Appendix A
Solution to the Invariance Equation

Section 2.3.4 describes the solution of the invariance equation (2.52) for the function $\Gamma(v, iD)$ in the free theory. The solution in the interacting theory is not simply obtained from the free one by replacing ∂ with D. Here we present a method of solution that is valid to any order in $1/M$. Since we use $\Gamma(v, iD)$ to construct the invariant Lagrangian, the existence of a solution for $\Gamma(v, iD)$ proves that a non-zero Lagrangian exists at any order in $1/M$. First, we will construct the general solution in Sect. A.1 and then explicitly apply this construction to the spin 1/2 theory up to order $1/M^3$ in Sect. A.2.

A.1 Series Solution for Γ

Recall Eq. (2.52) for Γ required to build explicitly invariant operators,

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1}W(\mathcal{B}, iD + Mv) = \Gamma(v, iD)\,, \tag{A.1}$$

where to first order in q we have $\mathcal{B}^{-1}v = v + q/M$. Let us expand in orders of $1/M$ and define

$$X \equiv \mathcal{B}^{-1}W = 1 + q^\mu X_\mu = 1 + q^\mu \left[\frac{1}{M}X_\mu^{(1)} + \frac{1}{M^2}X_\mu^{(2)} + \dots\right]\,, \tag{A.2a}$$

$$\Gamma = 1 + \frac{1}{M}\Gamma^{(1)} + \frac{1}{M^2}\Gamma^{(2)} + \dots\,. \tag{A.2b}$$

We note that the variation in Γ arises from the variations in v and in iD,

$$\delta\Gamma = \Gamma(v + q/M, iD - q) - \Gamma(v, iD) = q^\mu\left(-\frac{\partial}{\partial iD^\mu}\Gamma + \frac{1}{M}\frac{\partial}{\partial v^\mu}\Gamma\right)\,. \tag{A.3}$$

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Equating orders in $1/M$, we find

$$\frac{\partial}{\partial iD^\mu}\Gamma^{(n)} = \frac{\partial}{\partial v^\mu}\Gamma^{(n-1)} + \Gamma^{(n-1)}X^{(1)}_\mu + \Gamma^{(n-2)}X^{(2)}_\mu + \cdots + \Gamma^{(0)}X^{(n)}_\mu \equiv Y^{(n)}_\mu \,, \tag{A.4}$$

where we define $\Gamma^{(0)} = 1$. Note that Eq. (A.4) is understood to be contracted with q^μ so that pieces proportional to v^μ should be dropped. We can solve this equation for $\Gamma^{(n)}$ obtaining

$$\begin{aligned}\Gamma^{(n)} &= \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m!} iD_\perp^{\mu_1} iD_\perp^{\mu_2} \ldots iD_\perp^{\mu_m} \frac{\partial}{\partial iD^{\mu_1}}\frac{\partial}{\partial iD^{\mu_2}}\cdots\frac{\partial}{\partial iD^{\mu_{m-1}}} Y^{(n)}_{\mu_m} \\ &= iD_\perp^\mu Y^{(n)}_\mu - \frac{1}{2!} iD_\perp^\mu iD_\perp^\nu \frac{\partial}{\partial iD^\mu} Y^{(n)}_\nu + \ldots \,,\end{aligned} \tag{A.5}$$

provided that at each order, the $Y^{(n)}$ derived from the already determined $\Gamma^{(1)}, \ldots, \Gamma^{(n-1)}$ satisfy[1]

$$\frac{\partial}{\partial iD^{[\nu}} Y^{(n)}_{\mu]} = 0 \,, \tag{A.6}$$

where $A^{[\mu}B^{\nu]} = (A^\mu B^\nu - A^\nu B^\mu)/2$ denotes antisymmetrization. Using the definition of $Y^{(n)}$ we can show that this imposes constraints on $X^{(n)}$, for $n \geq 2$,

$$\frac{\partial}{\partial iD^{[\nu}} X^{(n)}_{\mu]} = -\frac{\partial}{\partial v^{[\mu}} X^{(n-1)}_{\nu]} + X^{(n-1)}_{[\mu} X^{(1)}_{\nu]} + X^{(n-2)}_{[\mu} X^{(2)}_{\nu]} + \ldots + X^{(1)}_{[\mu} X^{(n-1)}_{\nu]} \equiv Z^{(n)}_{\mu\nu} \,. \tag{A.7}$$

For Eq. (A.7) to have a solution, a consistency condition on $Z^{(n)}_{\mu\nu}$ requires that[2]

$$0 = v_\sigma \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial iD^\rho} Z^{(n)}_{\mu\nu} \,. \tag{A.8}$$

We can show by induction that Eq. (A.7) can be solved at each order. Since $X^{(1)}$ is dimensionless, it cannot depend on iD; hence $Z^{(2)}$ from (A.7) is also independent of iD and solves (A.8). Now assume that we have constructed solutions $X^{(n)}$ to Eq. (A.7) for $n = 1, \ldots, N-1$ (necessarily obeying the constraint (A.8)). Application of the Jacobi identity shows that the constraint (A.8) is then obeyed for $n = N$ and a solution to Eq. (A.7) can be found for $n = N$.

Let us find a solution to Eq. (A.7) that reduces to a given X_{free} for the non-interacting theory (e.g., $X_{\text{free}} = \mathcal{B}^{-1}W$ from (2.56)). First, note that the existence

[1]This is the analog of $\vec{\nabla} \times \vec{E} = \vec{0}$ for the existence of a solution ϕ of $\vec{\nabla}\phi = \vec{E}$ in electrostatics.

[2]This is the analog of $\vec{\nabla} \cdot \vec{B} = 0$ for the existence of a solution $\vec{A}$ of $\vec{\nabla} \times \vec{A} = \vec{B}$ in magnetostatics.

of the free case solution given in (2.57) implies that the $X^{(n)}$ defined in the free case from (2.56) must obey the constraint (A.7). Let us define naively covariantized quantities $\hat{X}^{(n)} = X^{(n)}_{\text{free}}\Big|_{\partial\to D}$, with a definite ordering prescription, e.g. as in (2.63), and define $\hat{Z}^{(n)}$ by

$$\hat{Z}^{(n)}_{\mu\nu} \equiv \frac{\partial}{\partial iD^{[\nu}}\hat{X}^{(n)}_{\mu]} \,. \tag{A.9}$$

A straightforward calculation then shows that (A.7) is solved by

$$X^{(n)}_{\mu} = \hat{X}^{(n)}_{\mu} + 2\sum_{m=1}^{n-1} \frac{(-1)^m}{(m+1)!} iD^{\nu_1}_{\perp} \cdots iD^{\nu_m}_{\perp} \frac{\partial}{\partial iD^{\nu_1}} \cdots \frac{\partial}{\partial iD^{\nu_{m-1}}} \left(Z^{(n)}_{\nu_m\mu} - \hat{Z}^{(n)}_{\nu_m\mu} \right) . \tag{A.10}$$

In the free case we have $Z^{(n)} = \hat{Z}^{(n)}$ and $X^{(n)}$ reduces to the free case solution. Having found a suitable $X^{(n)}$ satisfying (A.7) we may then proceed to build $\Gamma^{(n)}$ satisfying (A.4), and hence Γ satisfying (2.52).

Note that $Z^{(n)}$ has mass dimension $n-2$ so that $n = 4$ is the first order at which field strength dependent terms can cause $Z^{(n)} \neq \hat{Z}^{(n)}$. Correspondingly, our choice (A.10) ensures that field-strength dependent corrections to $X^{(n)} - \hat{X}^{(n)}$ can first appear at order $n = 4$. This can be explicitly seen in the solution for the spin 1/2 theory in the next section.

A.2 Explicit Solution for Γ in the Spin 1/2 Theory

To illustrate, let us calculate Γ for the spin 1/2 theory. Consider the free solution (2.56),

$$X_\mu(v, i\partial) = \frac{1}{2M}\gamma^{\perp}_{\mu} + \frac{1}{4M^2}\sigma^{\perp}_{\mu\nu}\partial^{\nu}\left[1 - \frac{iv\cdot\partial}{M} + \frac{1}{M^2}\left((iv\cdot\partial)^2 - \frac{1}{4}(i\partial_\perp)^2\right) + \dots\right], \tag{A.11}$$

and the arbitrary covariantization,

$$\hat{X}_\mu(v, iD) = \frac{1}{2M}\gamma^{\perp}_{\mu} + \frac{1}{4M^2}\sigma^{\perp}_{\mu\nu}D^{\nu}\left[1 - \frac{iv\cdot D}{M} + \frac{1}{M^2}\left((iv\cdot D)^2 - \frac{1}{4}(iD_\perp)^2\right) + \dots\right]. \tag{A.12}$$

A corresponding solution for Γ in the free theory is displayed in (2.57). Now let us follow the construction of the previous section order by order.

Order $1/M$: First, we determine,

$$Y_\mu^{(1)} = X_\mu^{(1)} = \hat{X}_\mu^{(1)} = \frac{\gamma_\mu^\perp}{2}\,. \tag{A.13}$$

This function clearly satisfies Eq. (A.6) so that we may solve for

$$\Gamma^{(1)} = \frac{1}{2} i\not{D}_\perp\,. \tag{A.14}$$

Order $1/M^2$: Continuing to the next order, we evaluate

$$Z_{\mu\nu}^{(2)} = -\frac{i}{4}\sigma_{\mu\nu}^\perp = \hat{Z}_{\mu\nu}^{(2)}\,, \tag{A.15a}$$

$$X_\mu^{(2)} = \frac{1}{4}\sigma_{\mu\nu}^\perp D^\nu = \hat{X}_\mu^{(2)}\,, \tag{A.15b}$$

$$Y_\mu^{(2)} = -\frac{1}{2}\gamma_\mu^\perp iv\cdot D - \frac{1}{4} iD_\mu^\perp\,. \tag{A.15c}$$

Solving for $\Gamma^{(2)}$ yields

$$\Gamma^{(2)} = -\frac{1}{8}(iD_\perp)^2 - \frac{1}{2} i\not{D}_\perp iv\cdot D\,. \tag{A.16}$$

Order $1/M^3$: At the next order, we find

$$Z_{\mu\nu}^{(3)} = \frac{i}{4}\sigma_{\mu\nu}^\perp iv\cdot D = \hat{Z}_{\mu\nu}^{(3)}\,, \tag{A.17a}$$

$$X_\mu^{(3)} = -\frac{1}{4}\sigma_{\mu\nu}^\perp D^\nu iv\cdot D = \hat{X}_\mu^{(3)}\,, \tag{A.17b}$$

$$\begin{aligned} Y_\mu^{(3)} = \frac{1}{2}\gamma_\mu^\perp (iv\cdot D)^2 + \frac{3}{8} iD_\mu^\perp iv\cdot D + \frac{1}{8} iv\cdot D iD_\mu^\perp - \frac{1}{2} i\not{D}_\perp iD_\mu^\perp \\ - \frac{1}{16}(iD_\perp)^2\gamma_\mu^\perp + \frac{1}{8} i\not{D}_\perp \sigma_{\mu\nu}^\perp D^\nu\,. \end{aligned} \tag{A.17c}$$

After some manipulations, the resulting $\Gamma^{(3)}$ is

$$\begin{aligned} \Gamma^{(3)} = \frac{1}{4}(iD_\perp)^2 iv\cdot D + \frac{i\not{D}_\perp}{2}\left[-\frac{3}{8} i\not{D}_\perp (iD_\perp)^2 + (iv\cdot D)^2\right] - \frac{g}{8} v^\alpha G_{\alpha\beta} D_\perp^\beta \\ -\frac{g}{16}\sigma_{\alpha\beta}^\perp G^{\alpha\beta} i\not{D}_\perp + \frac{g}{8}\Big[i\gamma_\perp^\beta \sigma_\perp^{\mu\alpha}[D_\mu, G_{\beta\alpha}] \\ -v^\alpha [D_\perp^\mu, G_{\alpha\mu}] - [D_\perp^\mu, G_{\mu\beta}^\perp]\gamma_\perp^\beta \Big]\,. \end{aligned} \tag{A.18}$$

Order $1/M^4$: Continuing to higher order we find

$$Z^{(4)}_{\mu\nu} = \hat{Z}^{(4)}_{\mu\nu} + \frac{g}{32}\left(-iG^{\perp}_{\mu\nu} + \sigma^{\perp}_{\mu\sigma}G^{\perp\sigma}_{\nu} - \sigma^{\perp}_{\nu\sigma}G^{\perp\sigma}_{\mu}\right) , \tag{A.19a}$$

$$X^{(4)}_{\mu} = \sigma^{\perp}_{\mu\nu}D^{\nu}\left[\frac{1}{4}(iv\cdot D)^2 - \frac{1}{16}(iD_{\perp})^2\right] + \frac{g}{32}iD^{\nu}_{\perp}\left(-iG^{\perp}_{\mu\nu} + \sigma^{\perp}_{\mu\sigma}G^{\perp\sigma}_{\nu} - \sigma^{\perp}_{\nu\sigma}G^{\perp\sigma}_{\mu}\right) . \tag{A.19b}$$

Note that $X^{(4)}_{\mu}$ differs from the trial solution $\hat{X}^{(4)}_{\mu}$. We may continue in this manner to construct $Y^{(4)}_{\mu}$ and $\Gamma^{(4)}$.

Appendix B
Integrals and Inputs for Weak Scale Matching

B.1 Self Energy Integrals and Standard Model Two-Point Functions

Here and in the following sections we use the notation

$$[c_\epsilon] = \frac{i\Gamma(1+\epsilon)}{(4\pi)^{2-\epsilon}}\,, \quad (dL) = \frac{d^dL}{(2\pi)^d}. \tag{B.1}$$

The self-energies in Sect. 3.3 and the $h\bar{\chi}\chi$ three-point functions in Sect. 4.2.1 require the following integrals,

$$\begin{aligned}
I_1(\delta, m) &= \int (dL) \frac{1}{v\cdot L - \delta + i0} \frac{1}{(L^2 - m^2 + i0)^2} \\
&= \frac{\partial}{\partial m^2} I_3(\delta, m) \\
&= [c_\epsilon] m^{-2\epsilon} \left\{ \frac{2}{\sqrt{m^2-\delta^2-i0}} \left[\arctan\left(\frac{\delta}{\sqrt{m^2-\delta^2-i0}} \right) - \frac{\pi}{2} \right] \right. \\
&\quad \left. + \mathcal{O}(\epsilon) \right\}, \\
I_2(\delta, m) &= \int (dL) v\cdot L \frac{1}{v\cdot L - \delta + i0} \frac{1}{(L^2 - m^2 + i0)^2} \\
&= \delta I_1(\delta, m) + \frac{i}{(4\pi)^2} B_0(0, m, m)
\end{aligned}$$

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$$= [c_\epsilon] m^{-2\epsilon} \left\{ \frac{1}{\epsilon} + \frac{2\delta}{\sqrt{m^2 - \delta^2 - i0}} \left[\arctan\left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] + \mathcal{O}(\epsilon) \right\},$$

$$I_3(\delta, m) = \int (dL) \frac{1}{v \cdot L - \delta + i0} \frac{1}{L^2 - m^2 + i0}$$

$$= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2\delta}{\epsilon} + 4\sqrt{m^2 - \delta^2 - i0} \left[\arctan\left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] - 4\delta + \mathcal{O}(\epsilon) \right\},$$

$$I_4(\delta_1, \delta_2, m) = \int (dL) \frac{1}{v \cdot L - \delta_1 + i0} \frac{1}{v \cdot L - \delta_2 + i0} \frac{1}{L^2 - m^2 + i0}. \tag{B.2}$$

For $I_4(\delta_1, \delta_2, m)$, let us specialize to $\delta_2 = 0$ or $\delta_1 = \delta_2$,

$$I_4(\delta, 0, m) = \frac{1}{\delta} \left[I_3(\delta, m) - I_3(0, m) \right]$$

$$= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2}{\epsilon} + \frac{4\sqrt{m^2 - \delta^2 - i0}}{\delta} \left[\arctan\left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] - 4 + \frac{2\pi m}{\delta} + \mathcal{O}(\epsilon) \right\},$$

$$I_4(\delta, \delta, m) = \frac{\partial}{\partial \delta} I_3(\delta, m)$$

$$= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2}{\epsilon} - \frac{4\delta}{\sqrt{m^2 - \delta^2 - i0}} \left[\arctan\left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] + \mathcal{O}(\epsilon) \right\}. \tag{B.3}$$

The two-point functions for the electroweak SM bosons appearing in (4.21) are obtained by summing the fermionic and bosonic contributions given below. Following Denner [32], we have

$$\Sigma^{AA\prime}(0) = -\frac{\alpha}{4\pi} \left\{ 3B_0(0, m_W, m_W) + 4m_W^2 B_0'(0, m_W, m_W) - \frac{4}{3} \sum_{f,i} \left[N_c^f Q_f^2 B_0(0, m_{f,i}, m_{f,i}) \right] \right\},$$

$$\frac{\Sigma^{AZ}(0)}{m_Z^2} = -\frac{\alpha}{4\pi} \left\{ -\frac{2c_W}{s_W} B_0(0, m_W, m_W) \right\},$$

$$
\begin{aligned}
\frac{\Sigma^{ZZ}(m_Z^2)_{\text{fermion}}}{m_Z^2} &= -\frac{\alpha}{4\pi}\Bigg\{\frac{2}{3}\left[-B_0(m_Z,0,0)+\frac{1}{3}\right]\sum_{f,i} N_c^f[(g_f^+)^2+(g_f^-)^2] \\
&+\frac{2}{3}N_c^t\Bigg[[(g_t^+)^2+(g_t^-)^2]\Bigg[-\left(1+\frac{2m_t^2}{m_Z^2}\right)B_0(m_Z,m_t,m_t) \\
&+B_0(m_Z,0,0)+\frac{2m_t^2}{m_Z^2}B_0(0,m_t,m_t)\Bigg] \\
&+\frac{3}{4s_W^2c_W^2}\frac{m_t^2}{m_Z^2}B_0(m_Z,m_t,m_t)\Bigg]\Bigg\}, \\
\frac{\Sigma^{ZZ}(m_Z^2)_{\text{boson}}}{m_Z^2} &= -\frac{\alpha}{4\pi}\frac{1}{s_W^2c_W^2}\Bigg\{\frac{1}{12}(4c_W^2-1)(12c_W^4+20c_W^2+1)B_0(m_Z,m_W,m_W) \\
&-\frac{1}{3}c_W^2(12c_W^4-4c_W^2+1)B_0(0,m_W,m_W)-\frac{1}{6}B_0(0,m_Z,m_Z) \\
&-\frac{1}{12}\left(\frac{m_h^4}{m_Z^4}-4\frac{m_h^2}{m_Z^2}+12\right)B_0(m_Z,m_Z,m_h)-\frac{1}{6}\frac{m_h^2}{m_Z^2}B_0(0,m_h,m_h) \\
&+\frac{1}{12}\left(1-\frac{m_h^2}{m_Z^2}\right)^2B_0(0,m_Z,m_h)-\frac{1}{9}(1-2c_W^2)\Bigg\}, \\
\frac{\Sigma^{WW}(m_W^2)_{\text{fermion}}}{m_W^2} &= -\frac{\alpha}{4\pi}\frac{1}{2s_W^2}\Bigg\{\frac{2}{3}\left[\frac{1}{3}-B_0(m_W,0,0)\right]\sum_{f,i}\frac{N_c^f}{2} \\
&+\frac{2}{3}N_c^t\Bigg[\frac{1}{2}\left(\frac{m_t^4}{m_W^4}+\frac{m_t^2}{m_W^2}-2\right)B_0(m_W,m_t,0)+B_0(m_W,0,0) \\
&+\frac{m_t^2}{m_W^2}B_0(0,m_t,m_t)-\frac{m_t^4}{2m_W^4}B_0(0,m_t,0)\Bigg]\Bigg\}, \\
\frac{\Sigma^{WW}(m_W^2)_{\text{boson}}}{m_W^2} &= -\frac{\alpha}{4\pi}\Bigg\{4B_0(m_W,m_W,\lambda)-\frac{4}{3}B_0(0,m_W,m_W)+\frac{2}{3}B_0(0,m_W,\lambda) \\
&+\frac{2}{9}+\frac{1}{12s_W^2}\Bigg[\frac{1}{c_W^4}(4c_W^2-1)(12c_W^4+20c_W^2+1)B_0(m_W,m_W,m_Z) \\
&-2(8c_W^2+1)B_0(0,m_W,m_W)-\frac{2}{c_W^2}(8c_W^2+1)B_0(0,m_Z,m_Z) \\
&+\frac{s_W^4}{c_W^4}(8c_W^2+1)B_0(0,m_W,m_Z)-\frac{2}{3}(1-4c_W^2)\Bigg] \\
&+\frac{1}{12s_W^2}\Bigg[-\left(\frac{m_h^4}{m_W^4}-4\frac{m_h^2}{m_W^2}+12\right)B_0(m_W,m_W,m_h) \\
&-2B_0(0,m_W,m_W)-2\frac{m_h^2}{m_W^2}B_0(0,m_h,m_h)
\end{aligned}
$$

$$+ \left(1 - \frac{m_h^2}{m_W^2}\right)^2 B_0(0, m_W, m_h) - \frac{2}{3}\Bigg]\Bigg\},$$

$$\Sigma^{HH\prime}(m_h^2)_{\text{fermion}} = -\frac{\alpha}{4\pi}\frac{3m_t^2}{2s_W^2 m_W^2}\left[(4m_t^2 - m_h^2)B_0'(m_h, m_t, m_t) - B_0(m_h, m_t, m_t)\right],$$

$$\Sigma^{HH\prime}(m_h^2)_{\text{boson}} = -\frac{\alpha}{4\pi}\Bigg\{ - \frac{1}{2s_W^2}\Bigg[\left(6m_W^2 - 2m_h^2 + \frac{m_h^4}{2m_W^2}\right) B_0'(m_h, m_W, m_W)$$

$$- 2B_0(m_h, m_W, m_W)\Bigg] - \frac{1}{4s_W^2 c_W^2}\Bigg[\left(6m_Z^2 - 2m_h^2 + \frac{m_h^4}{2m_Z^2}\right)$$

$$B_0'(m_h, m_Z, m_Z) - 2B_0(m_h, m_Z, m_Z)\Bigg] - \frac{9m_h^4}{8s_W^2 m_W^2} B_0'(m_h, m_h, m_h)\Bigg\}, \tag{B.4}$$

where the sums over indices f and i are for SM fermion flavors and generations, respectively. Above, N_c^f and Q_f respectively denote the number of colors and the electric charge of fermion f. We have also used

$$\alpha = \frac{g_2^2 s_W^2}{4\pi}, \quad g_f^+ = \frac{1}{8s_W^2 c_W^2}\left[c_V^{(f)2} + c_A^{(f)2}\right], \quad g_f^- = \frac{1}{8s_W^2 c_W^2}\left[c_V^{(f)2} - c_A^{(f)2}\right], \tag{B.5}$$

where

$$c_V^{(\ell)} = -1 + 4s_W^2, \quad c_A^{(\ell)} = 1, \quad c_V^{(\nu)} = 1, \quad c_A^{(\nu)} = -1, \tag{B.6}$$

with ℓ and ν denoting charged lepton and neutrino, respectively. The coefficients $c_V^{(f)}$ and $c_A^{(f)}$ for quarks can be found in (4.14). The basic integral appearing above is

$$\frac{i}{(4\pi)^2} B_0(M, m_0, m_1) = \int (dL) \frac{1}{L^2 - m_0^2 + i0}\frac{1}{(L+p)^2 - m_1^2 + i0}$$

$$= [c_\epsilon]\left[\frac{1}{\epsilon} + 2 - \log(m_0 m_1) + \frac{m_0^2 - m_1^2}{M^2}\log\frac{m_1}{m_0}\right.$$

$$\left. - \frac{m_0 m_1}{M^2}\left(\frac{1}{r} - r\right)\log r + \mathcal{O}(\epsilon)\right], \tag{B.7}$$

where $p^2 = M^2$ and

$$r = X + \sqrt{X^2 - 1}, \quad \frac{1}{r} = X - \sqrt{X^2 - 1}, \quad X = \frac{m_0^2 + m_1^2 - M^2 - i0}{2m_0 m_1}. \tag{B.8}$$

We find the following limits,

$$B_0(0, m, m) = (4\pi)^\epsilon \Gamma(1+\epsilon)\left[\frac{1}{\epsilon} - 2\log m + \mathcal{O}(\epsilon)\right],$$

$$
\begin{aligned}
B_0(0,m,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon} - 2\log m + 1 + \mathcal{O}(\epsilon)\right], \\
B_0(0,m_0,m_1) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon} - \frac{m_0^2}{m_0^2 - m_1^2}\log m_0^2 \right.\\
&\quad \left. + \frac{m_1^2}{m_0^2 - m_1^2}\log m_1^2 + 1 + \mathcal{O}(\epsilon)\right], \\
B_0(M,m,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon} + 2 - \frac{m^2}{M^2}\log m^2 \right.\\
&\quad \left. + \frac{m^2 - M^2}{M^2}\log(m^2 - M^2 - i0) + \mathcal{O}(\epsilon)\right], \\
B_0(M,0,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon} + 2 - \log(-M^2 - i0) + \mathcal{O}(\epsilon)\right], \\
\lim_{\lambda\to 0} B_0(m,m,\lambda) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon} + 2 - \log m^2 + \mathcal{O}(\epsilon)\right].
\end{aligned}
\tag{B.9}
$$

In the present application, only the real parts of the integrals are relevant. For the derivative of the integral we have,

$$
\begin{aligned}
B_0'(M,m,m) &\equiv \frac{\partial}{\partial p^2}B_0(M,m,m) \\
&= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{m^2}{M^4}\left(\frac{1}{r} - r\right)\log r - \frac{1}{M^2}\left(1 + \frac{r^2+1}{r^2-1}\log r\right) + \mathcal{O}(\epsilon)\right],
\end{aligned}
\tag{B.10}
$$

which has the following limits,

$$
\begin{aligned}
B_0'(0,m,m) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{6m^2} + \mathcal{O}(\epsilon)\right], \\
B_0'(M,0,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[-\frac{1}{M^2} + \mathcal{O}(\epsilon)\right].
\end{aligned}
\tag{B.11}
$$

B.2 Box Integrals

The integrals required for the two-boson exchange amplitudes in Sect. 4.2.3 may be written in terms of the integral operators $\mathcal{I}_{\text{even}}$ and $\mathcal{I}_{\text{odd}}$ defined in (4.53) as

$$
\begin{aligned}
J(m_V,M,\delta) &= \mathcal{I}_{\text{even}}(\delta,m_V)\frac{1}{L^2 - M^2 + i0}, \\
J^\mu(p,m_V,M,\delta) &= \mathcal{I}_{\text{even}}(\delta,m_V)\frac{1}{L^2 + 2L\cdot p - M^2 + i0}L^\mu \\
&= v\cdot p v^\mu J_1(m_V,M,\delta) + p^\mu J_2(m_V,M,\delta) + \mathcal{O}(p^3),
\end{aligned}
$$

$$J_-(p, m_V, M, \delta) = -\mathcal{I}_{\rm odd}(\delta, m_V) \frac{1}{L^2 + 2L \cdot p - M^2 + i0}$$
$$= v \cdot p J_-(m_V, M, \delta) + \mathcal{O}(p^3) ,$$
$$J_-^\mu(m_V, M, \delta) = -\mathcal{I}_{\rm odd}(\delta, m_V) \frac{1}{L^2 - M^2 + i0} L^\mu = v^\mu J_{1-}(m_V, M, \delta) . \tag{B.12}$$

Note that $J^\mu(p, m_V, M, \delta)$ and $J_-(p, m_V, M, \delta)$ vanish when p^μ vanishes since the integrands are then odd in L^μ. By standard manipulations, we may express the integrals J_1, J_2, J, and J_-, as

$$J_1(m_V, M, \delta) = -8[c_\epsilon](1+\epsilon) \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx \rho^2 (1-x)$$
$$\left[x m_V^2 + (1-x) M^2 + \rho^2 + 2\rho\delta - i0\right]^{-2-\epsilon} ,$$
$$J_2(m_V, M, \delta) = 4[c_\epsilon] \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx (1-x)$$
$$\left[x m_V^2 + (1-x) M^2 + \rho^2 + 2\rho\delta - i0\right]^{-1-\epsilon} ,$$
$$J(m_V, M, \delta) = -4[c_\epsilon] \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx$$
$$\left[x m_V^2 + (1-x) M^2 + \rho^2 + 2\rho\delta - i0\right]^{-1-\epsilon} ,$$
$$J_-(m_V, M, \delta) = 4[c_\epsilon] \frac{\partial}{\partial \delta} \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx (1-x)$$
$$\left[x m_V^2 + (1-x) M^2 + \rho^2 + 2\rho\delta - i0\right]^{-1-\epsilon} . \tag{B.13}$$

Let us introduce the integral

$$\hat{J}(m_V, M, \delta) = [c_\epsilon] \int_0^\infty d\rho \int_0^1 dx (1-x) \left[x m_V^2 + (1-x) M^2 + \rho^2 + 2\rho\delta - i0\right]^{-1-\epsilon} , \tag{B.14}$$

and write the above integrals in terms of $\hat{J}(m_V, M, \delta)$ as

$$J_2(m_V, M, \delta) = 4 \frac{\partial}{\partial m_V^2} \hat{J}(m_V, M, \delta) ,$$
$$J_-(m_V, M, \delta) = 4 \frac{\partial}{\partial \delta} \frac{\partial}{\partial m_V^2} \hat{J}(m_V, M, \delta) ,$$
$$J(m_V, M, \delta) = -4 \frac{\partial}{\partial m_V^2} \left[\hat{J}(m_V, M, \delta) + \hat{J}(M, m_V, \delta)\right] ,$$
$$J_1(m_V, M, \delta) = 4 \frac{\partial}{\partial m_V^2} \left[-\hat{J}(m_V, M, \delta) + \frac{\partial}{\partial A} \hat{J}(m_V, M, \delta/A)\Big|_{A=1} \right] . \tag{B.15}$$

For J_{1-}, we may use the identity (4.57) to write

$$\begin{aligned}
J_{1-}(m_V, M, \delta) &= -2\int (dL)\frac{1}{(L^2-m_V^2+i0)^2}\frac{1}{L^2-M^2+i0} - \delta J(m_V, M, \delta) \\
&= \frac{2[c_\epsilon]m_V^{-2-2\epsilon}}{\epsilon(1-\epsilon)}\left(1-\frac{M^2}{m_V^2}\right)^{-2}\left[\epsilon+\frac{M^2}{m_V^2}\left(1-\epsilon-\frac{m_V^{2\epsilon}}{M^{2\epsilon}}\right)\right] \\
&\quad - \delta J(m_V, M, \delta)\,. \qquad \text{(B.16)}
\end{aligned}$$

Having determined the above integrals in terms of $\hat{J}(m_V, M, \delta)$, it remains to compute this function. Let us write

$$\begin{aligned}
\hat{J}(m_V, M, \delta) &= -\frac{[c_\epsilon]}{\epsilon}\frac{\partial}{\partial M^2}\int_0^\infty d\rho\int_0^1 dx\big[xm_V^2+(1-x)M^2+\rho^2+2\rho\delta-i0\big]^{-\epsilon} \\
&= -\frac{[c_\epsilon]}{\epsilon}\frac{\partial}{\partial M^2}\int_0^\infty d\rho\frac{1}{m_V^2-M^2}\frac{1}{1-\epsilon} \\
&\quad \left\{[m_V^2+\rho^2+2\rho\delta-i0]^{1-\epsilon}-[M^2+\rho^2+2\rho\delta-i0]^{1-\epsilon}\right\} \\
&= -\frac{[c_\epsilon]}{\epsilon}\frac{\partial}{\partial M^2}\frac{1}{m_V^2-M^2}\frac{1}{1-\epsilon}\Big\{m_V^{3-2\epsilon}f_1(\delta/m_V, 1-\epsilon) \\
&\quad - M^{3-2\epsilon}f_1(\delta/M, 1-\epsilon)\Big\}\,, \qquad \text{(B.17)}
\end{aligned}$$

where

$$\begin{aligned}
f_1(\delta, a) &= \int_0^\infty d\rho(1+\rho^2+2\rho\delta-i0)^a \\
&= (1-\delta^2-i0)^{a+\frac{1}{2}}\frac{\sqrt{\pi}}{2}\frac{\Gamma(-a-\frac{1}{2})}{\Gamma(-a)} - \delta^{2a+1}\int_0^1 dx\left[\delta^{-2}-1+x^2-i0\right]^a\,. \qquad \text{(B.18)}
\end{aligned}$$

Although for the present application we require only $\delta > 0$, the expression is for general sign of δ. We presently need $f_1(\delta, a)$ for $a = 1-\epsilon$, and hence consider

$$\begin{aligned}
&\frac{\sqrt{\pi}}{2}\frac{\Gamma(-\frac{3}{2}+\epsilon)}{\Gamma(-1+\epsilon)} = -\frac{2\pi}{3}\epsilon + \frac{2\pi}{9}(6\log 2-5)\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \\
&\int_0^1 dx\left[\delta^{-2}-1+x^2-i0\right]^{1-\epsilon} = B^2+\frac{1}{3}+\epsilon\Big\{\frac{2}{9}+\frac{4}{3}B^2-\frac{4}{3}B^3\,\mathrm{arccot}\,B \\
&\quad -\left(B^2+\frac{1}{3}\right)\log(B^2+1)\Big\} + \epsilon^2\Big\{\frac{4}{27}+\frac{20}{9}B^2+\frac{4}{9}B^3(6\log 2B-5)\,\mathrm{arccot}\,B
\end{aligned}$$

$$+\frac{4}{3}B^3 i\left[\mathrm{Li}_2\left(\frac{1+iB}{1-iB}\right)-\mathrm{arccot}^2 B+\frac{\pi^2}{12}\right]+\frac{1}{2}\left(B^2+\frac{1}{3}\right)\log^2(B^2+1)$$
$$-\left(\frac{4}{3}B^2+\frac{2}{9}\right)\log(B^2+1)\Bigg\}+\mathcal{O}(\epsilon^3)\,, \tag{B.19}$$

where $B^2=1/\delta^2-1-i0$. For $B^2>0$, the bracket involving dilogarithm may be written

$$i\left[\mathrm{Li}_2\left(\frac{1+iB}{1-iB}\right)-\mathrm{arccot}^2 B+\frac{\pi^2}{12}\right]=-\mathrm{Im}\,\mathrm{Li}_2\left(\frac{1+iB}{1-iB}\right)$$
$$=-\mathrm{Cl}_2\left[\arccos\left(\frac{1-B^2}{1+B^2}\right)\right]\,, \tag{B.20}$$

where Cl_2 is the Clausen function of order two. The general expression is required for continuing to arbitrary mass parameters. Having determined $f_1(\delta,1-\epsilon)$, we may proceed to compute $\hat{J}(m_V,M,\delta)$ using (B.17), and then $J_2(m_V,M,\delta)$, $J(m_V,M,\delta)$, $J_-(m_V,M,\delta)$ and $J_1(m_V,M,\delta)$ using (B.15), and $J_{1-}(m_V,M,\delta)$ using (B.16).

For $M=0$, the expressions in (B.13), the expressions for $J_2(m_V,M,\delta)$, $J_1(m_V,M,\delta)$, and $J_-(m_V,M,\delta)$ in (B.15), and the expression for $J_{1-}(m_V,M,\delta)$ in (B.16), remain valid. The integral $J(m_V,0,\delta)$ is now given by

$$J(m_V,0,\delta)=-4[c_\epsilon]\frac{\partial}{\partial m_V^2}\left\{-\frac{1}{\epsilon}m_V^{-1-2\epsilon}\Big[f_1(\delta/m_V,-\epsilon)-f_0(\delta/m_V,-\epsilon)\Big]\right\}, \tag{B.21}$$

and the integral $\hat{J}(m_V,0,\delta)$ by

$$\hat{J}(m_V,0,\delta)=\frac{[c_\epsilon]m_V^{-2}}{\epsilon}\int_0^\infty d\rho\Bigg\{(\rho^2+2\rho\delta-i0)^{-\epsilon}$$
$$-\frac{m_V^{-2}}{1-\epsilon}\Big[(m_V^2+\rho^2+2\rho\delta-i0)^{1-\epsilon}-(\rho^2+2\rho\delta-i0)^{1-\epsilon}\Big]\Bigg\}$$
$$=\frac{[c_\epsilon]m_V^{-1-2\epsilon}}{\epsilon}\Bigg\{f_0(\delta/m_V,-\epsilon)-\frac{1}{(1-\epsilon)}\Big[f_1(\delta/m_V,1-\epsilon)$$
$$-f_0(\delta/m_V,1-\epsilon)\Big]\Bigg\}\,, \tag{B.22}$$

where $f_1(\delta,a)$ is given by (B.18) and

$$f_0(\delta,a)=\int_0^\infty d\rho(\rho^2+2\rho\delta-i0)^a=\frac{\delta^{1+2a}\Gamma(1+a)\Gamma\left(-a-\frac{1}{2}\right)}{2\sqrt{\pi}}\,. \tag{B.23}$$

We also need $f_1(\delta/m_V,a)$ for $a=-\epsilon$, which we may write as

$$f_1(\delta/m_V,-\epsilon)=\frac{1}{1-\epsilon}m_V^{-1+2\epsilon}\frac{\partial}{\partial m_V^2}\Big[m_V^{3-2\epsilon}f_1(\delta/m_V,1-\epsilon)\Big]\,. \tag{B.24}$$

At vanishing residual mass, $\delta = 0$, only the integrals $J(m_V, M, 0)$, $J_1(m_V, M, 0)$ and $J_2(m_V, M, 0)$ are required, and from (B.13) they can be easily represented in closed form,

$$J(m_V, M, 0) = [c_\epsilon] \frac{2\sqrt{\pi}}{(1-2\epsilon)} \frac{\Gamma(\frac{1}{2}+\epsilon)}{\Gamma(1+\epsilon)} \frac{m_V^{1-2\epsilon}}{(M^2 - m_V^2)^2} \left[1 + 2\epsilon - 2\left(\frac{M}{m_V}\right)^{1-2\epsilon} + (1-2\epsilon)\left(\frac{M}{m_V}\right)^2\right],$$

$$J_2(m_V, M, 0) = -J_1(m_V, M, 0) = [c_\epsilon] \frac{4\sqrt{\pi}}{(3-2\epsilon)(1-2\epsilon)} \frac{\Gamma(\frac{1}{2}+\epsilon)}{\Gamma(1+\epsilon)} \frac{m_V^{3-2\epsilon}}{(M^2 - m_V^2)^3} \left[1 + 2\epsilon - (3-2\epsilon)\left(\frac{M}{m_V}\right)^{1-2\epsilon} + (3-2\epsilon)\left(\frac{M}{m_V}\right)^2 - (1+2\epsilon)\left(\frac{M}{m_V}\right)^{3-2\epsilon}\right]. \tag{B.25}$$

The result $J_2(m_V, M, 0) = -J_1(m_V, M, 0)$ follows from the observation that when $\delta = 0$ the identity in (4.57) implies $v_\mu J^\mu(p, m_V, M, 0) = 0$. The case $\delta = M = 0$ is simply obtained by substitution in (B.25).

B.3 Heavy Particle Integrals with Electroweak Polarization Tensor Insertion

The two-boson exchange amplitudes for gluon matching require the integrals $H(n)$, $F(n)$, $H^{\mu\nu}(n)$, and $H^\mu(n)$ defined in (4.56). Let us parameterize the last two as

$$H^{\mu\nu}(n) = H_1(n) v^\mu v^\nu + H_2(n) g^{\mu\nu}, \quad H^\mu(n) = H_3(n) v^\mu. \tag{B.26}$$

Upon contracting the above expressions with v_μ and $g_{\mu\nu}$, we may solve for the relations

$$H_1(n) = \frac{1}{3-2\epsilon}\left[(4-2\epsilon) v_\mu v_\nu H^{\mu\nu}(n) - H^\mu_{\ \mu}(n)\right],$$
$$H_2(n) = \frac{1}{3-2\epsilon}\left[H^\mu_{\ \mu}(n) - v_\mu v_\nu H^{\mu\nu}(n)\right],$$
$$H_3(n) = v_\mu H^\mu(n). \tag{B.27}$$

Using the identities in (4.54) and (4.57), we further obtain

$$v_\mu H^\mu(n) = \delta H(n) + 2F(n),$$
$$v_\mu v_\nu H^{\mu\nu}(n) = \delta^2 H(n) + 2\delta F(n),$$
$$H^\mu_{\ \mu}(n) = \left[\frac{m_1^2}{x} + \frac{m_2^2}{(1-x)}\right] H(n) - \frac{H(n-1)}{x(1-x)}, \tag{B.28}$$

and hence the boson loops are completely specified by $H(n)$ and $F(n)$. In evaluating these functions it may be advantageous to relate to more basic integrals by means of derivatives. Let us write,

$$H(n) = 2\frac{\partial}{\partial m_V^2}\int (dL)\frac{1}{v\cdot L-\delta+i0}\frac{1}{L^2-m_V^2+i0}\Delta^{-n-\epsilon},$$

$$F(n) = \frac{\partial}{\partial m_V^2}\int (dL)\frac{1}{L^2-m_V^2+i0}\Delta^{-n-\epsilon}, \qquad \text{(B.29)}$$

with Δ as defined in (4.48). The singularity structure and evaluation of the above integrals can be classified into three cases, corresponding to zero, one, or two heavy fermions contributing to the electroweak polarization tensor. For pure states we obtain analytic expressions for all integrals, while for mixed states we encounter several integrals that require numerical evaluation of one Feynman parameter integral.

B.3.1 Case of Zero Heavy Fermions

Upon setting $m_1 = m_2 = 0$ in Δ and performing the integration in $d = 4-2\epsilon$ dimensions, we obtain

$$F(n) = [c_\epsilon]\frac{\Gamma(2-n-2\epsilon)\Gamma(n+2\epsilon)}{\Gamma(2-\epsilon)\Gamma(1+\epsilon)}[x(1-x)]^{-n-\epsilon}m_V^{-2n-4\epsilon},$$

$$H(n) = [c_\epsilon]\frac{4\Gamma(n+2\epsilon)}{\Gamma(n+\epsilon)\Gamma(1+\epsilon)}[x(1-x)]^{-n-\epsilon}\frac{\partial}{\partial m_V^2}I(n), \qquad \text{(B.30)}$$

where

$$I(n) = \int_0^1 dy\,(1-y)^{n-1+\epsilon}\int_0^\infty d\rho(\rho^2+2\rho\delta+ym_V^2-i0)^{-n-2\epsilon}. \qquad \text{(B.31)}$$

We may reduce to the case of $I(1)$ by noticing that

$$\begin{aligned} I(n+1) &= -\frac{m_V^{-2}}{n+2\epsilon}\int_0^1 dy(1-y)^{n+\epsilon}\frac{d}{dy}\int_0^\infty d\rho(\rho^2+2\rho\delta+ym_V^2-i0)^{-n-2\epsilon} \\ &= \frac{m_V^{-2}}{n+2\epsilon}\left[\int_0^\infty d\rho(\rho^2+2\rho\delta-i0)^{-n-2\epsilon}+(n+\epsilon)I(n)\right] \\ &= \frac{m_V^{-2}}{n+2\epsilon}\left[\delta^{1-2n-4\epsilon}\frac{\Gamma(1-n-2\epsilon)\Gamma\left(n-\frac{1}{2}+2\epsilon\right)}{2\sqrt{\pi}}+(n+\epsilon)I(n)\right]. \qquad \text{(B.32)} \end{aligned}$$

Finally, for $I(1)$ we require

$$I(1) = \delta^{-1-4\epsilon} \int_0^1 dy (1 + \epsilon \log(1-y) + \dots) \int_1^\infty d\rho (\rho^2 + \alpha^2)^{-1} \Big(1 - 2\epsilon \log(\rho^2 + \alpha^2) + \dots\Big), \tag{B.33}$$

where $\alpha = \left(y m_V^2/\delta^2 - 1 - i0\right)^{\frac{1}{2}}$. The relevant integrals are

$$\begin{aligned} \int_1^\infty d\rho \frac{1}{\rho^2 + \alpha^2} &= \frac{1}{\alpha} \arctan \alpha, \\ \int_1^\infty d\rho \frac{\log(\rho^2 + \alpha^2)}{\rho^2 + \alpha^2} &= \frac{1}{\alpha} \Bigg[2 \log(2\alpha) \arctan \alpha \\ &\quad - \frac{1}{2i} \left(\mathrm{Li}_2 \left(\frac{1 - i\alpha}{1 + i\alpha} \right) - \mathrm{Li}_2 \left(\frac{1 + i\alpha}{1 - i\alpha} \right) \right) \Bigg]. \end{aligned} \tag{B.34}$$

We perform the remaining integral over Feynman parameter y numerically.

B.3.2 Case of One Heavy Fermion

Let us set $m_1 = M$ (not to be confused with heavy WIMP mass M used elsewhere in the thesis) and $m_2 = 0$ in Δ, and consider separately the finite integrals for a- and c-type contributions, and the IR divergent integrals for b-type contributions.

Finite Integrals for *a*- and *c*-Type Contributions

For the finite a- and c-type contributions we may take $d = 4$. Let us evaluate the required integrals $F(2)$ and $H(1)$, and obtain the remaining integrals by differentiating with respect to M. We find

$$\begin{aligned} F(2) &= \frac{i}{(4\pi)^2} \frac{\partial}{\partial m_V^2} \left\{ \left[x(1-x) m_V \left(1 - \frac{M^2}{x m_V^2} \right) \right]^{-2} \left[-\log \frac{M^2}{x m_V^2} + \frac{M^2}{x m_V^2} - 1 \right] \right\}, \\ H(1) &= \frac{i}{(4\pi)^2} \frac{\partial}{\partial m_V^2} \left\{ 8 \left[x(1-x) m_V^2 \left(1 - \frac{M^2}{x m_V^2} \right) \right]^{-1} \left[\sqrt{m_V^2 - \delta^2} \arctan \right. \right. \\ &\quad \left. \left. \left(\sqrt{\frac{m_V^2}{\delta^2} - 1} \right) - \sqrt{\frac{M^2}{x} - \delta^2} \arctan \left(\sqrt{\frac{M^2}{x\delta^2} - 1} \right) - \frac{\delta}{2} \log \frac{x m_V^2}{M^2} \right] \right\}. \end{aligned} \tag{B.35}$$

The integrals have been obtained by breaking an integration region into pieces, e.g.,

$$\begin{aligned}
&\int_{\delta}^{\infty} d\rho \left[\log(\rho^2 + m_V^2 - \delta^2) - \log\left(\rho^2 + \frac{M^2}{x} - \delta^2\right)\right] \\
&= \lim_{\varepsilon\to 0}\int_{\delta}^{\infty} d\rho \left[\log(\rho^2 + m_V^2 - \delta^2 - i\varepsilon) - \log\left(\rho^2 + \frac{M^2}{x} - \delta^2 - i\varepsilon\right)\right] \\
&= \delta \lim_{\varepsilon\to 0}\int_{1}^{\infty} d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right)\right] \\
&= \delta \lim_{\varepsilon\to 0}\left\{\int_{0}^{\infty} d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right)\right]\right. \\
&\qquad \left. - \int_{0}^{1} d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right)\right]\right\}. \qquad \text{(B.36)}
\end{aligned}$$

Since the original integral is independent of ε, either choice of $\text{sgn}(\varepsilon)$ is correct provided it is used consistently in both terms. The continuation away from $\delta \to 0$ is thus obtained above by taking, e.g., $\delta \to \delta + i\varepsilon$ everywhere. For the evaluation of integrals over x involving $H(1)$, let us write

$$H(1) \equiv 2\frac{\partial}{\partial m_V^2}K(1) \equiv 2\frac{\partial}{\partial m_V^2}\left\{\frac{M^2}{xm_V^2 - M^2}k(1)\right\}. \qquad \text{(B.37)}$$

We then have

$$x^n K(1) = \left(\frac{M^2}{m_V^2}\right)^n K(1) + \frac{\left(\frac{M^2}{m_V^2}\right)^n - x^n}{\frac{M^2}{m_V^2} - x}\frac{M^2}{m_V^2}k(1)\,, \qquad \text{(B.38)}$$

so that all powers $x^n K(1)$ can be reduced to the case $n = 0$, in addition to the remaining straightforward integral involving a polynomial in x times $k(1)$, which in practice is evaluated numerically. The remaining integrals involving $F(2)$ are similarly straightforward to evaluate.

Infrared Divergent Integrals for *b*-Type Contributions

Let us now turn to the integrals for b-type contributions, where we work in $d = 4 - 2\epsilon$ spacetime dimensions to account for singular behavior at the endpoints of the x integration. We find,

$$F(1) = [c_\epsilon][x(1-x)]^{-1-\epsilon}\frac{\Gamma(1+2\epsilon)}{[\Gamma(1+\epsilon)]^2}\Bigg\{m_V^{-2-4\epsilon}\Bigg[\left(r^2-1\right)^{-2}\left(r^2\log r^2 - r^2 + 1\right)$$
$$+\,\epsilon\left(r^2-1\right)^{-2}\left(2r^2\log r^2 - r^2\log^2 r^2 - r^2 + 1 + r^2\mathrm{Li}_2\left(1-r^2\right)\right)\Bigg]$$
$$+\,m_V^{-2}\Bigg[\left(\frac{r^2}{x}-1\right)^{-2}\left(\frac{r^2}{x}\log\frac{r^2}{x} - \frac{r^2}{x} + 1\right) - \left(r^2-1\right)^{-2}$$
$$\left(r^2\log r^2 - r^2 + 1\right)\Bigg]\Bigg\}, \tag{B.39}$$

where $r \equiv M/m_V$. The first term in curly braces is obtained by taking $x = 1$ inside the $\int dy$ integral, and the second term is the remainder having no singularity in the final $\int dx$ integral at $x = 1$.

Similarly we find,

$$H(1) = [c_\epsilon][x(1-x)]^{-1-\epsilon}\frac{4\Gamma(1+2\epsilon)}{[\Gamma(1+\epsilon)]^2}\frac{\partial}{\partial m_V^2}\Bigg\{\delta^{-1-4\epsilon}\Bigg[Y_0(1) + \epsilon\Big(Y_1 + Y_2\Big)\Bigg]$$
$$+\,\delta^{-1}\Bigg[Y_0(x) - Y_0(1)\Bigg]\Bigg\}, \tag{B.40}$$

where

$$Y_0(x) = \frac{2}{r_V^2 - \frac{r_M^2}{x}}\Bigg\{\sqrt{r_V^2-1}\arctan\left(\sqrt{r_V^2-1}\right) - \sqrt{\frac{r_M^2}{x}-1}\arctan\left(\sqrt{\frac{r_M^2}{x}-1}\right)$$
$$-\,\frac{1}{2}\log\frac{xm_V^2}{M^2}\Bigg\}, \tag{B.41}$$

with $r_V \equiv m_V/\delta$ and $r_M \equiv M/\delta$. As in the discussion after (B.36), continuation away from $\delta = 0$ is given by taking $\delta \to \delta + i\varepsilon$ with arbitrary choice of $\mathrm{sgn}(\varepsilon)$. The remaining terms Y_1 and Y_2 are given by

$$Y_1 = \int_0^1 dy\int_0^\infty d\beta\left(r_M^2 - r_V^2\right)^{-1}\frac{d}{dy}\log^2\left[\beta^2 + 2\beta + yr_V^2 + (1-y)r_M^2\right]$$
$$= \left(r_M^2 - r_V^2\right)^{-1}\Bigg\{-4\pi\sqrt{r_V^2-1}\Big[1 - \log\left(2\sqrt{r_V^2-1}\right)\Big]$$
$$+\,4\pi\sqrt{r_M^2-1}\Big[1 - \log\left(2\sqrt{r_M^2-1}\right)\Big] - y_1\left(\sqrt{r_V^2-1}\right) + y_1\left(\sqrt{r_M^2-1}\right)\Bigg\},$$
$$Y_2 = \int_0^1 dy\,\log(1-y)\left(yr_V^2 + (1-y)r_M^2 - 1\right)^{-1}\arctan\left(\sqrt{yr_V^2 + (1-y)r_M^2 - 1}\right), \tag{B.42}$$

where

$$y_1(A) \equiv \int_0^1 dx \log^2(x^2 + A^2)\,. \tag{B.43}$$

For Y_2, we evaluate the remaining integral over Feynman parameter y numerically.

B.3.3 Case of Two Heavy Fermions

Let us set $m_1 = m_2 = M$ (not to be confused with heavy WIMP mass M used elsewhere in the thesis) in Δ, and work in $d = 4$ dimensions. Naming $x(1-x) \equiv z$, we find,

$$\begin{aligned}
F(1) &= \frac{i}{(4\pi)^2}\left[zm_V^2\left(1 - \frac{M^2}{zm_V^2}\right)^2\right]^{-1}\left[\frac{M^2}{zm_V^2}\log\frac{M^2}{zm_V^2} - \frac{M^2}{zm_V^2} + 1\right], \\
H(1) &= \frac{i}{(4\pi)^2}\frac{\partial}{\partial m_V^2}\left\{8\left[zm_V^2\left(1 - \frac{M^2}{zm_V^2}\right)\right]^{-1}\left[\sqrt{m_V^2 - \delta^2}\arctan\left(\sqrt{\frac{m_V^2}{\delta^2} - 1}\right)\right.\right. \\
&\quad \left.\left. - \sqrt{\frac{M^2}{z} - \delta^2}\arctan\left(\sqrt{\frac{M^2}{z\delta^2} - 1}\right) - \frac{\delta}{2}\log\frac{zm_V^2}{M^2}\right]\right\}.
\end{aligned} \tag{B.44}$$

The remaining integrals can be obtained by differentiating the above results with respect to M. In practice, we evaluate the remaining integral over Feynman parameter x (or z) numerically.

B.4 Numerical Inputs

Table B.1 Inputs to the numerical analysis

Parameter	Value	Reference	Parameter	Value	Reference
$\lvert V_{td}\rvert, \lvert V_{ts}\rvert$	$\sim$0	–	m_t	172 GeV	[87]
$\lvert V_{tb}\rvert$	$\sim$1	–	m_b	4.75 GeV	[87]
m_e	0.511 MeV	[13]	m_c	1.4 GeV	[87]
m_μ	106 MeV	[13]	m_s	93.5 MeV	[13]
m_τ	1.78 GeV	[13]	m_d	4.70 MeV	[13]
m_h	126 GeV	[1, 24]	m_u	2.15 MeV	[13]
m_W	80.4 GeV	[13]	c_W	m_W/m_Z	–
m_Z	91.188 GeV	[13]	$\alpha_s(m_Z)$	0.118	[13]

We use the inputs of Table B.1 in the numerical analysis of coefficients appearing in Fig. 4.6. Light fermion masses enter the analysis indirectly via the onshell renormalization scheme. The matching in (4.21) requires a limit of the photon two-point function which receives contributions from momentum regions of light (u, d and s) quark loops that are outside the domain of validity of QCD perturbation theory. A complete nonperturbative treatment of this function is not numerically relevant to the present analysis; for definiteness, we model these contributions using $\overline{\mathrm{MS}}$ light quark masses (cf. Table B.1) in the one-loop evaluation of the two-point function. Varying these mass inputs by an order of magnitude in either direction does not appreciably change the numerical matching coefficients of Fig. 4.6.

Appendix C
Inputs for Analysis of QCD Effects and Hadronic Matrix Elements

C.1 QCD Functions

The QCD beta function β, and the quark mass anomalous dimension γ_m, are defined as

$$\frac{\beta}{g} = \frac{d\log g}{d\log\mu} = -\beta_0\left(\frac{\alpha_s}{4\pi}\right) - \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 - \beta_2\left(\frac{\alpha_s}{4\pi}\right)^3 - \beta_3\left(\frac{\alpha_s}{4\pi}\right)^4 + \dots\,,$$

$$\gamma_m = \frac{d\log m_q}{d\log\mu} = \gamma_0\left(\frac{\alpha_s}{4\pi}\right) + \gamma_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \gamma_2\left(\frac{\alpha_s}{4\pi}\right)^3 + \gamma_3\left(\frac{\alpha_s}{4\pi}\right)^4 \dots\,, \tag{C.1}$$

where the ellipses denote terms higher order in α_s, and the required functions are

$$\begin{aligned}
\beta_0 &= 11 - 0.66667n_f \\
\beta_1 &= 102 - 12.6667n_f \\
\beta_2 &= 1428.50 - 279.611n_f + 6.01852n_f^2 \\
\beta_3 &= 29243 - 6946.30n_f + 405.089n_f^2 + 1.49931n_f^3
\end{aligned} \tag{C.2}$$

and

$$\begin{aligned}
\gamma_0 &= 8 \\
\gamma_1 &= 134.667 - 4.44445n_f \\
\gamma_2 &= 2498 - 292.367n_f - 3.45679n_f^2 \\
\gamma_3 &= 50659 - 9783.04n_f + 141.395n_f^2 + 2.96613n_f^3\,.
\end{aligned} \tag{C.3}$$

M.P. Solon, *Heavy WIMP Effective Theory*, Springer Theses,
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The strong coupling α_s is given as a function of scale as

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 t}\Bigg\{1 - \frac{\beta_1 \log t}{\beta_0^2 t} + \frac{\beta_1^2}{\beta_0^4 t^2}\left[\left[\log t - \frac{1}{2}\right]^2 - \frac{5}{4} + \frac{\beta_0\beta_2}{\beta_1^2}\right] - \frac{1}{\beta_0^6 t^3}\left[\beta_1^3\left[\log^3 t - \frac{5}{2}\log^2 t - 2\log t + \frac{1}{2}\right] + 3\beta_0\beta_1\beta_2 - \frac{1}{2}\beta_0^2\beta_3\right]\Bigg\}, \tag{C.4}$$

with

$$t = \log\left[\frac{\mu^2}{\Lambda(n_f)^2}\right], \quad \Lambda(5) = 0.213066\,, \quad \Lambda(4) = 0.297608\,, \quad \Lambda(3) = 0.339872\,. \tag{C.5}$$

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