GS}{\eta} q_{ik}^{Fc} = -\frac{GS}{\eta} w_{ik}^{Fi} \tag{E.69}$$

$$\zeta_k^{fc} = Sq_k^{sc} = Sw_k^{si} = \kappa S p_k^{pc} = \kappa \zeta_k^{pc} = Sw_k^{lc} = Sp_k^{fc} = -\frac{GS}{\eta}\zeta_k^{Fc} \tag{E.70}$$

E.8 Instantaneous Fluid Force

$f_{ik} = \delta_{ik}\delta(\vec{x} - \vec{\chi})\delta(t - \tau)$
2D

$$u_{ik}^{fi} = \kappa u_{ik}^{pi} = -\frac{\eta}{GS} q_{ik}^{fc} = -\frac{\eta}{GS} w_{ik}^{fi} = q_{ik}^{Fc} = w_{ik}^{Fi} \tag{E.71}$$

$$\sigma_{ijk}^{fi} = \kappa \sigma_{ijk}^{pi} = q_{kji}^{dc} = w_{kji}^{di} \tag{E.72}$$

$$p_k^{fi} = q_k^{si} = \kappa p_k^{pi} = c\,\zeta_k^{pi} = q_k^{lc} = w_k^{li} = \frac{1}{S}\zeta_k^{fi} = -\frac{G}{\eta}\zeta_k^{Fi} \tag{E.73}$$

$$w_{ik}^{fi} = -\frac{\kappa GS}{\eta} u_{ik}^{pi} = q_{ik}^{fc} = -\frac{GS}{\eta} u_{ik}^{fi} = -\frac{GS}{\eta} q_{ik}^{Fc} = -\frac{GS}{\eta} w_{ik}^{Fi} \tag{E.74}$$

$$q_{ik}^{fi} = -\frac{\kappa}{2\pi}\frac{1}{r^2}\left[(2r_{,i}r_{,k} - \delta_{ik})\delta(t-\tau) + 4c\frac{1}{r^2}(\delta_{ik} - 2r_{,i}r_{,k}\xi^2)\,\xi^4 e^{-\xi^2}\right] \tag{E.75}$$

$$\zeta_k^{fi} = Sq_k^{si} = \kappa Sp_k^{pi} = \kappa \zeta_k^{pi} = Sq_k^{lc} = Sw_k^{li} = Sp_k^{fi} = -\frac{GS}{\eta}\zeta_k^{Fi} \tag{E.76}$$

3D

$$u_{ik}^{fi} = \kappa u_{ik}^{pi} = -\frac{\eta}{GS} q_{ik}^{fc} = -\frac{\eta}{GS} w_{ik}^{fi} = q_{ik}^{Fc} = w_{ik}^{Fi} \tag{E.77}$$

$$\sigma_{ijk}^{fi} = \kappa \sigma_{ijk}^{pi} = q_{kji}^{dc} = w_{kji}^{di} \tag{E.78}$$

$$p_k^{fi} = q_k^{si} = \kappa p_k^{pi} = c\zeta_k^{pi} = q_k^{lc} = w_k^{li} = \frac{1}{S}\zeta_k^{fi} = -\frac{G}{\eta}\zeta_k^{Fi} \tag{E.79}$$

$$w_{ik}^{fi} = -\frac{\kappa GS}{\eta} u_{ik}^{pi} = q_{ik}^{fc} = -\frac{GS}{\eta} u_{ik}^{fi} = -\frac{GS}{\eta} q_{ik}^{Fc} = -\frac{GS}{\eta} w_{ik}^{Fi} \tag{E.80}$$

$$q_{ik}^{fi} = -\frac{\kappa}{4\pi}\frac{1}{r^3}\Bigg[(3r_{,i}r_{,k} - \delta_{ik})\delta(t-\tau) + \frac{8c}{\sqrt{\pi}}\frac{1}{r^2}(\delta_{ik} - 2r_{,i}r_{,k}\xi^2)\xi^5 e^{-\xi^2}\Bigg] \tag{E.81}$$

$$\zeta_k^{fi} = Sq_k^{si} = \kappa Sp_k^{pi} = \kappa \zeta_k^{pi} = Sq_k^{lc} = Sw_k^{li} = Sp_k^{fi} = -\frac{GS}{\eta}\zeta_k^{Fi} \tag{E.82}$$

E.9 Continuous Fluid Dodecapole

$\gamma = \nabla^2 \delta(\vec{x} - \vec{\chi})\,\mathrm{H}(t-\tau)$

2D

$$u_i^{oc} = -\frac{\eta}{2\pi G\kappa}\frac{r_{,i}}{r} e^{-\xi^2} \tag{E.83}$$

$$\sigma_{ij}^{oc} = -\frac{\eta}{\pi\kappa}\frac{1}{r^2}\left[\delta_{ij}(1+2\xi^2) - 2r_{,i}r_{,j}(1+\xi^2)\right]e^{-\xi^2} + \frac{2\eta}{\kappa}\delta_{ij}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.84}$$

$$p^{oc} = -\frac{1}{\pi\kappa}\frac{1}{r^2}\xi^2 e^{-\xi^2} - \frac{1}{\kappa}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.85}$$

$$w_i^{oc} = \frac{1}{2\pi c}\frac{r_{,i}}{r} e^{-\xi^2} + \frac{\partial\delta(\vec{x}-\vec{\chi})}{\partial x}(t-\tau)\,\mathrm{H}(t-\tau) \tag{E.86}$$

$$q_i^{oc} = \frac{2}{\pi}\frac{r_{,i}}{r^3}\xi^2 e^{-\xi^2} + \frac{\partial\delta(\vec{x}-\vec{\chi})}{\partial x}\mathrm{H}(t-\tau) \tag{E.87}$$

$$\zeta^{oc} = Sp^{oc} \tag{E.88}$$

3D

$$u_i^{oc} = -\frac{\eta}{4\pi G\kappa}\frac{r_{,i}}{r^2}\left[\operatorname{erfc}(\xi) + \frac{2}{\sqrt{\pi}}\,\xi e^{-\xi^2}\right] \tag{E.89}$$

$$\sigma_{ij}^{oc} = -\frac{\eta}{2\pi\kappa}\frac{1}{r^3}\left\{\left[\delta_{ij}(1+2\xi^2) - r_{,i}r_{,j}(3+2\xi^2)\right]\frac{2}{\sqrt{\pi}}\xi e^{-\xi^2}\right.$$
$$\left. +(\delta_{ij} - 3r_{,i}r_{,j})\operatorname{erfc}(\xi)\right\} + \frac{2\eta}{\kappa}\delta_{ij}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.90}$$

$$p^{oc} = -\frac{1}{\pi^{3/2}\kappa}\frac{1}{r^3}\xi^3 e^{-\xi^2} - \frac{1}{\kappa}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.91}$$

$$w_i^{oc} = \frac{1}{4\pi c}\frac{r_{,i}}{r^2}\left[\frac{2}{\sqrt{\pi}}\xi e^{-\xi^2} + \operatorname{erfc}(\xi)\right] + \frac{\partial\delta(\vec{x}-\vec{\chi})}{\partial x}(t-\tau)\,\mathrm{H}(t-\tau) \tag{E.92}$$

$$q_i^{oc} = \frac{2}{\pi^{3/2}}\frac{r_{,i}}{r^4}\xi^5 e^{-\xi^2} + \frac{\partial\delta(\vec{x}-\vec{\chi})}{\partial x}\,\mathrm{H}(t-\tau) \tag{E.93}$$

$$\zeta^{oc} = Sp^{oc} \tag{E.94}$$

E.10 Continuous Total Force

$F_{ik} = \delta_{ik}\delta(\vec{x}-\vec{\chi})\mathrm{H}(t-\tau)$

2D

$$u_{ik}^{Fc} = \frac{1}{8\pi G(1-\nu_u)}\left[r_{,i}r_{,k} - (3-4\nu_u)\delta_{ik}\ln r\right]\mathrm{H}(t-\tau)$$
$$+\frac{\eta^2}{8\pi G^2 S}\left[(\delta_{ik} - 2r_{,i}r_{,k})\frac{1-e^{-\xi^2}}{\xi^2} + \delta_{ik}\mathrm{E}_1(\xi^2)\right] \tag{E.95}$$

$$\sigma_{ijk}^{Fc} = \frac{1}{4\pi(1-\nu_u)}\frac{1}{r}\left[(1-2\nu_u)(\delta_{ij}r_{,k} - \delta_{jk}r_{,i} - \delta_{ik}r_{,j})\right.$$
$$\left. -2r_{,i}r_{,j}r_{,k}\right]\mathrm{H}(t-\tau)$$
$$+\frac{\eta^2}{2\pi GS}\frac{1}{r}\left[(4r_{,i}r_{,j}r_{,k} - \delta_{ik}r_{,j} - \delta_{jk}r_{,i} - \delta_{ij}r_{,k})\frac{1-e^{-\xi^2}}{\xi^2}\right.$$
$$\left. +2(\delta_{ij}r_{,k} - r_{,i}r_{,j}r_{,k})e^{-\xi^2}\right]$$
$$= u_{kji}^{dc} \tag{E.96}$$

$$p_k^{Fc} = u_k^{si} = u_k^{lc} \tag{E.97}$$

$$w_{ik}^{Fc} = \kappa u_{ik}^{pc} = u_{ik}^{fc} = -\frac{\eta}{GS}w_{ik}^{fc} \tag{E.98}$$

$$q_{ik}^{Fc} = \kappa u_{ik}^{pi} = -\frac{\eta}{GS} q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta}{GS} w_{ik}^{fi} = w_{ik}^{Fi} \tag{E.99}$$

$$\zeta_k^{Fc} = -\frac{\eta}{G} q_k^{sc} = -\frac{\eta}{G} w_k^{si} = -\frac{\eta\kappa}{G} p_k^{pc} = -\frac{\eta c}{G} \zeta_k^{pc} = -\frac{\eta}{G} w_k^{lc}$$
$$= -\frac{\eta}{G} p_k^{fc} = -\frac{\eta}{GS} \zeta_k^{fc} \tag{E.100}$$

3D

$$\begin{aligned} u_{ik}^{Fc} = {} & \frac{1}{16\pi G(1-\nu_u)} \frac{1}{r} \left[r_{,i} r_{,k} + (3 - 4\nu_u)\delta_{ik} \right] \mathrm{H}(t-\tau) \\ & + \frac{\eta^2}{16\pi G^2 S} \frac{1}{r} \left\{ (\delta_{ik} - 3 r_{,i} r_{,k}) \left[\frac{\mathrm{erf}(\xi)}{\xi^2} - \frac{2}{\sqrt{\pi}} \frac{e^{-\xi^2}}{\xi} \right] \right. \\ & \left. + 2(\delta_{ik} - r_{,i} r_{,k}) \mathrm{erfc}(\xi) \right\} \end{aligned} \tag{E.101}$$

$$\begin{aligned} \sigma_{ijk}^{Fc} = {} & \frac{1}{8\pi(1-\nu_u)} \frac{1}{r^2} \left[(1-2\nu_u)(\delta_{ij} r_{,k} - \delta_{jk} r_{,i} - \delta_{ik} r_{,j}) \right. \\ & \left. - 3 r_{,i} r_{,j} r_{,k} \right] \mathrm{H}(t-\tau) \\ & + \frac{\eta^2}{8\pi GS} \frac{1}{r^2} \left\{ 4(\delta_{ij} r_{,k} - r_{,i} r_{,j} r_{,k}) \left[\mathrm{erfc}(\xi) + \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right] \right. \\ & + (5 r_{,i} r_{,j} r_{,k} - \delta_{ij} r_{,k} - \delta_{ik} r_{,j} - \delta_{jk} r_{,i}) \\ & \left. \times \left[\frac{3\mathrm{erf}(\xi)}{\xi^2} - \frac{6}{\sqrt{\pi}} \frac{e^{-\xi^2}}{\xi} + 2\,\mathrm{erfc}(\xi) \right] \right\} = u_{kji}^{dc} \end{aligned} \tag{E.102}$$

$$p_k^{Fc} = u_k^{si} = u_k^{lc} \tag{E.103}$$

$$w_{ik}^{Fc} = \kappa u_{ik}^{pc} = u_{ik}^{fc} = -\frac{\eta}{GS} w_{ik}^{fc} \tag{E.104}$$

$$q_{ik}^{Fc} = \kappa u_{ik}^{pi} = -\frac{\eta}{GS} q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta}{GS} w_{ik}^{fi} = w_{ik}^{Fi} \tag{E.105}$$

$$\zeta_k^{Fc} = -\frac{\eta}{G} q_k^{sc} = -\frac{\eta}{G} w_k^{si} = -\frac{\eta\kappa}{G} p_k^{pc} = -\frac{\eta c}{G} \zeta_k^{pc} = -\frac{\eta}{G} w_k^{lc}$$
$$= -\frac{\eta}{G} p_k^{fc} = -\frac{\eta}{GS} \zeta_k^{fc} \tag{E.106}$$

E.11 Instantaneous Total Force

$F_{ik} = \delta_{ik}\delta(\vec{x} - \vec{\chi})\delta(t - \tau)$

2D

$$u_{ik}^{Fi} = \frac{1}{8\pi G(1-\nu_u)}\delta(t-\tau)\left[r_{,i}r_{,k} - (3-4\nu_u)\delta_{ik}\ln r\right] + \frac{\eta^2 c}{2\pi G^2 S}\frac{1}{r^2}\left[(\delta_{ik} - 2r_{,i}r_{,k})(1-e^{-\xi^2}) + 2r_{,i}r_{,k}\xi^2 e^{-\xi^2}\right] \tag{E.107}$$

$$\begin{aligned}\sigma_{ijk}^{Fi} &= \frac{1}{4\pi(1-\nu_u)}\delta(t-\tau)\frac{1}{r}\left[(1-2\nu_u)(\delta_{ij}r_{,k} - \delta_{jk}r_{,i} - \delta_{ik}r_{,j}) - 2r_{,i}r_{,j}r_{,k}\right] \\ &\quad + \frac{2\eta^2 c}{\pi G S}\frac{1}{r^3}\Big[(4r_{,i}r_{,j}r_{,k} - \delta_{ik}r_{,j} - \delta_{jk}r_{,i} - \delta_{ij}r_{,k}) \\ &\quad \times(1 - e^{-\xi^2} - \xi^2 e^{-\xi^2}) - 2(r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k})\xi^4 e^{-\xi^2}\Big] \\ &= u_{kji}^{di}\end{aligned} \tag{E.108}$$

$$p_k^{Fi} = u_k^{li} \tag{E.109}$$

$$w_{ik}^{Fi} = \kappa u_{ik}^{pi} = -\frac{\eta}{GS}q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta}{GS}w_{ik}^{fi} = q_{ik}^{Fc} \tag{E.110}$$

$$q_{ik}^{Fi} = \frac{\eta c}{2\pi G}\,\frac{1}{r^2}\Big[(2r_{,i}r_{,k} - \delta_{ik})\delta(t-\tau) + 4c\frac{1}{r^2}(\delta_{ik} - 2r_{,i}r_{,k}\xi^2)\xi^4 e^{-\xi^2}\Big] \tag{E.111}$$

$$\begin{aligned}\zeta_k^{Fi} &= -\frac{\eta}{G}q_k^{si} = -\frac{\eta\kappa}{G}p_k^{pi} = -\frac{\eta c}{G}\zeta_k^{pi} = -\frac{\eta}{G}q_k^{lc} = -\frac{\eta}{G}w_k^{li} \\ &= -\frac{\eta}{G}p_k^{fi} = -\frac{\eta}{GS}\zeta_k^{fi}\end{aligned} \tag{E.112}$$

3D

$$\begin{aligned}u_{ik}^{Fi} &= \frac{1}{16\pi G(1-\nu_u)}\delta(t-\tau)\frac{1}{r}\left[r_{,i}r_{,k} + (3-4\nu_u)\delta_{ik}\right] \\ &\quad + \frac{\eta^2 c}{4\pi G^2 S}\frac{1}{r^3}\left\{(\delta_{ik} - 3r_{,i}r_{,k})\left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}}\xi e^{-\xi^2}\right] + \frac{4}{\sqrt{\pi}}r_{,i}r_{,k}\xi^3 e^{-\xi^2}\right\}\end{aligned} \tag{E.113}$$

$$\sigma_{ijk}^{Fi} = \frac{1}{8\pi(1-\nu_u)}\delta(t-\tau)\frac{1}{r^2}\left[(1-2\nu_u)(\delta_{ij}r_{,k} - \delta_{jk}r_{,i} - \delta_{ik}r_{,j})\right.$$
$$\left. -3r_{,i}r_{,j}r_{,k}\right] + \frac{\eta^2 c}{2\pi GS}\frac{1}{r^4}\left\{\frac{8}{\sqrt{\pi}}(\delta_{ij}r_{,k} - r_{,i}r_{,j}r_{,k})\xi^5 e^{-\xi^2}\right.$$
$$-(\delta_{ij}r_{,k} + \delta_{ik}r_{,j} + \delta_{jk}r_{,i} - 5r_{,i}r_{,j}r_{,k})$$
$$\left.\left[3\,\mathrm{erf}(\xi) - \frac{2}{\sqrt{\pi}}(3+2\xi^2)\xi e^{-\xi^2}\right]\right\}$$
$$= u_{kji}^{di} \tag{E.114}$$

$$p_k^{Fi} = u_k^{li} \tag{E.115}$$

$$w_{ik}^{Fi} = \kappa u_{ik}^{pi} = -\frac{\eta}{GS}q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta}{GS}w_{ik}^{fi} = q_{ik}^{Fc} \tag{E.116}$$

$$q_{ik}^{Fi} = \frac{\eta c}{4\pi G}\frac{1}{r^3}\left[(3r_{,i}r_{,k} - \delta_{ik})\delta(t-\tau)\right.$$
$$\left. + \frac{8c}{\sqrt{\pi}}\frac{1}{r^2}(\delta_{ik} - 2r_{,i}r_{,k}\xi^2)\xi^5 e^{-\xi^2}\right] \tag{E.117}$$

$$\zeta_k^{Fi} = -\frac{\eta}{G}q_k^{si} = -\frac{\eta\kappa}{G}p_k^{pi} = -\frac{\eta c}{G}\zeta_k^{pi} = -\frac{\eta}{G}q_k^{lc} = -\frac{\eta}{G}w_k^{li}$$
$$= -\frac{\eta}{G}p_k^{fi} = -\frac{\eta}{GS}\zeta_k^{fi} \tag{E.118}$$

E.12 Continuous Solid Quadrupole and Hexapole

$F_{ik,k} = \delta_{,i}\,\mathrm{H}(t-\tau)$

2D

$$u_i^{qc} = -\frac{1-2\nu_u}{4\pi G(1-\nu_u)}\frac{r_{,i}}{r}\mathrm{H}(t-\tau) - \frac{\eta^2}{2\pi G^2 S}\frac{r_{,i}}{r}e^{-\xi^2} \tag{E.119}$$

$$\sigma_{ij}^{qc} = -\frac{1-2\nu_u}{2\pi(1-\nu_u)}\frac{1}{r^2}(\delta_{ij} - 2r_{,i}r_{,j}) - \frac{\nu}{1-\nu}\delta_{ij}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau)$$
$$-\frac{\eta^2}{\pi GS}\frac{1}{r^2}\left[\delta_{ij}(1+2\xi^2) - 2r_{,i}r_{,j}(1+\xi^2)\right]e^{-\xi^2} \tag{E.120}$$

$$p^{qc} = -\frac{\eta}{\pi GS}\frac{1}{r^2}\xi^2 e^{-\xi^2} \tag{E.121}$$

$$w_i^{qc} = \frac{\eta}{2\pi G}\frac{r_{,i}}{r}e^{-\xi^2} \tag{E.122}$$

$$q_i^{qc} = \frac{\eta c}{\pi G}\frac{r_{,i}}{r^2}\xi^2 e^{-\xi^2} \tag{E.123}$$

$$\zeta^{qc} = -\frac{\eta}{\pi G}\frac{1}{r^2}\xi^2 e^{-\xi^2} - \frac{\eta}{G}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.124}$$

3D

$$u_i^{qc} = -\frac{1-2\nu_u}{8\pi G(1-\nu_u)}\frac{r_{,i}}{r^2}\,\mathrm{H}(t-\tau) - \frac{\eta^2}{4\pi G^2 S}\frac{r_{,i}}{r^2}\left[\mathrm{erfc}(\xi) + \frac{2}{\sqrt{\pi}}\,\xi e^{-\xi^2}\right] \tag{E.125}$$

$$\sigma_{ij}^{qc} = -\frac{1-2\nu_u}{4\pi(1-\nu_u)}\frac{1}{r^3}(\delta_{ij}-3r_{,i}r_{,j}) - \frac{\nu}{1-\nu}\delta_{ij}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) - \frac{\eta^2}{2\pi GS}\frac{1}{r^3}\left\{\left[\delta_{ij}(1+2\xi^2) - r_{,i}r_{,j}(3+2\xi^2)\right]\frac{2}{\sqrt{\pi}}\xi e^{-\xi^2} + (\delta_{ij}-3r_{,i}r_{,j})\,\mathrm{erfc}(\xi)\right\} \tag{E.126}$$

$$p^{qc} = -\frac{\eta}{\pi^{3/2}GS}\frac{1}{r^3}\,\xi^3 e^{-\xi^2} \tag{E.127}$$

$$w_i^{qc} = \frac{\eta}{4\pi G}\frac{r_{,i}}{r^2}\left[\frac{2}{\sqrt{\pi}}\,\xi\, e^{-\xi^2} + \mathrm{erfc}(\xi)\right] \tag{E.128}$$

$$q_i^{qc} = \frac{\eta c}{\pi^{3/2}G}\frac{r_{,i}}{r^3}\xi^3 e^{-\xi^2} \tag{E.129}$$

$$\zeta^{qc} = -\frac{\eta}{\pi^{3/2}G}\frac{1}{r^3}\,\xi^3 e^{-\xi^2} - \frac{\eta}{G}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.130}$$

E.13 Continuous Solid Dilatation

$E_{ij} = (\ln r/2\pi)_{,ij}\,\mathrm{H}(t-\tau)$
2D

$$u_i^{cc} = \frac{1}{2\pi}\frac{r_{,i}}{r}\,\mathrm{H}(t-\tau) \tag{E.131}$$

$$\sigma_{ij}^{cc} = \frac{G}{\pi}\frac{1}{r^2}(\delta_{ij}-2r_{,i}r_{,j}) + \frac{2G\nu_u}{1-2\nu_u}\delta_{ij}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.132}$$

$$p^{cc} = w_i^{cc} = q_i^{cc} = \zeta^{cc} = 0 \tag{E.133}$$

$E_{ij} = -(1/4\pi r)_{,ij}\,\mathrm{H}(t-\tau)$
3D

$$u_i^{cc} = \frac{1}{4\pi}\frac{r_{,i}}{r^2}\,\mathrm{H}(t-\tau) \tag{E.134}$$

$$\sigma_{ij}^{cc} = \frac{G}{2\pi}\frac{1}{r^3}(\delta_{ij}-3r_{,i}r_{,j}) + \frac{2G\nu_u}{1-2\nu_u}\delta_{ij}\delta(\vec{x}-\vec{\chi})\,\mathrm{H}(t-\tau) \tag{E.135}$$

$$p^{cc} = w_i^{cc} = q_i^{cc} = \zeta^{cc} = 0 \tag{E.136}$$

E.14 Continuous Displacement Discontinuity

$E_{ijkl} = -\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(\vec{x} - \vec{\chi})\,\mathrm{H}(t-\tau)$
2D

$$u_{ikl}^{dc} = \sigma_{lki}^{Fc} \tag{E.137}$$

$$\begin{aligned}\sigma_{ijkl}^{dc} = & \frac{G}{2\pi\,(1-\nu_u)}\frac{1}{r^2}\mathrm{H}(t-\tau)\left[8r_{,i}r_{,j}r_{,k}r_{,l} - (1-2\nu_u)\right.\\ & \times\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl} + 2\delta_{kl}r_{,i}r_{,j} + 2\delta_{ij}r_{,k}r_{,l}\right)\\ & \left.-2\nu_u\left(\delta_{ij}\delta_{kl} + \delta_{lj}r_{,i}r_{,k} + \delta_{il}r_{,j}r_{,k} + \delta_{jk}r_{,i}r_{,l} + \delta_{ik}r_{,j}r_{,l}\right)\right]\\ & + \frac{\eta^2}{\pi S}\frac{1}{r^2}\left[\left(4\delta_{kj}r_{,i}r_{,l} + 4\delta_{ki}r_{,l}r_{,j} + 4\delta_{kl}r_{,i}r_{,j} + 4\delta_{lj}r_{,i}r_{,k} + 4\delta_{li}r_{,k}r_{,j}\right.\right.\\ & \left. + 4\delta_{ij}r_{,k}r_{,l} - \delta_{ij}\delta_{kl} - \delta_{kj}\delta_{il} - \delta_{ki}\delta_{jl} - 24r_{,i}r_{,j}r_{,k}r_{,l}\right)\frac{1-e^{-\xi^2}}{\xi^2}\\ & +2\left(2\delta_{ij}\delta_{kl} - 3\delta_{kl}r_{,i}r_{,j} - \delta_{lj}r_{,i}r_{,k} - \delta_{il}r_{,k}r_{,j}\right.\\ & \left.-\delta_{kj}r_{,i}r_{,l} - \delta_{ki}r_{,l}r_{,j} - 3\delta_{ij}r_{,l}r_{,k} + 8r_{,i}r_{,j}r_{,k}r_{,l}\right)e^{-\xi^2}\\ & \left.+ 4\left(\delta_{ij}\delta_{kl} - \delta_{kl}r_{,i}r_{,j} - \delta_{ij}r_{,l}r_{,k} + r_{,i}r_{,j}r_{,k}r_{,l}\right)\xi^2 e^{-\xi^2}\right]\end{aligned} \tag{E.138}$$

$$p_{kl}^{dc} = \sigma_{kl}^{si} = \sigma_{kl}^{lc} \tag{E.139}$$

$$w_{ikl}^{dc} = \kappa\sigma_{lki}^{pc} = \sigma_{lki}^{fc} \tag{E.140}$$

$$q_{ikl}^{dc} = \kappa\sigma_{lki}^{pi} = \sigma_{lki}^{fi} = w_{ikl}^{di} \tag{E.141}$$

$$\zeta_{kl}^{dc} = \frac{\eta}{\pi}\frac{1}{r^2}\left[2r_{,k}r_{,l} - \delta_{kl} - 2(\delta_{kl} - r_{,k}r_{,l})\xi^2\right]e^{-\xi^2} \tag{E.142}$$

3D

$$u_{ikl}^{dc} = \sigma_{lki}^{Fc} \tag{E.143}$$

$$\begin{aligned}\sigma_{ijkl}^{dc} = & \frac{G}{4\pi\,(1-\nu_u)}\frac{1}{r^3}\mathrm{H}(t-\tau)\left[15r_{,i}r_{,j}r_{,k}r_{,l} - (1-2\nu_u)\right.\\ & \times\left(\delta_{kj}\delta_{il} + \delta_{ki}\delta_{jl} - \delta_{ij}\delta_{kl} + 3\delta_{kl}r_{,i}r_{,j} + 3\delta_{ij}r_{,k}r_{,l}\right)\\ & \left.-\nu_u\left(2\delta_{ij}\delta_{kl} + 3\delta_{lj}r_{,i}r_{,k} + 3\delta_{il}r_{,k}r_{,j} + 3\delta_{kj}r_{,i}r_{,l} + 3\delta_{ki}r_{,l}r_{,j}\right)\right]\\ & + \frac{\eta^2}{4\pi S}\frac{1}{r^3}\left[\frac{8}{\sqrt{\pi}}\left(-2\delta_{ij}\delta_{kl} + 4\delta_{kl}r_{,i}r_{,j} + 4\delta_{ij}r_{,k}r_{,l} + \delta_{lj}r_{,i}r_{,k}\right.\right.\\ & \left. + \delta_{il}r_{,k}r_{,j} + \delta_{kj}r_{,i}r_{,l} + \delta_{ki}r_{,l}r_{,j} - 10r_{,i}r_{,j}r_{,k}r_{,l}\right)\xi e^{-\xi^2}\end{aligned}$$

$$+\frac{16}{\sqrt{\pi}}\left(-\delta_{ij}\delta_{kl}+\delta_{kl}r_{,i}r_{,j}+\delta_{ij}r_{,k}r_{,l}-r_{,i}r_{,j}r_{,k}r_{,l}\right)\xi^3 e^{-\xi^2}$$
$$+3\left(\delta_{lj}\delta_{ki}+\delta_{kj}\delta_{il}+\delta_{ji}\delta_{kl}-5\delta_{lj}r_{,i}r_{,k}-5\delta_{il}r_{,k}r_{,j}-5\delta_{kj}r_{,i}r_{,l}\right.$$
$$\left.-5\delta_{ki}r_{,l}r_{,j}-5\delta_{ij}r_{,k}r_{,l}-5\delta_{kl}r_{,i}r_{,j}+35r_{,i}r_{,j}r_{,k}r_{,l}\right)$$
$$\times\left(\frac{\mathrm{erf}(\xi)}{\xi^2}-\frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right)$$
$$+2\left(\delta_{kj}\delta_{il}+\delta_{ki}\delta_{jl}-3\delta_{ij}\delta_{kl}+3\delta_{kl}r_{,i}r_{,j}+3\delta_{ij}r_{,k}r_{,l}\right.$$
$$-3\delta_{il}r_{,k}r_{,j}-3\delta_{kj}r_{,i}r_{,l}-3\delta_{ki}r_{,l}r_{,j}-3\delta_{lj}r_{,i}r_{,k}$$
$$\left.\left.+15r_{,i}r_{,j}r_{,k}r_{,l}\right)\mathrm{erfc}(\xi)\right] \tag{E.144}$$

$$p_{kl}^{dc}=\sigma_{kl}^{si} = \sigma_{kl}^{lc} \tag{E.145}$$

$$w_{ikl}^{dc}=\kappa\sigma_{lki}^{pc} = \sigma_{lki}^{fc} \tag{E.146}$$

$$q_{ikl}^{dc}=\kappa\sigma_{lki}^{pi} = \sigma_{lki}^{fi} = w_{ikl}^{di} \tag{E.147}$$

$$\zeta_{kl}^{dc}=\frac{\eta}{2\pi}\frac{1}{r^3}\left\{(3r_{,k}r_{,l}-\delta_{kl})\left[\mathrm{erfc}(\xi)+\frac{2}{\sqrt{\pi}}\xi e^{-\xi^2}\right]\right.$$
$$\left.-\frac{4}{\sqrt{\pi}}(\delta_{kl}-r_{,k}r_{,l})\,\xi^3 e^{-\xi^2}\right\} \tag{E.148}$$

E.15 Instantaneous Displacement Discontinuity

$E_{ijkl}=-\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\delta(\vec{x}-\vec{\chi})\delta(t-\tau)$
2D

$$u_{ikl}^{di}=\sigma_{lki}^{Fi} \tag{E.149}$$

$$\sigma_{ijkl}^{di}=\frac{G}{2\pi(1-\nu_u)}\delta(t-\tau)\frac{1}{r^2}\left[8r_{,i}r_{,j}r_{,k}r_{,l}\right.$$
$$-(1-2\nu_u)\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}-\delta_{ij}\delta_{kl}+2\delta_{kl}r_{,i}r_{,j}+2\delta_{ij}r_{,k}r_{,l}\right)$$
$$\left.-2\nu_u\left(\delta_{ij}\delta_{kl}+\delta_{lj}r_{,i}r_{,k}+\delta_{il}r_{,j}r_{,k}+\delta_{jk}r_{,i}r_{,l}+\delta_{ik}r_{,j}r_{,l}\right)\right]$$
$$+\frac{4\eta^2 c}{\pi S}\frac{1}{r^4}\left\{\left(4\delta_{kj}r_{,i}r_{,l}+4\delta_{ki}r_{,l}r_{,j}+4\delta_{kl}r_{,i}r_{,j}+4\delta_{lj}r_{,i}r_{,k}\right.\right.$$
$$+4\delta_{li}r_{,k}r_{,j}+4\delta_{ij}r_{,k}r_{,l}-\delta_{ij}\delta_{kl}-\delta_{kj}\delta_{il}-\delta_{ki}\delta_{jl}$$
$$\left.-24r_{,i}r_{,j}r_{,k}r_{,l}\right)\left[1-(1+\xi^2)e^{-\xi^2}\right]$$

$$+2\left(6r_{,i}r_{,j}r_{,k}r_{,l} - \delta_{kl}r_{,i}r_{,j} - \delta_{lj}r_{,i}r_{,k} - \delta_{li}r_{,k}r_{,j} - \delta_{jk}r_{,i}r_{,l}\right.$$
$$\left. -\delta_{ki}r_{,l}r_{,j} - \delta_{ij}r_{,k}r_{,l}\right)\xi^4 e^{-\xi^2}$$
$$\left. +4\left(r_{,i}r_{,j}r_{,k}r_{,l} + \delta_{ij}\delta_{kl} - \delta_{ij}r_{,k}r_{,l} - \delta_{kl}r_{,i}r_{,j}\right)\xi^6 e^{-\xi^2}\right\} \tag{E.150}$$

$$p_{kl}^{di} = \sigma_{kl}^{li} \tag{E.151}$$

$$w_{ikl}^{di} = \kappa\sigma_{lki}^{pi} = \sigma_{lki}^{fi} = q_{ikl}^{dc} \tag{E.152}$$

$$q_{ikl}^{di} = \frac{2\eta c}{\pi}\delta(t-\tau)\frac{1}{r^3}(\delta_{il}r_{,k} + \delta_{kl}r_{,i} + \delta_{ik}r_{,l} - 4r_{,i}r_{,k}r_{,l})$$
$$-\frac{8\eta c^2}{\pi}\frac{1}{r^5}\left[(\delta_{il}r_{,k} + \delta_{ik}r_{,l} - 3\delta_{kl}r_{,i})\right.$$
$$\left. +2(\delta_{kl}r_{,i} - r_{,i}r_{,k}r_{,l})\xi^2\right]\xi^6 e^{-\xi^2} \tag{E.153}$$

$$\zeta_{kl}^{di} = \frac{4\eta c}{\pi}\frac{1}{r^4}\left[\delta_{kl}(1-2\xi^2) + 2r_{,k}r_{,l}\xi^2\right]\xi^4 e^{-\xi^2} \tag{E.154}$$

3D

$$u_{ikl}^{di} = \sigma_{lki}^{Fi} \tag{E.155}$$

$$\sigma_{ijkl}^{di} = \frac{G}{4\pi(1-\nu_u)}\frac{1}{r^3}\delta(t-\tau)\left[15r_{,i}r_{,j}r_{,k}r_{,l}\right.$$
$$-(1-2\nu_u)\left(\delta_{kj}\delta_{il} + \delta_{ki}\delta_{jl} - \delta_{ij}\delta_{kl} + 3\delta_{kl}r_{,i}r_{,j} + 3\delta_{ij}r_{,k}r_{,l}\right)$$
$$\left. -2\nu_u\left(\delta_{ij}\delta_{kl} + \delta_{lj}r_{,i}r_{,k} + \delta_{il}r_{,j}r_{,k} + \delta_{jk}r_{,i}r_{,l} + \delta_{ik}r_{,j}r_{,l}\right)\right]$$
$$+\frac{\eta^2 c}{\pi S}\frac{1}{r^5}\left\{8\left(7r_{,i}r_{,j}r_{,k}r_{,l} - \delta_{ij}\delta_{kl} - \delta_{kl}r_{,i}r_{,j} - \delta_{ij}r_{,k}r_{,l}\right.\right.$$
$$\left. -\delta_{lj}r_{,i}r_{,k} - \delta_{il}r_{,k}r_{,j} - \delta_{kj}r_{,i}r_{,l} - \delta_{ki}r_{,l}r_{,j}\right)\xi^5 e^{-\xi^2}$$
$$+16\left(r_{,i}r_{,j}r_{,k}r_{,l} + \delta_{ij}\delta_{kl} - \delta_{kl}r_{,i}r_{,j} - \delta_{ij}r_{,k}r_{,l}\right)\xi^7 e^{-\xi^2}$$
$$-\left(\delta_{lj}\delta_{ki} + \delta_{kj}\delta_{il} + \delta_{ji}\delta_{kl} - 5\delta_{lj}r_{,i}r_{,k} - 5\delta_{il}r_{,k}r_{,j} - 5\delta_{kj}r_{,i}r_{,l}\right.$$
$$\left. -5\delta_{ki}r_{,l}r_{,j} - 5\delta_{ij}r_{,k}r_{,l} - 5\delta_{kl}r_{,i}r_{,j} + 35r_{,i}r_{,j}r_{,k}r_{,l}\right)$$
$$\left.\times\left[3\,\mathrm{erf}(\xi) - \frac{2}{\sqrt{\pi}}(3+2\xi^2)\xi e^{-\xi^2}\right]\right\} \tag{E.156}$$

$$p_{kl}^{di} = \sigma_{kl}^{li} \tag{E.157}$$

$$w_{ikl}^{di} = \kappa\sigma_{lki}^{pi} = \sigma_{lki}^{fi} = q_{ikl}^{dc} \tag{E.158}$$

$$q_{ikl}^{di} = \frac{3\eta c}{2\pi}\delta(t-\tau)\frac{1}{r^4}(\delta_{il}r_{,k} + \delta_{kl}r_{,i} + \delta_{ik}r_{,l} - 5r_{,i}r_{,k}r_{,l})$$
$$-\frac{8\eta c^2}{\pi^{3/2}}\frac{1}{r^6}\left[(\delta_{il}r_{,k} + \delta_{ik}r_{,l} - 4\delta_{kl}r_{,i})\right.$$
$$\left.+2(\delta_{kl}r_{,i} - r_{,i}r_{,k}r_{,l})\xi^2\right]\xi^7 e^{-\xi^2} \tag{E.159}$$

$$\zeta_{kl}^{di} = \frac{8\eta c}{\pi^{3/2}}\frac{1}{r^5}\left[\delta_{kl}(1-\xi^2) + r_{,k}r_{,l}\xi^2\right]\xi^5 e^{-\xi^2} \tag{E.160}$$

Appendix F
Biography

F.1 Leonardo da Vinci (1452–1519)

Leonardo da Vinci, the scientist, inventor, and artist, was born an illegitimate son of a notary and a peasant girl, at a small town Vinci, in Tuscany, Italy. At age 14, he apprenticed with a famous artist in Florence to be a painter. In 1482, he entered into the service of the Duke of Milan. During the 17 years in Milan, Leonardo not only did painting and sculpting, but also designed churches, fortresses, canals, flying machines, tanks, various war devices, and submarines.

Unlike many of his contemporaries, da Vinci took the fresh approach in science by actually observing nature and asking deceptively simple scientific questions like: “How do birds fly?” “What causes cyclones?” As his curiosity took him in ever wilder directions, da Vinci always used this method of scientific inquiry: close observation, repeated testing of the observation, and precise illustration of the subject or phenomenon with brief explanatory notes. The result was volumes of remarkable notes on an amazing variety of topics, from the nature of the sun, moon and stars to the formation of fossils. During his lifetime, the range of topics of his inquiry included anatomy, zoology, botany, geology, optics, aerodynamics, and hydrodynamics, among others.

Between 1490 and 1495 da Vinci developed his habit of recording his studies in meticulously illustrated notebooks. These studies and sketches were collected into various codices and manuscripts, including the *Codex Leicester* [41], in which the text was written in its mirror image. In this codex, he presented his view that the Earth is a circulatory system like human body:

> The Earth is a living body. Its soul is its ability to grow. This soul, which also provides the Earth with its bodily warmth, is located in the inner fires of the Earth, which emerge at several places as baths, sulfur mines or volcanoes. Its flesh is the soil, its bones are the strata of rock, its cartilage is the tufa, its blood is the underground streams, the reservoir of blood around its heart is the ocean, the systole and diastole of the blood in the arteries and veins appear on the Earth as the rising and sinking of the oceans.

A.H.-D. Cheng, *Poroelasticity*, Theory and Applications of Transport in Porous Media 27, DOI 10.1007/978-3-319-25202-5

Other ideas and observations relating to porous medium include, on soil foundation settlement: [42, 43]

> As to the foundations of the component parts of temples and other public buildings, the depths of the foundations must bear the same proportions to each other as the weight of material which is to be placed upon them.
>
> Every part of the depth of Earth in a given space is composed of layers, and each layer is composed of heavier or lighter materials, the lowest being the heaviest. And this can be proved, because these layers have been formed by the sediment from water carried down to the sea, by the current of rivers which flow into it. The heaviest part of this sediment was that which was first thrown down, and so on by degrees; and this is the action of water when it becomes stagnant, having first brought down the mud whence it first flowed. And such layers of soil are seen in the banks of rivers, where their constant flow has cut through them and divided one slope from the other to a great depth; where in gravelly strata the waters have run off, the materials have, in consequence, dried and been converted into hard stone, and this happened most in what was the finest mud; whence we conclude that every portion of the surface of the Earth was once at the center of the Earth, and viceversa &c.
>
> The heaviest part of the foundations of buildings settles most, and leaves the lighter part above it separated from it.
>
> And the soil which is most pressed, if it be porous yields most.

on slope stability and landslide:

> Parallel fissures constantly occur in buildings which are erected on a hill side, when the hill is composed of stratified rocks with an oblique stratification, because water and other moisture often penetrates these oblique seams carrying in greasy and slippery soil; and as the strata are not continuous down to the bottom of the valley, the rocks slide in the direction of the slope, and the motion does not cease till they have reached the bottom of the valley, carrying with them, as though in a boat, that portion of the building which is separated by them from the rest. The remedy for this is always to build thick piers under the wall which is slipping, with arches from one to another, and with a good scarp and let the piers have a firm foundation in the strata so that they may not break away from them.
>
> In order to find the solid part of these strata, it is necessary to make a shaft at the foot of the wall of great depth through the strata; and in this shaft, on the side from which the hill slopes, smooth and flatten a space one palm wide from the top to the bottom; and after some time this smooth portion made on the side of the shaft, will show plainly which part of the hill is moving.

and on porous bone and sound transmission in inner ear:

> In the same way as to the sense of hearing, it would have sufficed if the voice had merely sounded in the porous cavity of the indurated portion of the temporal bone which lies within the ear, without making any farther transit from this bone to the common sense, where the voice confers with and discourses to the common judgment.

F.2 Robert Hooke (1635–1703)

Robert Hooke was perhaps the single greatest experimental scientist of the seventeenth century. His interests spanned from physics and astronomy, to chemistry, biology, and geology, to architecture and naval technology. He collaborated or competed with scientists as diverse as Robert Boyle (1627–1691), Christian Huygens (1629–1695), Antonie van Leeuwenhoek (1632–1723), Christopher Wren (1632–1723), and Isaac Newton (1643–1727).

Robert Hooke was born on 18 July 1635 in the small town of Freshwater, on the Isle of Wight, England [62]. His father, John Hooke, was the curator of the church of Freshwater. Robert Hooke was born in poor health, and was expected to die; but he hung on and gradually grew stronger. His father hoped that he might train Robert for a career in the church, as he did for Robert's three brothers. However, study gave Robert headaches; hence John gave up on his son's education. Left to follow his own inclination, Robert developed an aptitude for building mechanical devices. In fact, throughout his life, Robert had developed the ability to master other people's skills without formal teaching. For example, at young age, he watched John Hoskins, a famous painter of miniatures, at work, and decided that he could do it himself. Later, his skill with drawing became an invaluable tool, when he needed to communicate his mechanical and microscopical discoveries, as well as in his career as an architect and surveyor.

After his father passed away, Robert went to London, at age of 13. He first entered the Westminster School, and from there went to Oxford, where some of the best scientists in England were working at the time. Hooke impressed them with his skills at designing experiments and building equipment, and soon became an assistant to Robert Boyle in his study the physics of gases. In 1662 Hooke was named Curator of Experiments of the newly formed Royal Society of London—meaning that he was responsible for demonstrating new experiments at the Society's weekly meetings. He later became Gresham Professor of Geometry at Gresham College, London. He died in London on March 3, 1703.

Among other accomplishments, Hooke invented the universal joint, the iris diaphragm, an early prototype of the respirator; invented or improved meteorological instruments such as the barometer, anemometer, and hygrometer; and so on. He invented the anchor escapement and the balance spring, which made accurate clocks possible, and fought with Christiaan Huygens on the priority and patent of spring driven watches. He also disputed with Isaac Newton on the priority claim of the inverse square law of gravitation force, as well as the nature of light (suggesting that it is a wave, rather than particle, as Newton proposed). He served as Chief Surveyor and worked with Christopher Wren to rebuild London after the Great Fire of 1666.

In *De Potentia Restitutivâ, or of Spring* [61], Hooke wrote:

> Take a wire string of 20, or 30, or 40 ft long, and fasten the upper part thereof to a nail, and to the other end fasten a Scale to receive the weights: Then with a pair of Compasses take the distance of the bottom of the scale from the ground or floor underneath, and set down the said distance, then put in weights into the scale and measure the several stretchings of the said string, and set them down. Then compare the several stretchings of the said string, and you will find that they will always bear the same proportions one to the other that the weights do that made them.

Hooke also described experiments with helical springs, with watch springs coiled into spirals, and also with

> a piece of dry wood that will bend and return, if one end thereof be fixed in a horizontal posture, and to the other end be hanged weights to make it bend downwards.

This suggested linear relation between the force and the deformation is the so-called *Hooke's Law* [80].

F.3 Jean Baptiste Joseph Fourier (1768–1830)

Joseph Fourier was born in Auxerre, France, the ninth of the twelve children of his father's second marriage. One of his letters showed that he really wanted to make a major impact in mathematics: "*Yesterday was my 21st birthday; at that age Newton and Pascal had already acquired many claims to immortality.*" In 1790 Fourier became a teacher at the Benedictine College, where he had studied earlier. Soon after, he was entangled in the French Revolution and joined the local revolutionary committee. He was arrested in 1794, and almost went to the guillotine. Only the political changes resulted in his being released. In 1794 Fourier was admitted to the newly established Ecole Normale in Paris, where he was taught by Joseph-Louis Lagrange (1736–1813), Pierre-Simon Laplace (1749–1827), and Gaspard Monge (1746–1818). In 1797 he succeeded Lagrange in being appointed to the Chair of Analysis and Mechanics.

In 1798, Fourier joined Napoleon's army in its invasion of Egypt as a scientific advisor. It was there that he recorded many observations that later led to his work in heat diffusion. Fourier returned to Paris in 1801. Soon Napoleon appointed him as the Prefect of Isére, headquartered at Grenoble. Among his achievements in this administrative position included the draining of swamps of Bourgoin and the construction of a new highway between Grenoble and Turin. Some of his most important scientific contributions came during this period (1802–1814). In 1807 he completed his memoir *On the Propagation of Heat in Solid Bodies* in which he not only expounded his idea about heat conduction, which was later called Fourier's Law, but also outlined his new method of mathematical analysis, which we now call Fourier analysis. This memoir however was never published, because one of its examiner, Lagrange, objected to his use of trigonometric series to express initial temperature.

Fourier was elected to the Académie des Sciences in 1817. In 1822 he published *The Analytical Theory of Heat* [47], ten years after its winning the Institut de France competition of the Grand Prize in Mathematics in 1812. The judges however criticized that he had not proven the completeness of the trigonometric (Fourier) series. The proof will come years later by *Johann Peter Gustav Lejeune Dirichlet* (1805–1859).

F.4 Thomas Young (1773–1829)

Thomas Young was born to a Quaker family, as the eldest of ten children. At the age of fourteen, Young not only had learned Greek and Latin, but also was acquainted with a dozen modern languages. From 1787–1792, he earned his living by working as tutor for a rich family. In 1792, Young began to study medicine, first in London, and then in Edinburgh. In 1796, he obtained a Doctor degree in medicine from Göttingen. He started to practice as a physician in London in 1799. Young published many of his first academic articles anonymously to protect his reputation as a physician.

In 1801 Young was appointed professor of natural philosophy at the Royal Institution. In two years he delivered 91 lectures. These lectures were published in 1807 [87], which covered a broad range of subjects and contained a number of anticipations of later theories, such as Maxwell's equation and Einstein's special relativity. In 1802, he was appointed foreign secretary of the Royal Society, of which he had been elected a fellow in 1794. He resigned his professorship in 1803, fearing that its duties would interfere with his medical practice. Thomas Young died in London on 10 May 1829 [72].

Like Leonardo da Vinci and Robert Hooke, Thomas Young was a renaissance man, whose expertise spans a significant number of different subject areas [74]. He had made notable scientific contributions to the fields of vision, light, solid mechanics, energy, physiology, language, musical harmony and Egyptology. In Young's own judgment, of his many achievements the most important was to establish the wave theory of light, contradicting the particle theory of Newton. Through theory and experiment, he had investigated the reflection, refraction, and interference of visible light, and measured its wavelengths.

Young has also been called the founder of physiological optics. In his lectures [87] he presented the hypothesis that color perception depends on the presence in the retina of three kinds of nerve fibers which respond respectively to red, green and violet light. Young developed the theory of capillary phenomena on the principle of surface tension, which was further developed by Laplace to become the Young-Laplace equation. In physiology Young made an important contribution to hemodynamics. In an *Encyclopedia Britannica* article "Languages", Young compared the grammar and vocabulary of 400 languages. In a separate work in 1813, he introduced the term Indo-European languages. Young was also one of the first who deciphered Egyptian hieroglyphs. In 1814 Young had translated the demotic text of the Rosetta Stone, an ancient Egyptian granodiorite stele inscribed with a decree issued at Memphis, Egypt, in 196 BC.

In his lectures [87], Young described the deformation of prismatic bars, and introduced the notion of modulus of elasticity (Young's modulus), as the proportionality constant between the stress and the strain. Prior to Young's contribution, engineers who applies Hooke's law in the form of $F = kx$, to identify the deformation x of a body subject to a force F, relied on a constant k, which is dependent on both the geometry (the cross-sectional area and the length of prismatic bar) and material under consideration. Young's modulus depends only on the material. Young observed similar elastic deformation for shear force and torsion. Describing the experiments of tension and compression of bars, Young drew the attention to the fact that longitudinal deformation is always accompanied by some change in the lateral dimensions (Poisson's ratio). Introducing Hooke's law, he observed that after certain limit, the deformation became inelastic, and resulted in permanent set. He also presented results of deflection of beams, and buckling, and looked at the fracture created by impact force [80].

F.5 Claude Louis Marie Henri Navier (1785–1836)

Navier's father was a lawyer, who was a member of the National Assembly in Paris during the time of the French Revolution. He died when Navier was only eight years old. Navier was left to the care of his granduncle Emiland Gauthey, who was considered the leading civil engineer in France at that time. In 1802, Navier passed the competitive entrance exam to enter École Polytechnique, where he was taught analysis by Jean Baptiste Joseph Fourier (1768–1830). In 1804 Navier entered the École des Ponts et Chaussées (School of Bridges and Highways) and graduated as one of the top students.

It was not long after Navier's graduation that his granduncle died, and Navier was asked to take on the task of editing Gauthey's works on bridges and channels, which was published in three volumes. To bring the work up to date, Navier added editorial notes in many places, which provided the theoretical mechanics basis to the empirical work.

Navier started teaching at the École des Ponts et Chaussées in 1819 and took charge of the applied mechanics courses. In 1824, he was elected a member of Académie des Sciences, and became professor of calculus and mechanics in 1830. For his engineering contributions, He was known as the first to introduce theoretical analysis of suspension bridges. His report "*Memorandum of Suspension Bridges*" [69] was for a long time the most important book for the design of suspension bridges [73, 80].

Navier was the first to investigate the general equations of equilibrium and vibrations of elastic solids and present them in the differential equation form [68]. He set out from the hypothesis that the elastic reactions arise from variations in the intermolecular forces. He assumed that the force between two molecules, whose distance is slightly increased, is proportional to the product of the increase in the distance and some function of the initial distance. His method consists in forming an expression for the component in any direction of all the forces, which act upon a displaced molecule, and thence the equations of motion of the molecule. The equations are thus obtained in terms of the displacements of the molecule. The solid is assumed to be isotropic, and the equation obtained contains a single elastic constant in this form [67, 80]:

$$C\left(\nabla^2\vec{u} + 2\nabla e\right) + \vec{F} = 0 \tag{F.1}$$

Navier also developed the equation of longitudinal vibration. This equation however only predicts the longitudinal wave. It takes a second elastic constant in order to support a transverse wave. The correct form of the above equation, known as the Navier-Cauchy equation, containing two elastic constants, rather than one, was later obtained by Cauchy (see Sect. F.6).[1]

Navier's lecture note at École des Ponts et Chaussées [70], first published in 1826, made a great advancement in elasticity. The state-of-the-art of mechanic of materials until that time focused on the ultimate strength (failure) of materials and structures. Navier stressed the importance of the *elastic* analysis, which can be applied to functioning structures. In that book, he established the differential equation for the deflection of beams, developed method of analyzing indeterminate structures, presented theory of thin shells, and treated retaining walls, arches, plates, and trusses. However, the role of shear stress and shear strength was not well understood at that time, not until after the death of Navier [80].

Navier also derived the Navier-Stokes equations of fluid dynamics, although he did not fully understand the physics behind it, that is, he did not recognizing the role of shear stress in a fluid. Later George Gabriel Stokes (1819–1903) looked into the internal friction in fluids in motion and deduced the same equation of motion that carries both of their names. Navier was also the first to investigate the statically indeterminate system by examining the bending of a beam resting on supports [70].

F.6 Augustin-Louis Cauchy (1789–1857)

Cauchy was born in Paris during the difficult time of French Revolution. Cauchy's father was active in the education of young Augustin-Louis. Laplace and Lagrange were frequent visitors at the Cauchy family home, and Lagrange particularly took interest in Cauchy's mathematical ability.

In 1807 Cauchy entered École des Ponts et Chaussées to study engineering. At the age of 20, he was appointed as a Junior Engineer to work on the construction of Port Napoléon in Cherbourg. In 1812, he became ill and decided to return to Paris to seek a teaching position. His initial attempts in seeking academic appointment were

[1] In modern day Molecular Dynamics simulation, it requires not only a potential that is a function of distance between two atoms, which mimics the attractive and repulsive van der Waals forces, but also a potential that is a function of the angle between two molecular bonds, in order to create the elastic effects of both a bulk modulus and a shear modulus. Navier's assumption was equivalent to the former, but not the latter.

unsuccessful, although he continued to publish important pieces of mathematical work. In 1814 he published the memoir on definite integrals that later became the basis of his theory of complex functions. In 1815 Cauchy was appointed Assistant Professor of Analysis at the École Polytechnique. In 1816 he won the Grand Prix of the Académie des Sciences for a work on waves, and was later admitted to the Académie. In 1817, he was appointed Chair of Mathematical Physics at the Collège de France.

Cauchy was staunchly Catholic and was politically a royalist. By 1830 the political events in Paris forced him to leave Paris for Switzerland. He soon lost all his positions in Paris. In 1831 Cauchy went to Turin and later accepted an offer to become a Chair of Theoretical Physics. In 1833 Cauchy went from Turin to Prague, and returned to Paris in 1838. He regained his position at the Académie but not his teaching positions because he had refused to take an oath of allegiance to the new regime. Due to his political and religious views, he continued to have difficulty in getting appointment.

Cauchy was probably next to Leonhard Euler (1707–1783) the most published author in mathematics, having produced five textbooks and over 800 articles. Cauchy and his contemporary Carl Friedrich Gauss (1777–1855) were credited for introducing rigor into modern mathematics. The formulation of elementary calculus in modern textbooks is essentially what Cauchy expounded in his three great treatises [34–36]. Cauchy was also credited for setting the mathematical foundation for complex variable and elasticity.

For elasticity, Cauchy first tried Navier's molecular hypothesis, and then abandoned it [80]. Considering elastic solid as a continuum, he had proven the theorems (1) that the stress at any point can be expressed by means of three principal stresses, on three planes at right angles to each other, and normal to the planes across which they act, and (2) that the strain at any point can be reduced to three principal extensions, of three mutually perpendicular line-elements. He made assumptions that (1) the principal stresses are linear functions of the principal extensions, and (2) in an isotropic solid, the principal planes of stress are normal to the principal axes of extension. These assumptions led to the equations of equilibrium of an isotropic solid with two constants [67], which can be expressed as

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla e + \vec{F} = 0 \tag{F.2}$$

known as the Navier-Cauchy equation.

F.7 George Green (1793–1841)

George Green was virtually unknown as a physicist-mathematician during his lifetime. His most influential work, known as Green's theorem, was discovered posthumously by William Thomson (Lord Kelvin).

As the son of a semi-literate, but well-to-do Nottingham baker and miller, Green was sent to a private academy at the age of eight, and left school at nine. This was the only formal education that he received until adulthood. For the next twenty years after leaving primary school, no one knew how, and from whom, Green could have acquainted himself to the advanced mathematics of his day in a backwater place like Nottingham. Even the whole country of England in those days was scientifically depressed as compared to the continental Europe. Hence it was a mystery how Green could have produced as his first publication such a masterpiece without any guidance [32, 33, 57].

The next time there existed a record about Green was in 1823. At the age of thirty, he joined the Nottingham Subscription Library as a subscriber. In the library he had some access to books and journals. The next five years was not easy for Green; he had to work full time in the mill, had two daughters born (he had seven children with Jane Smith, but never married her, due to social barrier), and his mother died in 1825. Despite these difficulties in life and his flimsy mathematical background, in 1828 he produced one of the most important mathematical works of all times—*An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* [58]. In the introduction section, Green stated:

> The object of this Essay is to submit to Mathematical Analysis the phenomena of the equilibrium of the Electric and Magnetic Fluids, and to lay down some general principles equally applicable to perfect and imperfect conductors; but, before entering upon the calculus, it may not be amiss to give a general idea of the method that has enabled us to arrive at results, remarkable for their simplicity and generality, which it would be very difficult if not impossible to demonstrate in the ordinary way.

In the "method that has enabled us to arrive at results", Green introduced the concept of electric and magnetic potentials, whose derivatives produce the physical quantity of force, and the phrase *potential function*. He derived the three Green's identities, as shown in the modern notations below:

$$\int\int\int \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right)\, dV = \int\int \phi\frac{\partial\psi}{\partial n}\, dS \tag{F.3}$$

$$\int\int\int \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\, dV = \int\int \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\, dS \tag{F.4}$$

$$\phi = \frac{1}{4\pi}\int\int \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial\,(1/r)}{\partial n}\right]\, dS \tag{F.5}$$

where dS denotes a surface integral, and dV a volume integral. Particularly, the third identity (F.5) gives the general solution of the so-called *Dirichlet problem*, namely the solution of Laplace equation with Dirichlet-type boundary condition. The singular solution of a partial differential equation, such as the $1/r$ in (F.5), has been called *Green's function*.

As Green did not feel that he had the proper educational credential and social status to submit the Essay to the customary publication channels at that time, the journals of the *Royal Society* and the *Cambridge Philosophical Society*, he privately published the Essay through a Nottingham publisher. The essay had 51 subscribers, who were mostly local doctors, schoolmasters, clergymen, and owners of local factories, and could hardly understand a word of it. Green was disappointed by the lack of response. One subscriber, Sir Edward Bromhead, however, was impressed by Green's prowess in mathematics. He offered to refer Green's future papers to the Societies and recommended Green to attend Cambridge University.

Several years later, Green finally enrolled at Caius College of Cambridge University at the age of forty. From 1833 to 1836, Green wrote three more papers, two on electricity published by the Cambridge Philosophical Society, and one on hydrodynamics published by the Royal Society of Edinburgh. After graduating in 1837, he stayed at Cambridge for a few years to work on his own mathematics and to wait for an appointment. From 1838 to 1839 he had two papers in hydrodynamics, two papers on reflection and refraction of light, and two papers on sound [59]. In 1839, he was elected to a Parse Fellowship at Caius College, a junior position. Due to poor health, he had to return to Nottingham in 1840. He died in 1841 at the age of forty-seven. At the time of his death, his Essay, never formally published, was virtually unknown and lost.

At the year of Green's death, William Thomson (Lord Kelvin) was admitted to Cambridge. While studying the subject of electricity as a part of preparation for his Senior Wrangler exam, he first noticed the existence of Green's paper in a footnote of a paper by Robert Murphy. He started to look for a copy, but no one knew about it. After his graduation in 1845, and before his departure to France to enrich his education, he mentioned it to his tutor William Hopkins. It so happened that Hopkins had three copies. Thomson was immediately excited about what he had read in the paper. He brought the article to Paris and showed it to Jacques Charles François Sturm and Joseph Liouville, leading mathematicians at that time. Later Thomson republished Green's essay, rescuing it from sinking into permanent obscurity.

Green's 1828 essay had profoundly influenced Thomson and James Clerk Maxwell in their study of electrodynamics and magnetism, and was considered the founder of the 'Cambridge School' of natural philosophers, of which Thomson, Stokes, Rayleigh, and Maxwell were the most illustrious members in the later half of the nineteenth century [86]. The methodology has also been applied to many other fields of physics such as acoustics, elasticity, and hydrodynamics. During the bicentennial celebration of Green's birth in 1993, physicists Julian Schwinger and Freeman Dyson delivered speeches on the role of Green's functions in the development of 20th century quantum electrodynamics.

F.8 Gabriel Lamé (1795–1870)

Lamé was born in Tours, France. He and his colleague Benoît Paul Émile Clapeyron (1799–1864) both studied at École Polytechnique, and later graduated at École des Mines. In 1820, right after graduation, Lamé and Clapeyron were sent by the French government to Saint Petersburg to teach at the Institut et Corps du

Genie des Voies de Communication (Institute of Corps of Engineers of Ways of Communication), as a part of the plan of Russian Emperor Alexander I to import Western Europe scientific knowledge and its applications to military techniques and industrial development to Russia.

During their 12 year stay in Russia, Lamé and Clapeyron not only taught, but also supervised the construction of bridges and roads. They also wrote the important memoir "*Memorandum on the inner equilibrium of homogeneous solid*" [64]. In this work they demonstrated the equivalence between Navier's molecular and Cauchy's continuum approach in deriving the equilibrium equation, presented the graphical way of representing the state of stress at a point, later known as Lamé's stress ellipsoid, derived the solutions of internally and externally pressurized hollow cylinder and sphere, and simple torsion of a shaft, and for the first time, introduced cylindrical coordinates in their investigation of an infinite cylinder. This work contained all the known theoretical solutions dealing with the deformation of isotropic elastic materials at that time. It also became the basis for the famous book by Lamé in 1852 [63], "*Lectures on the mathematical theory of elasticity of solid bodies*", which is considered the first book on elasticity [80]. In this book, Lamé recognized that Cauchy's two constants version of elasticity is the correct version, over Navier's [81, 82].

In 1832, the relation between the French and Russian governments deteriorated, and Lamé returned to Paris. in the same year Lamé was elected professor of physics at École Polytechnique. Subsequently, he held various engineering posts and professorships in Paris, including a tour as the chief engineer of mines, during which he led the construction of the first two railroads of France, from Paris to Versailles and Paris to St. Germain, and chair of mathematical physics and probability at the Sorbonne. He was elected to the Académie des Sciences in 1843.

In addition to being a prominent engineer, elasticity, Lamé was also an accomplished mathematician. He contributed in differential geometry, curvilinear coordinates, and number theory. He also tackled the Fermat's last theorem with $n = 7$. In fact, Lamé was one of the few mathematicians who were ever praised by Gauss, the greatest mathematician.

F.9 Henry-Philter Gaspard Darcy (1803–1858)

Darcy was born in Dijon, France. His father was a minor civil servant, and died when Darcy was 14. His mother rose to the task of insuring his and his brother's education. She had only a small city pension, but was able to borrow money for tutors and obtained city scholarships for her sons to attend college. In 1821, at the age 18, Darcy entered École Polytechnique, Paris. Two years later Darcy was admitted to École des Ponts et Chaussées. After graduation, he was assigned by the Imperial Corps des Ponts et Chaussées to a position in Jura, and then transferred to Dijon [27, 48].

In 1828, Darcy was assigned to a deep well drilling project that found water but could not provide an adequate supply for the town. Under his own initiative, Henry set out to provide a clean, dependable water supply to the city from more conventional surface water sources. The effort eventually produced a system that delivered 8 m^3/min of water from the Rosolic Spring through 12.7 km of covered aqueduct to an enclosed 5,700 m^3 reservoir located near the Porte Guillaume and another reservoir at Montmusard. Pressurized distribution lines totaling 28,000 m were laid in underground galleries and provided water to major buildings and 142 public street hydrants spaced 100 m apart throughout the city. The entire system was enclosed and gravity driven, thus it required no pumps or filters [29]. For this work, Darcy was awarded the Legion of Honor.

By 1848 Darcy was Chief Engineer for the Department of Côte-d'Or, but at the beginning of the Second Republic, due to political pressures, he was forced to leave Dijon. He was however soon appointed to the higher position of Chief Director for Water and Pavements, Paris. In 1855 due to failing health, he resigned from normal duties but was allowed to return to Dijon to carry out research of his own interest. In 1856 Darcy published "*Les Fontaines Publiques de la Ville de Dijon*" [39], with the famous Appendix D, in which he conducted sand column experiments for water filtering that established Darcy's law. In 1857, Darcy published "*Recherches Expérimentales Relatives au Mouvement de l'eau Dans les Tuyaux*" [40]. This presented his research on pipe flow resistance and was the first work to suggest the existence of the boundary layer in fluid flow. This publication provides the basis for his credit in the Darcy-Weisbach equation, even though Weisbach published before him in 1845. Darcy died of pneumonia in 1858 at age 54 in Paris.

F.10 Achille Ernest Oscar Joseph Delesse (1817–1881)

Delesse was a French geologist and mineralogist. Born in Metz in 1817, he was orphaned at young age and was raised by his two aunts. He was admitted to École Polytechnique at age 20, and graduated first in class two years after. As a student in mining engineering, he received the assignment of cataloging a collection of geological maps, which initiated his life interest in mineralogy. He entered École des Mines for his doctoral work, and conducted his studies by traveling to England, Germany, Poland, and Ireland. His early work on rocks earned him an appointment as Professor of Mineralogy and Geology at the University of Besançon in 1845. At the same time, he worked as a mining engineer for the city. He returned to Paris as a substitute teacher of geology at the Sorbonne and was appointed Associate Professor of Geology at the École Normale Superieure. He was also appointed Inspector of Quarries of the Seine [44].

Delesse's career had a brilliant start. However, it became stagnant during the Second Empire for political reasons. Delesse was a staunch Republican and he did not hide it. In 1864 he lost his appointment as Inspector. To compensate for the termination, he was called to Ecole des Mines de Paris where he created a course in agriculture, drainage and irrigation. After the fall of the Empire, the Third Republic accelerated his career. Delesse finally became Chief Engineer of Mines in 1870, at age of 53, which is very late for a brilliant graduate of the Polytechnic. He became Inspector General of Mines in 1878, in charge of the southeast of France. In 1875 he was the President of the First International Congress of Geography. In 1877 he was appointed Professor of Mineralogy and Geology at the National Agronomic Institute. In 1879, his career culminated in the election to the Académie des Sciences. Unfortunately, it was not long that he could enjoy this honor. Saddened by the loss of his father, a fatigue of overtook him. His health rapidly declined, and he passed away in 1881, aged 64 [28].

Delesse was an excellent chemist. He analyzes thousands of rocks, soils, and water and cataloged them with infinite patience. He made a fundamental contribution to the understanding of metamorphism through the petrography of igneous and metamorphic rocks, especially microcrystalline. His book on the lithology of the seabed [45] is still regarded as the starting point of modern oceanography [28]. Delesse made fundamental scientific impact in many other areas. For example, his research on the nitrogen content of the rocks is considered the beginning of the organic geochemistry. Delesse was among the first to practice the method of mapping at various scales, including at national level and in three different disciplines such as geology, agronomy and hydrogeology. In addition, he had introduced the economic factor, such as the cash income of the land and contour of underground deposits.

For the purpose of this book, his argument that the volume fraction of a mineral in a rock can be determined from its surface fraction, known as Delesse's law, played an important role in the volume averaging theory of porous medium presented in Chap. 4.

F.11 Josiah Willard Gibbs (1839–1903)

Josiah Willard Gibbs was an American theoretical physicist, chemist, and mathematician. He was born in New Haven, Connecticut, in 1839, and died in the same city, in 1903 [31]. Quoting from the science writer Bryson [30], "*Gibbs is perhaps the most brilliant person that most people have never heard of. Modest to the point of near invisibility, he passed virtually the whole of his life, apart from three years spent studying in Europe, within a three-block area bounded by his house and the Yale campus in New Haven, Connecticut.*"

In 1863, Gibbs received the first Ph.D. degree in engineering granted in the U.S., for a thesis entitled "On the form of the teeth of wheels in spur gearing," in which he used geometrical techniques to investigate the optimum design for gears. This was also only the fifth Ph.D. granted in the US in any subject. In 1866, Gibbs embarked on his trip to Europe, which he spent in Paris, Berlin and Heidelberg. In the universities, he attended lectures by some of the most prominent scientists at that time, such as mathematicians Joseph Liouville, Karl Weierstrass and Leopold

Kronecker, Chemists Gustav Magnus and Robert Bunsen, and physicists Gustav Kirchhoff and Hermann von Helmholtz. As the American science community was in the backwater in those days, this only trip that Gibbs ventured outside the small town in Connecticut transformed his scope in science.

After returning from Europe, Gibbs was appointed Professor of Mathematical Physics, the first such professorship in the United States and a position he held for the rest of his life. During the first ten years, Yale did not even pay him a salary. (As the only son from his parents, he inherited a modest fortune when he was young.) While at Yale, his courses attracted an average of slightly over one student a semester. His written work was difficult to follow and employed a private form of notation that many found incomprehensible. But buried among his arcane formulations were insights of the loftiest brilliance.

Rather surprisingly his appointment to the chair at Yale came before he had published any work. Perhaps it is also surprising that Gibbs did not publish his first work until 1873 when he was 34 years old. Gibbs's important 1873 papers were "*Graphical methods in the thermodynamics of fluids*" [51] and "*A method of geometrical representation of the thermodynamic properties of substances by means of surfaces*" [52]. In 1876 Gibbs published the first part of the work, for which he is most famous, "*On the equilibrium of heterogeneous substances*," publishing the second part of this work in 1878 [53]. In this series of papers, he elucidated the thermodynamic principles of *nearly everything* [30], "on gases, mixtures, surfaces, solids, phase changes, chemical reactions, electrochemical cells, sedimentation, and osmosis" [38]. Gibbs's Equilibrium has been called "the Principia of thermodynamics," but Gibbs chose to publish these landmark observations in the *Transactions of the Connecticut Academy of Arts and Sciences*, a journal that managed to be obscure even in Connecticut, which is why both Max Planck and Einstein had repeated Gibbs's work in their early part of career without knowing its existence.

Actually Gibbs's work did not go totally unnoticed. His work had impressed James Clerk Maxwell so much that he constructed a three dimensional model of Gibbs's thermodynamic surface and sent the model to Gibbs. Unfortunately, the prospects of collaboration between Maxwell and Gibbs were cut short by Maxwell's early death in 1879, aged 48. The joke later circulated in New Haven that "only one man lived who could understand Gibbs's papers, that was Maxwell, and now he is dead" [75].

Together with Maxwell and Ludwig Boltzmann, Gibbs is considered as a founder of *statistical mechanics* [54]. He coined the term to identify the branch of theoretical physics that accounts for the observed thermodynamic properties of systems in terms of the statistics of large ensembles of particles .

Gibbs's work on vector analysis was also of major importance in mathematics. British scientists, including Maxwell, had relied on William Rowan Hamilton's quaternions in order to express the dynamics of physical quantities. He proposed defining distinct dot and cross products for pairs of vectors and introduced the now common notation for them. He was also largely responsible for the development of the vector calculus techniques still used today in electrodynamics and fluid mechanics.

F.12 Karl von Terzaghi (1883–1963)

Karl von Terzaghi, generally known as the father of soil mechanics, was born in 1883 as the first child of Army Lieutenant-Colonel Anton von Terzaghi and Amalia Eberle in Prague. Upon Anton's retirement from the army, the family moved to Graz, Austria. Anton passed away when Karl was 7 years old, and his maternal grandfather Karl Eberle became the head of the Terzaghi household. Karl Terzaghi was strongly influenced in his character development by his grandfather, who was a graduate of the Vienna Technical University, a successful mechanical engineer, and later in his life, an organizer of international banking.

From age 10 to 17, Terzaghi attended military boarding schools. Although his family anticipated that he pursue a military officer career, Terzaghi however discovered that he much preferred science to "mindless soldier games". In 1900, Terzaghi entered the Technical University in Graz to study mechanical engineering. At the university he very quickly lost interest in the lectures of mechanical engineering, which were purely professional. He rarely attended classes; rather he took the advantage of the various science and humanistic courses offered in various other faculties of the university, and pursued with broad interest in philosophy, experimental psychology, art history, plant physiology, astronomy, and especially geology [55].

At one of the university dinner gathering Terzaghi found himself sat next to Ferdinard Wittenbauer, his teacher of applied mechanics. When questioned why he did not attend classes, Terzaghi responded that he would rather read the "prescriptions" for himself. Wittenbauer challenged Terzaghi to read the original text of Lagrange's *Analytical Mechanics* in French. The high pace of developing ideas of Lagrange's mind offered Terzaghi for the first time the "unforgettable revelation" of "science in the making". Wittenbauer became Terzaghi's mentor and a "fatherly friend and confessor" [55].

Terzaghi graduated from Graz with honors in 1904. After fulfilling a compulsory one year military service, he returned to the university for one year and combined the study of geology with courses on subjects such as highway and railway engineering. His first job was as a junior design engineer for the firm Adol Baron Pittle, Vienna. The firm was becoming more involved in the relatively new field of hydroelectric power generation, and Karl became involved in the geological problems the firm faced. By 1908, he was managing a construction site, workers, and the design and construction of steel reinforced structures. He embarked on ambitious and challenging project to construct a hydroelectric dam in Croatia and St. Petersburg. During six months in Russia, he developed some novel graphical methods for the design of industrial tanks, which he submitted as a thesis for his PhD at the university. His growing list of achievements began to open more opportunity to him. He resolved to go to the United States of America, which he did in 1912. In the US, on his own, he undertook an engineering tour of major dam construction sites in the West.

When World War I broke out, Terzaghi found himself drafted into the army as an officer directing a 250 man engineering battalion, which further increased to leading 1000 men. After a short stint managing an airfield, he became a professor in the Royal Ottoman College of Engineering in Istanbul (now Istanbul Technical University). There he began a very productive period, in which he began his lifelong work of bringing true engineering understanding to soil as an engineering material whose properties could be measured in standardized ways. He set up a laboratory using only the most rudimentary of equipment, and began his revolution. His measurements and analysis of the force on retaining walls were first published in English in 1919, and was quickly recognized as an important new contribution to the scientific understanding of the fundamental behavior of soils.

At the end of the war, he was forced to resign his post at Ottoman College, but found a new post at Robert College in Istanbul. There he again constructed a laboratory out of the simplest equipment. This time he studied various experimental and quantitative aspects of the permeability of soils to water and was able to work out some theories to explain the observations. In 1924 he published much of his research in *Erdbaumechanik* [77], which revolutionized the field to great acclaim. It resulted in a job offer from the Massachusetts Institute of Technology (MIT), which he immediately accepted.

One of his first tasks in the USA was to bring his work to the attention of engineers. This he proceeded to do by writing a series of articles for the *Engineering News-Record* [78]. He once more set up a new laboratory geared to making measurements on soils with instruments of his own devising. He entered a new phase of prolific publication and a rapidly growing and lucrative involvement as an engineering consultant on many large-scale projects.

In 1928, Terzaghi returned to Europe and accepted a chair at the Vienna Technical University. His teaching workload was relatively light, so he continued his experimental investigations, and became especially interested in the problems of the settling of foundations, and of grouting. In 1936, Terzaghi's pioneer work was publicly recognized when the first International Conference on Soil Mechanics was held at Harvard University in his honor.

The political turmoil in Austria began to interfere with his work, and in 1938 Terzaghi emigrated to the United States and took up a post at Harvard University. Before the end of the war, he consulted on the Chicago Subway system, and the Newport News shipways construction, among others. He remained as a part-timer at Harvard University until his retirement in 1953 at the mandatory age of 70. He died in 1963.

Goodman [56] commented: "Among Terzaghi's publications, reports and lectures, one finds seminal contributions across a wide terrain, including: classification methods for soils and rocks; capillary phenomena in soils; the theory and documentation of consolidation and settlement; piping and its prevention; design and construction of earth, rock and concrete dams on all kinds of foundations; anchorages for suspension bridges in soils; field and laboratory measurement of pore pressures and soil properties; use of flow nets in two and three dimensions; design of drainage wells and tunnels; design to avoid scour of river and waterfront structures;

earth pressure variations on walls and bulkheads; engineering in terrain underlain by permafrost; pile foundations; soil improvement by compaction, pile-driving, grouting and incorporation of geotextiles; soil and rock tunneling; engineering geology; sinkhole formation and collapse; regional subsidence due to oil-field operations; and landslides."

F.13 Yacov Il'ich Frenkel (1894–1952)

Yakov (Jacob) Frenkel was born on February 10, 1894 in the southern Russian city Rostov-on-Don. Since his early years he showed a talent in music, fine arts, and science. Being a student in the May Gymnasium at St. Petersburg, Frenkel wrote a 300-page exposition paper on the origin of the Earth's magnetism and atmospheric electricity. The paper was shown to Abram Fedorovich Ioffe, then a professor at St. Petersburg Polytechnic Institute, and the two met for the first time. In 1913 Frenkel graduated from gymnasium with the highest honor (Gold Medal).

Worrying that he might not be admitted to the university (in Tsar's time, admission quota existed for persons of the Jewish religion), he briefly toured the US for the opportunity in North American universities. Fortunately, he was accepted by the Physics and Mechanics Department at the St. Petersburg University, where he completed the six-year course study in three years.

The physics school in St. Petersburg was associated with A. F. Ioffe, who was an excellent scientist, teacher, and scientific manager—he was "Daddy Ioffe" as the first generation of Russian physicists after the Revolution affectionately called him. In 1916, Ioffe organized the first of the seminars that later became the trademark of Russian physics. Frenkel took part in the Ioffe seminar from its very beginning together with Lev Davidovich Landau, Nikolai Nikolayevich Semenov, and Pyotr Leonidovich Kapitsa, all Nobel Prize winners later in their career.

Right after the Revolution, in 1918, Frenkel left St. Petersburg and took part in the organization of Tavrichesky University in Yalta, Crimea. Living conditions in Crimea at that time were terrible. Frenkel was also jailed for two months during that time for political reasons. In 1921 Frenkel returned to St. Petersburg and started to work in St. Petersburg Polytechnic Institute. In the four years between 1922 and 1925, he published five monographs: *The Structure of Matter* in 3 parts, *The Theory of Relativity*, *Electrical Theory of Solid Bodies*, a popular science book *Electricity and Matter*, and a textbook *Course of Vector and Tensor Analyses*. He was merely thirty years old.

In 1925, upon the recommendation of Paul Ehrenfest, Frenkel was awarded a scholarship by the Rockefeller Foundation to visit Western Europe. He spent almost a year in Germany, France, and England. He worked with Max Born in Göttingen and was acquainted with Einstein in Berlin. It was also during this time that he published the first volume of his two-volume book *Electrodynamics*. In 1927 Frenkel was one of the participants in the International Physics Congress in Como, Italy. He presented his work on the quantum theory of electroconductivity of metals, which became an important part of the modern solid state physics. Frenkel in 1927 also calculated the theoretical stiffness of metals from its crystalline structure and showed that the stiffness was orders of magnitude greater than that of real metals. One of the major successes of Frenkel during these years was the explanation of the effect of spontaneous magnetization of ferromagnetic materials and the development of the theory of magnetic domain. In 1926 Frenkel introduced the key idea of defects of crystalline structure, known as Frenkel defects.

In 1929 Frenkel was elected a Corresponding Member of the Academy of Sciences of the USSR, among the first group of young scientists admitted to the Academy after the Revolution. In 1930, Frenkel received an invitation from the University of Minnesota to spend a year in the US as a Visiting Professor. During his visit to the US, Frenkel developed the theory of light absorption in solid bodies. In these papers he introduced the idea and the term "exciton" for the description of the waves of excitation. Frenkel in 1936 introduced the original theory of fission of atomic nuclei. In 1939, several months ahead of Niels Bohr and John Wheeler, Frenkel proposed the so called "drop model" of the atomic nucleus.

Geophysics was a field of Frenkel's early interest, and in 1944, Frenkel returned to it. He frequently visited the Institute of Theoretical Geophysics in Moscow traveling from Kazan'. In the Institute, he became interested in the work of A. G. Ivanov, who had discovered in 1939 that the propagation of seismic waves in soil was accompanied by the appearance of a potential difference between electrode probes inserted into the ground. Ivanov considered the phenomenon as being caused by the pressure difference between two points in wet soil resulting from the propagation of longitudinal waves. Frenkel took a different view. He correctly recognized the need to model the wet soil as a two-phase material. He formulated the continuum mechanical theory; and by mathematical analysis set up the characteristic equations. He then stated [49]: "*We shall not write down the expressions for its roots and shall only remark that … one of them corresponds to waves with a very small damping, and the other to waves with a very large damping*." Hence Frenkel discovered the existence of the second compressional wave. However, Frenkel's interest was in the seismoelectric effect; hence he dismissed the mechanical effect as unimportant by stating: "*The waves of the second kind are thus really non-existent*," because it would not propagate to a large distance. For the electrical effect, Frenkel recognized the presence of electrolytes in liquid, which created electric double layers on the interface of soil particles and the liquid. The second wave, despite its non-propagating nature, allowed the solid and fluid phases to move relative to each other. The relative movement of electrical charges created local electrical currents, whose variation in turn caused the emission of electromagnetic waves.

After the war, Frenkel returned to Leningrad and resumed his teaching and research at the Polytechnic Institute. In 1945, Frenkel was honored with the Labor Red Banner Order. Two years later, his Kinetic Theory of Liquids was awarded the First Grade State Prize. However, at that time there existed the first hint of a change in the socialism policy; and the first gust of cold wind soon reached Frenkel. The ensuing political persecution affected not only Frenkel, but also many other prominent scientists. Frenkel's work was criticized for not contributing to the construction of the society of great socialism. His contributions in quantum mechanics and the theory of relativity were labeled as servility to Western science. Frenkel's work was greatly affected and his health deteriorated toward the end of his life. He died in 1952, not quite 58 years old.

In 1994, the Physico-Technical Institute of the now St. Petersburg Polytechnic State University, where Frenkel worked from 1921 to 1952, celebrated his centennial. Frenkel's memorial plaque now adorns the wall of the main building of the Institute [50, 66].

F.14 Maurice Anthony Biot (1905–1985)

Biot was born in Antwerp, Belgium on May 25th 1905. The war in 1914–1918 and the siege of Antwerp caused the Biot family to travel first to London, then Paris, and finally settling in Chambéry, France. These moves matured the young Biot and exposed him to several languages. Later returning to Antwerp, Biot concluded his secondary school. In 1924 he was admitted at the Université catholique de Louvain (UCL). It was at this time that Biot showed his insatiable appetite for knowledge. While pursuing his studies in Engineering, Biot was awarded a Bachelor degree in Philosophy, and also attended courses in Economics. He obtained a Mining Engineering degree in 1929, and an Electrical Engineering degree in 1930. (See Figs. F.1 and F.2 for photos of Biot at various stages of life.) Biot was awarded a Doctor of Science degree in 1931.

The sponsorship of the Belgian American Educational Foundation allowed Biot to spend the next two years in the US at the California Institute of Technology (Caltech). It was at Caltech where he first met and worked with Theodore von Kármán, who had arrived in the US in 1929. Biot acquired a Ph.D. in Aeronautical Sciences in 1932 by defending his dissertation on "Transient oscillations in elastic system" [3] in which he introduced the concept of response spectrum for earthquake analysis. The methodology brought great simplifications to the analysis of structures under transient loading and has since been used as a tool in earthquake-proof design. It was during the same period that he published his first papers on a new approach to the nonlinear theory of elasticity accounting for the effect of initial stress.

At the beginning of 1934, he obtained a fellowship from the Belgian Scientific Foundation which allowed him to pay short but interesting visits at the universities of Delft, Louvain, Cambridge, Zürich, and Göttingen. The few weeks spent at Göttingen made him aware of the remarkable advances of Germany in the field of Physics and Aeronautics. The atmosphere surrounding the new Nazi regime electrified the German students. Upon his return, Biot saw this as an ill omen but was regarded as a pessimist.

In 1934, Biot started his academic career as a teacher of applied mathematics at Harvard University. In June 1935, he returned to Pasadena as an Advanced Fellow of the Belgian American Educational Foundation. By 1936, Biot was elected to the faculty at his Alma Mater UCL, where he taught Elasticity and Analytical

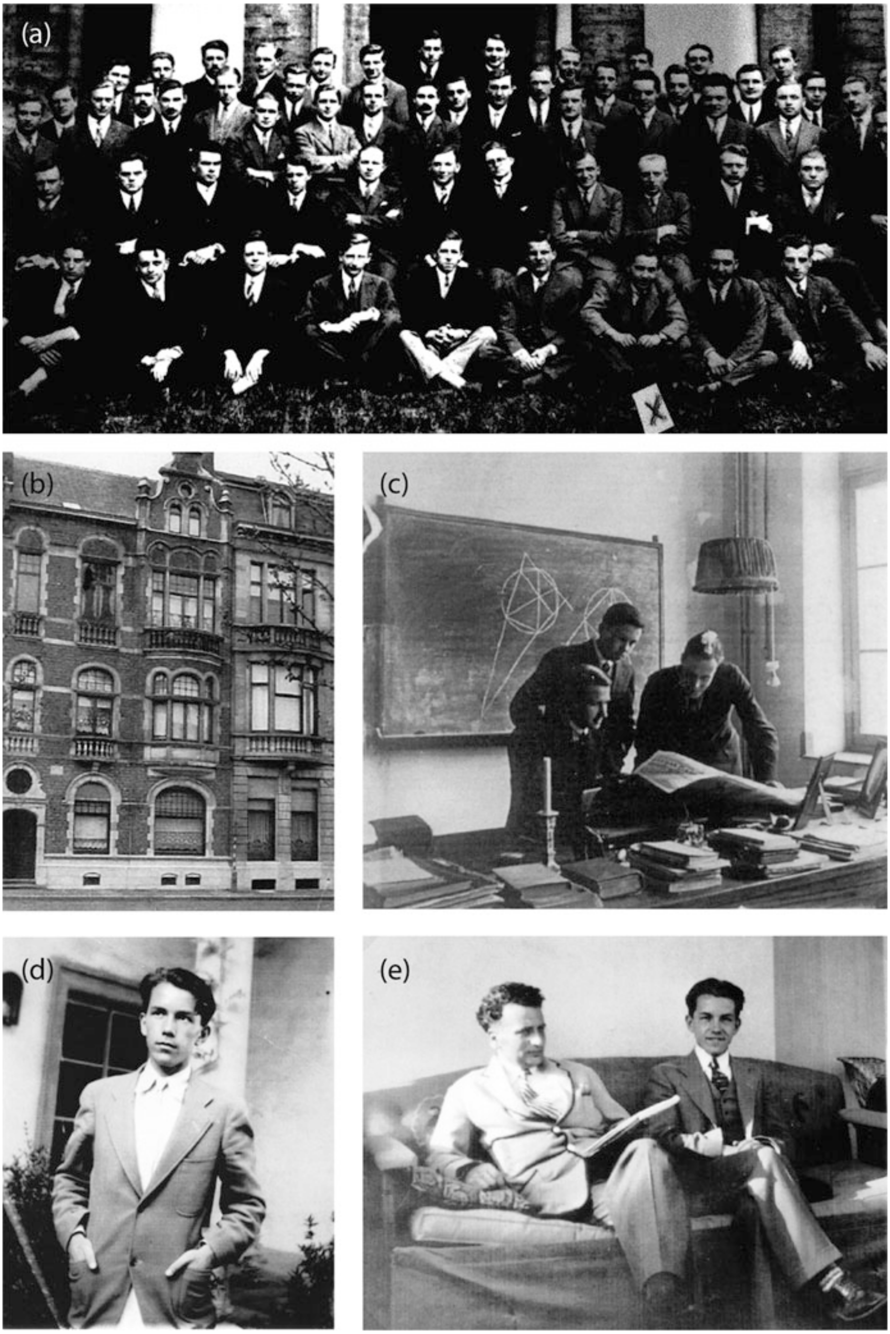

Fig. F.1 (**a**) Biot (marked with X) earning Mining Engineering degree, Louvain (1929); (**b**) Family house, Louvain (1924–1940); (**c**) Science preparatory school in Brussels (standing in the back) (1923); (**d**) Arriving at Caltech at age 26 (1931); and (**e**) At von Kármán's house (1932) (All photos courtesy of Mme Biot)

Fig. F.2 (**a**) Lieutenant Commander, heading the Structural Section of the Bureau of Aeronautics, Washington D.C. (1943); (**b**) Crossing the Rhine as a member of the Combined Allied Field Task (1944); (**c**) London Airport (1944); (**d**) At 300 Central Park West apartment, New York (1964); (**e**) Biot (right) with G.I. Taylor in Madrid IUTAM Symposium (1955); and (**f**) At 117 Avenue Paul Hymans apartment, Bruxelles (All photos courtesy of Mme Biot)

Mechanics. From 1937 to 1946, Biot was an Assistant Professor of Theoretical Mechanics and Physical Mathematics at Columbia University.

It was in 1940 that the monograph *Mathematical Methods in Engineering* [85] was written with von Kármán. Its translation into nine languages is evidence of its influence on several generations of engineers. Later in his career he wrote two more books: *Mechanics of Incremental Deformations* [19] and *Variational Principles in Heat Transfer* [21].

Unfortunately, war was brewing in Europe. Poland was overrun in 1939, and then came Holland and Belgium in 1940. France requested an armistice a few days later. Caltech at that time had undertaken a research contract with the OSRD (Office of Scientific Research and Development). Robert Millikan, President of Caltech, requested that Columbia grant a leave of absence to Biot, so that his special knowledge on vibration problems in high-speed aircraft can be utilized. It was during this time that Biot developed a three-dimensional theory of aircraft flutter.

When US entered the war, Biot decided to enlist. Assigned to the US Navy he was send to Key West where his hydro-mechanical knowledge was needed in anti-submarine warfare. In 1943, as Lieutenant Commander, he headed the Structural Dynamics Section of the Bureau of Aeronautics, US Navy, in Washington, D.C. As member of the CAFT (Combined Allied Field Task) he was in London on D-Day. He crossed the English Channel and joined the US 6th Army as the Allied pushed to the outskirts of Paris. As the US Army advanced in south Germany, Biot's mission was to gather scientific intelligence on aircraft and missile technology. He was responsible for the capture of a number of research institutes complete with files and personnel.

After the war, Biot returned to Columbia. In 1946 Brown University offered him the position of Professor in Applied Physics and Sciences, which he held until 1952. By 1951, Biot had produced a large number of scientific works for Shell Development Co., Cornell Aeronautical Laboratory, and for the US Air Force. After 1952 Biot worked largely alone as a consultant for various governmental agencies and industrial laboratories. From 1969 to 1982, Biot was a consultant for Mobil Research and Development Corporation in Dallas, in the area of Rock Mechanics.

Relocated in Brussels since 1970, Biot continued his research till his last day. It was on one of his last trips to the US that Biot felt the early signs of his illness, which would suddenly deprive him of his life on the 12th of September 1985, at the age of eighty.

Biot's interest in the mechanics of porous media dated back to 1935. Inspired by Terzaghi's 1923 work on the consolidation of clay [76], Biot published his first paper on generalized consolidation theory [6]. This publication was followed by a series of papers in the Journal of Applied Physics, starting with the classical paper entitled "*General theory of three-dimensional consolidation*" in 1941 [8], which established a rigorous, coupled theory for the mechanical response of a porous elastic material in the quasi-static range. The dynamics, or acoustics, theory was established by several papers in the Journal of the Acoustical Society of America [13, 14]. These papers constitute the foundations of what is now referred to as Biot's theory of poroelasticity.

In addition to poroelasticity, the work and original contributions that distinguished Biot's career cover an unusually broad range of science and technology, including applied mechanics, sound, heat, thermodynamics, aeronautics, geophysics, earthquake engineering [84], and electromagnetism.

As early as 1932, working under von Kármán, Biot was the first to analyze the dynamics problem of how to estimate the maximum response of oscillators to transient excitation in his dissertation [3–5]. In the subsequent 10 years [9, 10], the method developed into what is known in present day as the *response spectrum method* in earthquake engineering [83].

Biot's interest in the non-linear effects of initial stress and the inelastic behavior of solids [7, 18, 19] led to his mathematical theory of folding of stratified rocks [15, 17]. He verified his approach in the laboratory and successfully applied it to explain the dominant features of geological structures [26].

Biot also made significant contribution in the irreversible thermodynamics by using the variational approach [12, 25], and was the first to introduce the dissipation function and the minimum dissipation principle in thermodynamics to account for the dissipation phenomenon [11, 16, 21]. The variational principle allowed the new formulation of viscoelasticity [11], heat transfer [20], mass diffusion [23], chemical reaction [22], and thermorheology [24]. Biot's contribution is recognized in a quote by Nowacki in his book *Thermoelasticity* [71]: "*the real development of the theory occurred in the last twenty years. The starting point was the paper by M.A. Biot [12] in which on the basis of the thermodynamics of irreversible processes the basic relations and equations were derived and the variational theorems of thermoelasticity were formulated.*"

The honors that Biot received during his lifetime included the Timoshenko Medal of the American Society of Mechanical Engineers (ASME), the von Kármán Medal of the American Society of Civil Engineers (ASCE), and an Honorary Fellow of the Acoustical Society of America. He was also a member of the US National Academy of Engineering [37, 46].

To honor Biot's pioneering contribution, a Biot Conference on Poromechanics was convened in 1998 [79] in his home country Belgium. It subsequently held once every four years [1, 2, 60, 65] in locations in the US, France, and Austria. The ASCE established a Maurice A. Biot Medal in 2003 to annually award "*an individual who has made outstanding research contributions to the mechanics of porous materials*."

References

1. Abousleiman Y, Cheng AHD, Ulm FJ (eds) (2005) Poromechanics III–Biot centennial (1905–2005). Balkema, Leiden, 828pp
2. Auriault JL, Geindreau C, Royer P, Bloch JF, Boutin C, Lewandowska L (eds) (2002) Poromechanics II. Balkema, Lisse/Exton, 955pp
3. Biot MA (1932) Transient oscillations in elastic system. Doctoral thesis, California Institute of Technology

4. Biot MA (1933) Theory of elastic systems vibrating under transient impulse with an application to earthquake-proof buildings. Proc Natl Acad Sci 19(2):262–268
5. Biot MA (1934) Theory of vibration of buildings during earthquakes. Zeitschrift für Angewandte Mathematik und Mechanik 14(4):213–223
6. Biot MA (1935) Le problème de la consolidation des matières argileuses sous une charge (The problem of consolidation of clay material under load). Annales de la Société Scientifique de Bruxelles, Serie B 55:110–113
7. Biot MA (1939) Non-linear theory of elasticity and the linearized case for a body under initial stress. Philos Mag 27(183):468–489
8. Biot MA (1941) General theory of three-dimensional consolidation. J Appl Phys 12(2):155–164
9. Biot MA (1941) A mechanical analyzer for the prediction of earthquake stresses. Bull Seismol Soc Am 31(2):151–171
10. Biot MA (1942) Analytical and experimental methods in engineering seismology. Proc Am Soc Civil Eng 68:365–409
11. Biot MA (1955) Variational principles in irreversible thermodynamics with application to viscoelasticity. Phys Rev 97(6):1463–1469
12. Biot MA (1956) Thermoelasticity and irreversible thermodynamics. J Appl Phys 27(3):240–253
13. Biot MA (1956) Theory of propagation of elastic waves in a fluid-saturated porous solid.1. Low-frequency range. J Acoust Soc Am 28(2):168–178
14. Biot MA (1956) Theory of propagation of elastic waves in a fluid-saturated porous solid. 2. Higher frequency range. J Acoust Soc Am 28(2):179–191
15. Biot MA (1957) Folding instability of a layered viscoelastic medium under compression. Proc R Soc Lond Ser A – Math Phys Sci 242(1231):444–454
16. Biot MA (1959) New thermomechanical reciprocity relations with application to thermal stress analysis. J Aerosp Sci 26(7):401–408
17. Biot MA (1961) Theory of folding of stratified viscoelastic media and its implications in tectonics and orogenesis. Geol Soc Am Bull 72(11):1595–1620
18. Biot MA (1963) Internal buckling under initial stress in finite elasticity. Proc R Soc Lond Ser A – Math Phys Sci 273(135):306–328
19. Biot MA (1965) Mechanics of incremental deformations. Wiley, New York/London, 504pp
20. Biot MA (1966) Generalized variational principles for convective heat transfer and irreversible thermodynamics. J Math Mech 15(2):177–186
21. Biot MA (1970) Variational principles in heat transfer. Oxford University Press, Oxford, 185pp
22. Biot MA (1977) New fundamental concepts and results in thermodynamics with chemical applications. Chem Phys 22(2):183–198
23. Biot MA (1977) Variational Lagrangian-thermodynamics of nonisothermal finite strain mechanics of porous solids and thermomolecular diffusion. Int J Solids Struct 13(6):579–597
24. Biot MA (1981) Generalized Lagrangian thermodynamics of thermorheology. J Therm Stress 4(3–4):293–320
25. Biot MA (1984) New variational-Lagrangian irreversible thermodynamics with application to viscous flow, reaction-diffusion, and solid mechanics. Adv Appl Mech 24:1–91
26. Biot MA, Odé H, Roever WL (1961) Experimental verification of the theory of folding of stratified viscoelastic media. Geol Soc Am Bull 72(11):1621–1631
27. Biswas AK (1970) History of hydrology. North-Holland, Amsterdam, 336pp
28. Boulain J, Trichet J (1997) Delesse Achille (1817–1881) et ses cartes thématiques (Delesse Achille (1817–1881) and thematic maps), Travaux du Comité Français d'Histoire de la Géologie, Troisième série–T.XI
29. Brown GO (2002) Henry Darcy and the making of a law. Water Resour Res 38(7):12pp
30. Bryson WM (2004) A short history of nearly everything. Broadway Books, New York, 544pp
31. Bumstead HA (1903) Josiah Willard Gibbs. Am J Sci Ser 4 16(4):187–202
32. Cannell DM (1999) George Green: an enigmatic mathematician. Am Math Mon 106(2):136–151

33. Cannell DM (2001) George Green: mathematician and Physicist, 1793–1841. The background to his life and work, 2nd edn. SIAM, Philadelphia, 316pp
34. Cauchy A-L (1821) Cours d'analyse de l'École royale polytechnique. 1. analyse algébrique (Course of analysis at Royal Polytechnic School. 1. Algebraic Analysis). Imprimerie Royale, Paris, 576pp
35. Cauchy A-L (1823) Résumé des leçons données à l'École Royale Polytechnique, sur le calcul infinitésimal (Lecture note at the Royal Polytechnic School, the calculus). Imprimerie Royale, Paris, 172pp
36. Cauchy A-L (1829) Leçon sur le calcul différentiel (Lesson on calculus). Imprimerie de Béthune, Paris, 289pp
37. Cheng AHD, Detournay E, Abousleiman Y (1998) Poroelasticity—Maurice A. Biot memorial issue: preface. Int J Solids Struct 35(34–35):4513–4516
38. Cropper WH (2001) Great physicists: the life and times of leading physicists from Galileo to Hawking. Oxford University Press, 500pp
39. Darcy H (1856) Les Fontaines Publiques de la Ville de Dijon (The public fountains of the city of Dijon). Dalmont, Paris, 647pp
40. Darcy H (1857) Recherches Expérimentales Relatives au Mouvement de l'eau Dans les Tuyaux (Experimental research relating to the movement of water in pipes). Mallett-Bachelier, Paris, 268pp
41. da Vinci L (1506–1510) Codex Leicester, (unpublished manuscript).
42. da Vinci L (1970) The notebooks of Leonardo da Vinci, vol 1 (Trans Richter JP, 1888). Dover, New York, 396pp
43. da Vinci L (1970) The notebooks of Leonardo da Vinci, vol 2 (Trans Richter JP, 1888). Dover, 514pp
44. de Lapparent A (1897) Achille Ernest Oscar Joseph Delesse (1817–1881). In Centenary book (Ecole Polytechnique), vol 1. Gauthier-Villars, Paris
45. Delesse A (1872) Carte Lithologique des Mers de France et des Mers Principales de Globe (Lithological map of the seas of France and the main seas of the globe). Lacroix, Paris, 479pp
46. Delmer A, Jaumotte A (1990) Note on Maurice Anthony Biot. Académie Royale de Belgique, Brussels
47. Fourier JBJ (1822) Théorie Analytique de la Chaleur (The analytical theory of heat), Chez Firmin Didot. Père et Fils, Paris
48. Freeze RA (1994) Henry Darcy and the fountain of Djion. Ground Water 35(1):23–30
49. Frenkel J (1944) On the theory of seismic and seismoelectric phenomena in moist soil. J Phys USSR 13(4):230–241
50. Frenkel VY (1996) Yakov Il'ich Frenkel, his work, life and letters. Birkhäuser, Basel, 323pp
51. Gibbs JW (1873) Graphical methods in the thermodynamics of fluids. Trans Conn Acad Arts Sci 2:309–342
52. Gibbs JW (1873) A method of geometrical representation of the thermodynamic properties of substances by means of surfaces. Trans Conn Acad Arts Sci 2:382–404
53. Gibbs JW (1876, 1878) On the equilibrium of heterogeneous substances. Trans Conn Acad Arts Sci 3:108–248, 343–524
54. Gibbs JW (1902) Elementary principles in statistical mechanics, developed with especial reference to the rational foundation of thermodynamics. Charles Scribner's Sons, New York/Edward Arnold, London, 207pp
55. Goodman RE (1999) Karl Terzaghi: the engineer as artist. ASCE, Reston, 340pp
56. Goodman RE (2002) Karl Terzaghi's legacy in geotechnical engineering. Geo-Strata, GeoInstitute, ASCE
57. Grattan-Guinness I (1995) Why did George Green write his essay of 1828 on electricity and magnetism? Am Math Mon 102(5):387–396
58. Green G (1828) An essay on the application of mathematical analysis to the theories of electricity and magnetism. T. Wheelhouse, Nottingham
59. Green G (1871) Mathematical papers of George Green. N.M. Ferrers (Reprinted by Chelsea, New York, 1970)

60. Hellmich C, Pichler B, Adam D (eds) (2013) Poromechanics V. ASCE, Reston, 2565pp
61. Hooke R (1678) De Potentia Restitutivâ, Or of spring, explaining the power of springy bodies. Martyn, London, 56pp
62. Inwood S (2003) The forgotten genius: the biography of Robert Hooke 1635–1703. McAdam/Cage, San Francisco, 482pp
63. Lamé G (1852) Leçons sur la théorie mathématique de l'élasticité des corps solides (Lectures on the mathematical theory of elasticity of solid bodies). Bachelier, Paris, 335pp
64. Lamé G, Clapeyron BPE (1833) Mémoire sur l'équilibre intérieur des corps solides homogènes (Memorandum on the inner equilibrium of homogeneous solid). *Mémoires présentés par divers savans* 4:465–562
65. Ling HI, Smyth A, Betti R (eds) (2009) Poromechanics IV. DEStech Publications, Lancaster, 1151pp
66. Lopatnikov SL, Cheng AHD (2005) If you ask a physicist from any country: a tribute to Yacov Il'ich Frenkel. J Eng Mech ASCE 131(9):875–878
67. Love AEH (1892) A treatise on the mathematical theory of elasticity, vol 1. Cambridge University Press, Cambridge , 354pp
68. Navier C-L (1821) Memoire sur les lois de l'équilibre et du mouvement des corps solides élastiques (Memorandum on the laws of equilibrium and motion of elastic solids). Mem Acad Sci Inst Fr 2:375–393
69. Navier C-L (1823) Rapport à Monsieur Becquey, …Directeur Général des Ponts; …et Mémoire sur les Ponts Suspendus (Report to Mr. Becquey …Director General of Bridges, …Memorandum of Suspension Bridges).
70. Navier C-L (1864) Résumé des leçons données à l'École des ponts et chaussées sur l'application de la mécanique à l'établissement des constructions et des machines (Lecture note at the school of bridges and highways on the application of mechanics to the establishment of structures and machines), 3rd edn. Dunod, Paris, 851pp
71. Nowacki W (1986) Thermoelasticity, 2nd edn. Pergamon, Oxford/New York, 566pp
72. Peacock G (1855) Life of Thomas Young, M.D., F.R.S., &c., and one of the eight foreign associates of the national institute of France. John Murray, London, 514pp
73. Picon A (1988) Navier and the introduction of suspension bridges in France. Constr Hist 4:21–34
74. Robinson A (2006) The last man who knew everything: Thomas Young, the anonymous genius who proved Newton wrong and deciphered the Rosetta Stone, among other surprising feats. Plume, New York, 304pp
75. Rukeyser M (1988) Willard Gibbs. Ox Bow Press, Woodbridge, 465pp
76. Terzaghi K (1923) Die Berechnung der Durchlässigkeitsziffer des Tones aus dem Verlauf der hydrodynamischen Spannungserscheinungen (A method of calculating the permeability of clay from the history of hydrodynamic stress variation). Sitzungsber. d. Akad. d. Wiss., Wien Math.-Naturwiss. Kl., Abt. IIa 132(3/4):125–138
77. Terzaghi K (1925) Erdbaumechanik auf Bodenphysikalischer Grundlage (Soil mechanics on soil physical basis). F. Deuticke, Leipzig u. Wien, 399pp
78. Terzaghi K (1926), Principles of soil mechanics: I-phenomena of cohesion of clay, II-compressive strength of clay, III-determination of permeability of clay, IV-settlement and consolidation of clay, V-physical differences between sand and clay, VI-elastic behavior of sand and clay, VII-friction in soil and in clay, VIII-future development and problems. Eng News-Record 95(19–23, 25–27):742–746, 796–800, 832–836, 874–878, 912–915, 987–990, 1026–1029, 1064–1068
79. Thimus JF, Abousleiman Y, Cheng AHD, Coussy O, Detournay E (eds) (1998) Poromechanics–a tribute to Maurice A. Biot. Balkema, Rotterdam/Brookfield, 648pp
80. Timoshenko SP (1983) History of strength of materials. Dover, 452pp
81. Todhunter I (1886) A history of the theory of elasticity and of the strength of materials from Galilei to the present time, vol I: Galilei to Saint-Venant (1639–1850). Cambridge University Press, Cambridge, 924pp

82. Todhunter I (1893) A history of the theory of elasticity and of the strength of materials from Galilei to the present time, vol II: Saint-Venant to Lord Kelvin. Cambridge University Press, Cambridge, 762pp
83. Trifunac MD (2006) Biot response spectrum. Soil Dyn Earthq Eng 26(6–7):491–500
84. Trifunac MD (2011) Maurice Anthony Biot: a pioneer of earthquake engineering. Ingegneria Sismica 38(1):7–20
85. von Kármán T, Biot MA (1940) Mathematical methods in engineering, an introduction to the mathematical treatment of engineering problems. McGraw-Hill, New York/London, 505pp
86. Whittaker E (1910) History of the theories of Aether and electricity. Nelson, London.
87. Young T (1807) A course of lectures on natural philosophy and the mechanical arts, 2 vols. Joseph Johnson, London, 738pp

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