Appendix A
Papers in Translation

A.1 Fourier on Fourier Series

[From Fourier, *The analytical Theory of Heat*, tr. A Freeman.]

§220. We see by this that the coefficients a, b, c, d, e, f, &c., which enter into the equation

$$\frac{1}{2}\pi\phi(x) = a\sin x + b\sin 2x + c\sin 3x + d\sin 4x + \&c,$$

and which we found formerly by way of successive eliminations, are the values of definite integrals expressed by the general term $\int \sin kx\phi(x)dx$, k being the number of the term whose coefficient is required. This remark is important, because it shews how even entirely arbitrary functions may be developed in series of sines of multiple arcs. In fact, if the function $\phi(x)$ be represented by the variable ordinate of any curve whatever whose abscissa extends from $x = 0$ to $x = \pi$, and if on the same part of the axis the known trigonometric curve, whose ordinate is $y = \sin x$, be constructed, it is easy to represent the value of any integral term. We must suppose that for each abscissa x, to which corresponds one value of $\phi(x)$, and one value of $\sin x$, we multiply the latter value by the first, and at the same point of the axis raise an ordinate equal to the product $\phi(x)\sin x$. By this continuous operation a third curve is formed, whose ordinates are those of the trigonometric curve, reduced in proportion to the ordinates of the arbitrary curve which represents $\phi(x)$. This done, the area of the reduced curve taken from $x = 0$ to $x = \pi$ gives the exact value of the coefficient of $\sin x$; and whatever the given curve may be which corresponds to $\phi(x)$, whether we can assign to it an analytical equation, or whether it depends on no regular law, it is evident that it always serves to reduce in any manner whatever the trigonometric curve; so that the area of the reduced curve has, in all possible cases, a definite value, which is the value of the coefficient of $\sin x$ in the development of the function. The same is the case with the following coefficient b, or $\int \phi(x)\sin 2x dx$.

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In general, to construct the values of the coefficients a, b, c, d, &c, we must imagine that the curves, whose equations are

$$y = \sin x,\ \ y = \sin 2x,\ \ y = \sin 3x,\ \ y = \sin 4x,\ \ \&c.$$

have been traced for the same interval on the axis of x, from $x = 0$ to $x = \pi$; and then that we have changed these curves by multiplying all their ordinates by the corresponding ordinates of a curve whose equation is $y = \phi(x)$. The equations of the reduced curves are

$$y = \sin x\, \phi(x),\ \ y = \sin 2x\, \phi(x),\ \ y = \sin 3x\, \phi(x), \&c.$$

The areas of the latter curves, taken from $x = 0$ to $x = \pi$, are the values of the coefficients a, b, c, d, &c., in the equation

$$\frac{1}{2}\pi\phi(x) = a \sin x + b \sin 2x + c \sin 3x + d \sin 4x + \&c,$$

§221. We can verify the foregoing equation (D), (Art. 220), by determining directly the quantities a_1, a_2, a_3, a_4, &c.., in the equation

$$\phi(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + a_4 \sin 4x\&c.;$$

for this purpose, we multiply each member of the latter equation by $\sin kxdx$, k being an integer, and take the integral from $x = 0$ to $x = \pi$, whence we have

$$\int \phi(x) \sin kx = a_1 \int \sin x \sin kx + a_2 \int \sin 2x \sin kx + \&c.$$
$$+ a_j \int \sin jx \sin kxdx + \cdots \&c.; \tag{A.1}$$

Now it can easily be proved, 1st, that all the integrals, which enter into the second member, have a nul value, except only the term $a_k \int \sin kx \sin kxdx$; 2nd, that the value of $\int \sin kx \sin kxdx$; $\frac{\pi}{2}$; whence we derive the value of a_j, namely

$$\frac{2}{\pi} \int \phi(x) \sin kxdx.a$$

The whole problem is reduced to considering the value of the integrals which enter into the second member, and to demonstrating the two preceding propositions. The integral

$$2 \int \sin jx \sin kxdx,$$

taken from $x = 0$ to $x = \pi$, in which k and j are integers, is

$$\frac{1}{k-j}\sin(k-j)x - \frac{1}{k-j}\sin(k+j)x + C.$$

Since the integral must begin when $x = 0$ the constant C is nothing, and the numbers k and j being integers, the value of the integral will become nothing when $x = \pi$; it follows that each of the terms, such as

$$a_1 \int \sin x \sin kx, \;\; a_2 \int \sin 2x \sin kx, \;\; a_3 \int \sin 3x \sin kx dx \&c.$$

vanishes, and that this will occur as often as the numbers k and j are different. The same is not the case when the numbers k and j are equal, for the term $\frac{1}{k-j}\sin(k-j)x$ to which the integral reduces, becomes $\frac{0}{0}$, and its value is π. Consequently we have thus we obtain, in a very brief manner, the values of $a_1, a_2, a_3, \ldots, a_j$, &c., namely,

$$a_1 = \tfrac{2}{\pi}\int \phi(x)\sin x dx, \quad a_2 = \tfrac{2}{\pi}\int \phi(x)\sin 2x dx$$
$$a_3 = \tfrac{2}{\pi}\int \phi(x)\sin 3x dx, \quad a_4 = \tfrac{2}{\pi}\int \phi(x)\sin 4x dx$$

Substituting these we have

$$\frac{1}{2}\pi\phi(x) = \sin x \int \phi(x)\sin x dx + \sin 2x \int \phi(x)\sin 2x dx + \&c.$$
$$+ \sin kx \int \phi(x)\sin kx dx + \&c. \tag{A.2}$$

§222. The simplest case is that in which the given function has a constant value for all values of the variable x included between 0 and π; in this case the integral $\int \sin kx dx$ is equal to $\frac{2}{k}$, if the number k is odd, and equal to 0 if the number k is even.

Hence we deduce the equation

$$\frac{1}{4}\pi = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \&c.,$$

which has been found before.

It must be remarked that when a function has been developed in a series of sines of multiple arcs, the value of the series

$$a \sin x + b \sin 2x + c \sin 3x + d \sin 4x + \&c.$$

is the same as that of the function $\phi(x)$ so long as the variable x is included between 0 and π; but this equality ceases in general to hold good when the value of x exceeds the number π.

Suppose the function whose development is required to be x, we shall have, by the preceding theorem,

$$\frac{1}{2}\pi x = \sin x \int x \sin x dx + \sin 2x \int x \sin 2x dx + \&c.$$
$$+ \sin kx \int x \sin kx dx + \&c. \tag{A.3}$$

The integral $\int_0^\pi x \sin kx dx$ is equal to $\pm\frac{\pi}{k}$; the indices 0 and π which are connected with the sign $\int$, shew the limits of the integral; the sign $+$ must be chosen when k is odd, and the sign $-$ when k is even. We have then the following equation,

$$\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \&c.$$

§223. We can develope also in a series of sines of multiple arcs functions different from those in which only odd powers of the variable enter. To instance by an example which leaves no doubt as to the possibility of this development, we select the function $\cos x$, which contains only even powers of x, and which may be developed under the following form:

$$a \sin x + b \sin 2x + c \sin 3x + d \sin 4x + e \sin 5x + \&c.,$$

although in this series only odd powers of the variable enter.

We have, in fact, by the preceding theorem,

$$\frac{1}{2}\pi \cos x = \sin x \int \cos x \sin x dx + \sin 2x \int \cos x \sin 2x dx$$
$$+ \sin 3x \int \cos x \sin 3x dx \&c. \tag{A.4}$$

The integral $\int \cos x \sin kx dx$ is equal to zero when k is an odd number, and to $\frac{2k}{k^2-1}$ when k is an even number. Supposing successively $k = 2, 4, 6, 8$, etc., we have the always convergent series

$$\frac{1}{4}\pi \cos x = \frac{2}{1.3}\sin 2x + \frac{4}{3.5}\sin 4x + \frac{6}{5.7}\sin 6x + \&c.\ ;$$

or,

$$\cos x = \frac{2}{\pi}\left\{\left(\frac{1}{1} + \frac{1}{3}\right)\sin 2x + \left(\frac{1}{3} + \frac{1}{3}\right)\sin 4x + \left(\frac{1}{5} + \frac{1}{7}\right)\sin 6x + \&c.\right\}.$$

This result is remarkable in this respect, that it exhibits the development of the cosine in a series of functions, each one of which contains only odd powers. If in the preceding equation x be made equal to $\frac{1}{4}\pi$, we find

$$\frac{1}{4}\frac{\pi}{\sqrt{2}} = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \&c.\right).$$

This series is known ([Euler] *Introd. ad analysin. infinit.* cap. x.).

[In §224 Fourier showed that "a similar analysis may be employed for the development of any function whatever in a series of cosines of multiple arcs." He concluded that "This and the preceding theorem suit all possible functions, whether their character can be expressed by known methods of analysis, or whether they correspond to curves traced arbitrarily."]

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§230. If we apply these principles to the problem of the motion of vibrating strings, we can solve difficulties which first appeared in the researches of Daniel Bernoulli. The solution given by this geometrician assumes that any function whatever may always be developed in a series of sines or cosines of multiple arcs. Now the most complete of all the proofs of this proposition is that which consists in actually resolving a given function into such a series with determined coefficients.

In researches to which partial differential equations are applied, it is often easy to find solutions whose sum composes a more general integral; but the employment of these integrals requires us to determine their extent, and to be able to distinguish clearly the cases in which they represent the general integral from those in which they include only a part. It is necessary above all to assign the values of the constants, and the difficulty of the application consists in the discovery of the coefficients. It is remarkable that we can express by convergent series, and, as we shall see in the sequel, by definite integrals, the ordinates of lines and surfaces which are not subject to a continuous law. We see by this that we must admit into analysis functions which have equal values, whenever the variable receives any values whatever included between two given limits, even though on substituting in these two functions, instead of the variable, a number included in another interval, the results of the two substitutions are not the same. The functions which enjoy this property are represented by different lines, which coincide in a definite portion only of their course, and offer a singular species of finite osculation. These considerations arise in the calculus of partial differential equations; they throw a new light on this calculus, and serve to facilitate its employment in physical theories.

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§235. It follows from that which has been proved in this section, concerning the development of functions in trigonometrical series, that if a function $f(x)$ be proposed, whose value in a definite interval from $x = 0$ to $x = X$ is represented by the ordinate of a curved line arbitrarily drawn; we can always develope this function in a series which contains only sines or only cosines, or the sines and cosines of multiple arcs, or the cosines only of odd multiples. [...] These trigonometric series, arranged according to cosines or sines of multiples of arcs, belong to elementary analysis, like the series whose terms contain the successive powers of the variable. The coefficients of the trigonometric series are definite areas, and those of the series of powers are functions given by differentiation, in which, moreover, we assign to the variable a definite value. We could have added several remarks concerning the use and properties of trigonometrical series; but we shall limit ourselves to enunciating briefly those which have the most direct relation to the theory with which we are concerned.

1st. The series arranged according to sines or cosines of multiple arcs are always convergent; that is to say, on giving to the variable any value whatever that is not imaginary, the sum of the terms converges more and more to a single fixed limit, which is the value of the developed function.

Fourier's Proofs

Fourier deferred the proofs of his theorems largely to the end of his book, when he availed himself of infinitely large and infinitely small quantities, writing (in §417) of one proof that

> it supposes that notion of infinite quantities which has always been admitted by geometers. It would be easy to offer the same proof under another form, examining the changes which result from the continual increase of the factor p under the symbol $\sin p(\alpha - x)$. These considerations are too well known to make it necessary to recall them.

This was the proof of equation (B), which he went on to turn into the equation (B') he referred to in the extract that follows:

$$f(x)\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha f(\alpha)\frac{2\sin p(\alpha - x)}{\alpha - x}, \quad (p = \infty).$$

He added

> Above all, it must be remarked that the function $f(x)$, to which this proof applies, is entirely arbitrary , and not subject to any continuous law.

418. The theorem expressed by Eq. (II) Art. 234 must be considered under the same point of view. This equation serves to develope an arbitrary function $f(x)$ in a series of sines or cosines of multiple arcs. The function $f(x)$ denotes a function completely arbitrary, that is to say a succession of given values, subject or not to a common law, and answering to all the values of x included between 0 and any magnitude X.

The value of this function is expressed by the following equation,

$$f(x) = \frac{1}{2\pi}\sum\int_{a}^{b} d\alpha f(\alpha)\cos\frac{2ix}{X}(x - \alpha) \qquad (A).$$

423. The propositions expressed by Eqs. (A.) and (B'), Arts. 418 and 417, may be considered under a more general point of view. The construction indicated in Arts. 415 and 416 applies not only to the trigonometrical function

$$\frac{\sin(p\alpha - px)}{\alpha - x};$$

but suits all other functions, and supposes only that when the number p becomes infinite, we find the value of the integral with respect to α, by taking this integral between extremely near limits. Now this condition belongs not only to trigonometrical functions, but is applicable to an infinity of other functions. We thus arrive at the expression of an arbitrary function $f(x)$ under different very remarkable forms; but we make no use of these transformations in the special investigations which occupy us.

With respect to the proposition expressed by Eq. (A), Art. 418, it is equally easy to make its truth evident by constructions, and this was the theorem for which we employed them at first, It will be sufficient to indicate the course of the proof.

In Eq. (A), namely,

$$f(x) = \frac{1}{2\pi}\int_{-X}^{+X} d\alpha f(\alpha) \sum_{-\infty}^{+\infty} \cos 2ix \frac{\alpha - x}{X} ;$$

we can replace the sum of the terms arranged under the sign $\sum$ by its value, which is derived from known theorems. We have seen different examples of this calculation previously, Sect. III, Chap. III. It gives as the result if we suppose, in order to simplify the expression, $2\pi = X$, and denote α by r,[1]

$$\sum_{-j}^{+j} \cos jr = \cos jr + \sin jr \frac{\sin r}{\operatorname{versin} r}.$$

We must then multiply the second member of this equation by $d\alpha f(\alpha)$, suppose the number j infinite, and integrate from $\alpha = -\pi$ to $\alpha = +\pi$. The curved line, whose abscissa is α and ordinate cos jri being conjoined with the line whose abscissa is α and ordinate $f(\alpha)$, that' is to say, when the corresponding ordinates are multiplied together, it is evident that the area of the curve *produced*, taken between any limits, becomes nothing when the number j increases without limit. Thus the first term $\cos jr$ gives a nul result.

The same would be the case with the term $\sin jr$, if it were not multiplied by the factor $\frac{\sin r}{\operatorname{versin} r}$; but on comparing the three curves which have a common abscissa α, and as ordinates $\sin jr$, $\frac{\sin r}{\operatorname{versin} r}$, $f\alpha)$, we see clearly that the integral

$$\int d\alpha f(\alpha) \sin jr \frac{\sin r}{\operatorname{versin} r}$$

has no actual values except for certain intervals infinitely small, namely when the ordinate $\frac{\sin r}{\operatorname{versin} r}$ becomes infinite. This will take place if r or $\alpha - x$ is nothing; and

[1] $\operatorname{versin} r = 1 - \cos r$.

in the interval in which α differs infinitely little from x, the value of $f(\alpha)$ coincides with $f(x)$/. Hence the integral becomes

$$2f(x)\int_{)}^{\infty} dr \sin jr \frac{r}{\frac{1}{2}r^2}, \text{ or } \frac{1}{2}f(x)\int_{0}^{\infty} \frac{dr}{r} \sin jr\,,$$

which is equal to $2\pi f(x)$, Arts. 415 and 356. Whence we conclude the previous Eq. (A).

When the variable x is exactly equal to $-\pi$ or $+\pi$, the construction shews what is the value of the second member of the Eq. (A), $[\frac{1}{2}f(-\pi)$ *or* $\frac{1}{2}f(\pi)]$.

If the limits of integrations are not $-\pi$ and $+\pi$, but other numbers a and b, each of which is included between $-\pi$ and $+\pi$, we see by the same figure what the values of x are, for which the second member of Eq. (A) is nothing.

If we imagine that between the limits of integration certain values of $f(\alpha)$ become infinite, the construction indicates in what sense the general proposition must be understood. But we do not here consider cases of this kind, since they do not belong to physical problems.

If instead of restricting the limits $-\pi$ and $+\pi$, we give greater extent to the integral, selecting more distant limits a' and b', we know from the same figure that the second member of equation (A) is formed of several terms and makes the result of integration finite, whatever the function $f(x)$ may be.

We find similar results if we write $2\pi\frac{\alpha-x}{X}$ instead of r, the limits of integration being $-X$ and $+X$.

It must now be considered that the results at which we have arrived would also hold for an infinity of different functions of $\sin jr$. It is sufficient for these functions to receive values alternately positive and negative, so that the area may become nothing when j increases without limit. We may also vary the factor $\frac{\sin r}{\text{versin}\, r}$ as well as the limits of integration, and we may suppose the interval to become infinite. Expressions of this kind are very general, and susceptible of very different forms. We cannot delay over these developments, but it was necessary to exhibit the employment of geometrical constructions; for they solve without any doubt questions which may arise on the extreme values, and on singular values; they would not have served to discover these theor.ems, but they prove them and guide all their applications.

A.2 Dirichlet on Fourier Series

Dirichlet, On the convergence of trigonometric series which serve to represent an arbitrary function between two given limits. *Journal für die reine und angewandte Mathematik*, 1829, 4, 157–169, in *Werke*, I, 117–133.

The series of sines and cosines, by means of which one can represent an arbitrary function in a given interval, enjoy among other remarkable properties that of being convergent. This property has not escaped the illustrious geometer who has opened a new domain in the applications of analysis, in introducing there the manner of expressing arbitrary functions that is under consideration; it is found in the Memoir which contains his first researches on heat. But no one, as far as I know, has given a general demonstration. I only know one work on this topic, which is by M. Cauchy and which appeared in the Memoirs of the Paris Academy of Sciences for the year 1823. The author of this work himself admitted that his proof was defective for certain functions for which however the convergence was indubitable. A close examination of the said memoir has led me to believe that the demonstration which is expounded there is not even sufficient in the cases to which the author believes it is applicable. Before entering into this matter I am going to state in a few words the objections to which the proof of M. Cauchy appears to me to be subject. The path which this famous geometer has taken in this research requires that one considers that values of the function $\phi(x)$ that one wishes to develop, when one replaces the variable x in it by a quantity of the form $u + v\sqrt{-1}$. The consideration of these values seems foreign to the question, and in any case one does not easily see what one must understand by the result of a similar substitution when the function one is given cannot be expressed by an analytic formula. I give this objection with even more confidence because the author seems to share my opinion on this point. In fact, in several of his works he insists on the necessity of defining in a precise manner the sense one attaches to a similar substitution even when this is done to a function with a regular analytic law. Above all one finds in the Memoir which is published in volume 19 of the *Journal de l'École Polytechnique* page 567 and following, some remarks on the difficulties that imaginary quantities give rise to when placed in the signs of arbitrary functions. What ever one makes of this first observation, there is still another objection to the demonstration that M. Cauchy gives that seems to leave no doubt about its insufficiency. The consideration of imaginary quantities leads the author to a result on the decrease of terms in a series that is far from proving that the terms form a convergent sequence. The result under considerataion can be stated as follows, supposing that the interval considered extends from zero to π.

"The ratio of the term of rank n to the quantity $A\frac{\sin nx}{n}$ (A being a determinate constant, depending on the extreme values of the function) differs from unity taken positively by a quantity which diminishes indefinitely as n becomes greater."

From this result and from the fact that the series which has $A\frac{\sin nx}{n}$ as its general term is convergent, the author concludes that the general trigonometric series is too. But this conclusion is not permissible, because it is easy to arrange that two series (at least when, as happens here, the terms do not always have the same sign) can be the one convergent and the other divergent, although the ratio of the two terms of the same rank differs as little as one wishes from unity taken positively when the terms are of a very high rank.

One sees a very simple example in the two series, the one having the general terms $\frac{(-1^n}{\sqrt{n}}$ and the other $\frac{(-1^n)}{\sqrt{n}}\left(1+\frac{(-1^n}{\sqrt{n}}\right)$. The first of these series is convergent, the second on the contrary is divergent, because on subtracting the first from it one obtains the divergent series

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\cdots$$

and however the ratio of the corresponding two terms, which is $1 \pm \frac{1}{\sqrt{n}}$, converges to unity as n increases.

I am now going to enter into the matter, beginning with an examination of the simplest cases, to which all the others can be reduced. Let us denote by h a positive number less than or at most equal to $\pi/2$ and by $f(\beta)$ a function of β that is continuous between the limits 0 and h; I understand by that a function that has a finite and definite value for every value of β between 0 and h, and is also such that the difference $f(\beta+\varepsilon) - f(\beta)$ decreases without limit when ε becomes smaller and smaller. Suppose finally that the function is always positive between 0 and h and that it decreases constantly from 0 to h in such a way that if p and q are two numbers between 0 and h, $f(p) - f(q)$ always has the opposite sign to $p - q$. That done, let us consider the integral

$$\int_0^h \frac{\sin k\beta}{\sin \beta} f(\beta) d\beta \qquad (1)$$

in which k is a positive quantity, and let us see what this integral becomes as k increases.[2] To do this, we divide it into several others, the first taken from $\beta = 0$ to $\beta = \pi/k$, the second from $\beta = \pi/k$ to $\beta = 2\frac{\pi}{k}$, and so on, the penultimate one having the limits $\frac{(r-1)\pi}{k}$ and $r\frac{\pi}{k}$, and the last the limits $r\frac{\pi}{k}$ and h, $r\frac{\pi}{k}$ denoting the greatest multiple of π/k which is contained in h. It is easy to see that these new integrals, which are $r+1$ in number, are alternately positive and negative, the function under the integral sign evidently being positive between the first limits, negative between the limits of the second, and so on. It is no less easy to convince oneself that each one is smaller than the one before, abstraction being made of sign.[3] In fact ν denoting an integer less than r, the expressions:

$$\int_{(\nu-1)\pi/k}^{\nu\pi/k} \frac{\sin k\beta}{\sin \beta} f(\beta) d\beta, \quad \int_{\nu\pi/k}^{(\nu+1)\pi/k} \frac{\sin k\beta}{\sin \beta} f(\beta) d\beta$$

represent two consecutive integrals. In the second, let us replace β by $\pi/k + \beta$; the integral then changes into this:

$$\int_{(\nu-1)\pi/k}^{\nu\pi/k} \frac{\sin(k\beta+\pi)}{\sin(\beta+\pi/k)} f(\beta + \frac{\pi}{k}) d\beta$$

or, which comes to the same thing

$$-\int_{(\nu-1)\pi/k}^{\nu\pi/k} \frac{\sin k\beta}{\sin(\beta+\pi/k)} f(\beta+\pi/k) d\beta.$$

[2]Dirichlet wrote i for k.

[3]The 19th Century phrase for 'in absolute value'.

The two integrals which we thus wish to compare having therefore the same limits, one sees without difficulty that the second has a numerical value less than the first. For that it is sufficient to note that it follows from the supposition made about the function $f(\beta)$:

$$f\left(\frac{\pi}{k}+\beta\right) < f(\beta),$$

and that on the other hand

$$\sin\left(\frac{\pi}{k}+\beta\right) > \sin\beta,$$

the arcs β and $\frac{\pi}{k}+\beta$ each being less that $\pi/2$, from which the inequality follows that:

$$\frac{f(\beta)}{\sin\beta} > \frac{f(\beta+\frac{\pi}{k})}{\sin(\beta+\pi/k)},$$

which holding for all values of β between the limits $(\nu-1)\frac{\pi}{k}$ and $\nu\pi/k$ shows, as we have said, that each integral is bigger than the one before, abstraction being made of sign. This circumstance holds a fortiori when one compares the penultimate one with the last one, noting that the difference of the limits $r\frac{\pi}{k}$ and h of the last is less than π/k, the common difference of all the others.

Now let us examine in a little more detail the integral of rank ν, which is:

$$\int_{(\nu-1)\pi/k}^{\nu\pi/k}\frac{\sin k\beta}{\sin\beta}f(\beta)d\beta.$$

as the function of β which is found under the integral sign is a product of factors $\frac{\sin k\beta}{\sin\beta}$ and $f(\beta)$, which are both continuous functions of β between the limits of integration, and as on the other hand the first of these factors always has the same sign between these same limits, one concludes, in virtue of a known theorem, that the integral under consideration is equal to the integral of the first factor multiplied by a quantity lying between the greatest and the least value of the other factor. As the second factor decreases from the first limit to the second, the quantity under consideration lies between $f((\nu-1)\frac{\pi}{k})$ and $f(\nu\pi/k)$. Denoting this by ρ_ν, our integral will be equivalent to:

$$\rho_\nu\int_{(\nu-1)\pi/k}^{\nu\pi/k}\frac{\sin k\beta}{\sin\beta}d\beta.$$

the integral which is still contained in this expression depends both on ν and k. It is positive or negative according as $\nu-1$ is even or odd; we shall nonetheless denote it by K_ν, abstraction being made of sign. We shall however need to know the limit to which this converges when, ν remaining constant, k becomes greater and greater.

To discover this limit, let us replace β by γ/k, γ being a new variable. We shall then have:

$$\int_{(\nu-1)\pi}^{\nu\pi} \frac{\sin\gamma}{\sin(\gamma/k)} d\gamma/k.$$

In this form, it is evident that it converges to the limit

$$\int_{(\nu-1)\pi}^{\nu\pi} \frac{\sin\gamma}{\gamma} d\gamma,$$

which, to abbreviate, we denote by k_ν, abstraction being made of sign.

One knows that the integral $\int_0^\infty \frac{\sin\gamma}{\gamma}$ has a finite value and is equal to $\pi/2$. This integral can be divided into an infinity of others, the first taken from $\gamma = 0$ to $\gamma = \pi$, the second from $\gamma = \pi$ to $\gamma = 2\pi$, and so on. These new integrals are alternately positive and negative, each of them has a numerical value less than the one before, and that of rank ν is k_ν, abstraction being made of sign. The proposition we have cited reduces therefore to saying that the infinite sequence:

$$k_1 - k_2 + k_3 - k_4 + k_5 - etc. \quad (2)$$

is convergent and has a sum equal to $\frac{\pi}{2}$.

The terms of this sequence always decrease, and it follows from a known proposition that the sum of the first n terms is greater or less than $\pi/2$ according as n is odd or even, and that this sum, which one can denote by S_n, differs from $\pi/2$ by a quantity less than the following term k_{n+1}.

Let us now return to integral (1) and seek to find the limit to which it converges as k increases indefinitely. In making the number k thus increase, the integrals into which we have decomposed integral (1) change their values continually and at the same time their number will increase; it is necessary to know the result of this double change when it is continued indefinitely. For this we take an integer m (which we shall suppose is even for simplicity) and suppose that m remains fixed while k increases. The number r, which increases continually with k will eventually finish by surpassing the fixed number m, however large it is chosen.

This done, we divide the integrals into two groups whose sum is equivalent to integral (1). The first group will contain the first m of these integrals, the second will be composed of all that follow. One has, for the sum of the first group

$$K_1\rho_1 - K_2\rho_2 + K_3\rho_3 - K_4\rho_4 + \cdots - K_m\rho_m \quad (3)$$

and the second, the number of whose terms increases continually with k has for its first terms:

$$K_{m+1}\rho_{m+1} - K_{m+2}\rho_{m+2} + \cdots \qquad (4)$$

Let us consider these two groups separately. As the number k increases indefinitely, the sum (3) converges to a limit that it is easy to determine. In fact, the quantities $\rho_1, \rho_2, \ldots, \rho_m$, which are contained the first between $f(0)$ and $f(\frac{\pi}{k})$, the second between $f(\frac{\pi}{k})$ and $f(2\frac{\pi}{k})$, and the last between $f((m-1)\frac{\pi}{k})$ and $f(m\frac{\pi}{k})$ each converge to the limit $f(0)$ as, m remaining fixed, k increases without limit. On the other hand we have seen that the quantities:

$$K_1, K_2, \ldots, K_m$$

converge in the same circumstances to the respective limits:

$$k_1, k_2, \ldots, k_m.$$

Therefore the sum (3) converges to the limit:

$$(k_1 - k_2 + k_3 - \cdots - k_m)\, f(0) = S_m f(0),$$

which is to say that the difference between the sum (3) and $S_m f(0)$ will always finish, abstraction being made of sign, by being constantly less that ω, ω denoting a positive quantity as small as one wishes.

Let us now consider the sum (4) the number of terms of which increases continually. These terms being alternately positive and negative and each of them having a numerical value less than the term before, as we have seen above, and considering the integrals that the terms represent, if follows from a known principle[4] that this sum, whatever the number of terms, is positive like its first term $K_{m+1}\rho_{m+1}$and has a value less than this term. Now, this first term converging to the limit $k_{m+1} f(0)$, it follows that the sum (4) will always be less than $k_{m+1} f(0)$ increased by a positive quantity ω' as small as one wishes. In combining this result with the one we obtained for the sum (3) only a moment ago, one sees that integral (1), which is the sum of expressions (3) and (4) will always finish by differing from $S_m f(0)$ by a quantity less, abstraction being made of sign, than $\omega + \omega' + k_{m+1} f(0)$, ω and ω' being quantities of arbitrary smallness. On the other hand S_m differs from $\frac{\pi}{2}$ by a quantity numerically less than $k_m + 1$; therefore the integral will always finish by differing from $\frac{\pi}{2} f(0)$ by a quantity less than $\omega + \omega' + 2k_{m+1} f(0)$, abstraction being made of sign.

As m can be chosen so great that k_{m+1} will be less than any given quantity, it follows that integral (1) will always finish when k increases without limit, by constantly differing from $(\pi/2) f(0)$ by a quantity less, abstraction being made of sign, than any number as small as you wish. It is therefore proved that integral (1) converges to the limit $(\pi/2) f(0)$ for increasing values of k.

Let us now suppose that the function $f(\beta)$, instead of always decreasing from 0 to h, is constant and equal to unity. In this case one can determine the limit to which integral (1) converges by the same considerations as we have already used; one can

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[4]The principle that we are going to apply can be stated in this way. The letters $A, A', A'', \ldots$ denoting an arbitrary number of positive quantities such that $A > A' > A''$ etc., the quantity $A - A' + A'' - A'''+$ etc. is positive and less than A. This follows immediately from the fact that the preceding quantity can always be put in one of the two following forms: $(A - A') + (A'' - A''') + \cdots$ or $A - (A' - A'') - (A''' - A^{iv}) + \cdots$.

see this at once by recalling that the preceding demonstration is based on the fact that the integrals into which we have decomposed integral (1) form a decreasing sequence. Now this decrease depends on two things, the decrease in the factor $f(\beta)$ and the increase in the divisor $\sin\beta$. If $f(\beta)$ becomes a constant number, the increase in $\sin\beta$ will always suffice to make each integral in the series smaller than the one before. One thus finds, always supposing h to be positive and at most equal to $\frac{\pi}{2}$ that the integral $\int_0^h \frac{\sin k\beta}{\sin\beta}d\beta$ converges to the limit $\frac{\pi}{2}$. It follows that the integral $\int_0^h c\frac{\sin k\beta}{\sin\beta}d\beta$, in which c is a positive or negative constant, converges to the limit $c\frac{\pi}{2}$.

We have supposed that the function $f(\beta)$ was decreasing and positive between the limits 0 and h. The first circumstance continuing to hold, that is to say the function being such that $f(p) - f(q)$ has the opposite sign to $p - q$ for values of p and q between 0 and h, let us suppose that $f(\beta)$ is not always positive. One can take a positive constant c so great that $c + f(\beta)$ always takes a positive sign from $\beta = 0$ to $\beta = h$. The integral $\int_0^h \frac{\sin k\beta}{\sin\beta}d\beta$ being equal to the difference of these:

$$\int_0^h (c + f(\beta))\frac{\sin k\beta}{\sin\beta}d\beta - \int_0^h c\frac{\sin k\beta}{\sin\beta}d\beta$$

the limit will be the difference of the limits to which these latter converge. Now these latter belong to the case previously examined ($c + f(\beta)$ being a decreasing and positive function) and converge to the limits $[c + f(0)]\frac{\pi}{2}$ and $c\frac{\pi}{2}$, whence the first converges to the limit $\frac{\pi}{2}f(0)$.

Let us now consider a function $f(\beta)$ increasing from 0 to h. In this case $-f(\beta)$ will be a decreasing function. The integral $\int_0^h -\frac{\sin k\beta}{\sin\beta}d\beta$ will therefore converge to the limit $-\frac{\pi}{2}f(0)$, and consequently the integral $\int_0^h \frac{\sin k\beta}{\sin\beta}d\beta$ to the limit $\frac{\pi}{2}f(0)$.

Putting these results together. One has this statement: Whatever be the function $f(\beta)$, provided that it remains continuous between the limits 0 and h (h being positive and at most equal to $\frac{\pi}{2}$), and whether it increases or decreases from the first of these limits to the second, the integral $\int_0^h f(\beta)\frac{\sin k\beta}{\sin\beta}d\beta$ will finish by constantly differing from $\frac{\pi}{2}f(0)$ by a quantity less than any number assignable when one lets k increase beyond any positive limit.

Let us denote by g a positive number different from 0 and less than h, and let us suppose that the function remains continuous and increases or decreases from g to h. The integral $\int_0^h f(\beta)\frac{\sin k\beta}{\sin\beta}d\beta$ will then converge to a limit that it is easy to discover. One can proceed by considerations analogous to those that we have applied to integral (1); but it is simpler to reduce this new case to those we have considered in what has preceded. The function being given only from g to h it is entirely arbitrary for values of β between 0 and g. Suppose that one understands by $f(\beta)$, for values of β between 0 and g, a continuous function that increases or decreases from 0 to g according as $f(\beta)$ increases or decreases from g to h; suppose moreover that $f(g-\varepsilon)$ differs infinitely little from $f(g + \varepsilon)$ if ε decreases without limit; having satisfied these conditions in an arbitrary way, which one can always do in infinitely many ways, the function $f(\beta)$ satisfies from 0 to h the conditions expressed in statement (5). The integrals:

$$\int_0^g f(\beta)\frac{\sin k\beta}{\sin\beta}d\beta \quad \int_0^h f(\beta)\frac{\sin k\beta}{\sin\beta}d\beta$$

will therefore converge, both of them, to the limit $\frac{\pi}{2}f(0)$. Whence one concludes that the integral $\int_g^h f(\beta)\frac{\sin k\beta}{\sin\beta}d\beta$, which is the difference of the preceding, has zero as its limit.

This new result can be put together in a single statement with the one we have obtained above. One will then have: The letter h denoting a positive quantity at most equal to $\frac{\pi}{2}$ and g being a quantity that is also positive and in addition less than h, the integral

$$\int_g^h f(\beta)\frac{\sin k\beta}{\sin\beta}d\beta$$

in which the function $f(\beta)$ is continuous between the limits of integration and is either always increasing or always decreasing from $\beta = g$ to $\beta = h$, will converge to a certain limit when the number k becomes greater and greater. This limit is equal to zero except in the case when g has the value null, in which case it has the value $\frac{\pi}{2}f(0)$.

It is evident that this result will only be lightly modified if the function $f(\beta)$ interrupts the continuity at $\beta = g$ or $\beta = h$, that is to say if $f(g)$ were different from $f(g+\varepsilon)$ and $f(h)$ from $f(h-\varepsilon)$, ε denoting an infinitely small positive quantity, provided that the values $f(g)$ and $f(h)$ do not become infinite. It is only necessary in this case to replace $f(0)$ by $f(\varepsilon)$ in the preceding statement, considering that $f(\varepsilon)$ is equal to $f(0)$.

We are now ready to prove the convergence of periodic series which express arbitrary functions between given limits. The path which we shall follow will lead us to establish the convergence and at the same time to determine their values. Let $\phi(x)$ be a function of x, having a finite and definite value for each value of x lying between $-\pi$ and π, and suppose that it is required to develop this function in a series of sines and cosines of multiple arcs of x. The series which solves this question is, as one knows:

$$\frac{1}{2\pi}\int \phi(\alpha)d\alpha + \frac{1}{\pi}\left\{\begin{array}{l}\cos x\int\phi(\alpha)\cos\alpha d\alpha + \cos 2x\int\phi(\alpha)\cos 2\alpha d\alpha + \cdots \\ \sin x\int\phi(\alpha)\sin\alpha d\alpha + \sin 2x\int\phi(\alpha)\sin 2\alpha d\alpha + \cdots\end{array}\right\}, \quad (7)$$

the integrals which determine the constant coefficients being taken from $\alpha = -\pi$ to $\alpha = \pi$ and x denoting an arbitrary quantity lying between $-\pi$ and π. (*Théorie de la Chaleur* No. 232 ff.).

Let us consider the first $2n + 1$ terms of this series (n being an integer) and see to what limit the sum of these terms converges when n becomes greater and greater. This sum can be put in the following form:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(\alpha)d\alpha\left[\frac{1}{2}+\cos(\alpha-x)+\cos 2(\alpha-x)+\cdots+\cos n(\alpha-x)\right],$$

or, on summing the sequence of cosines:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(\alpha)\frac{\sin(n+\frac{1}{2})(\alpha-x)}{2\sin\frac{1}{2}(\alpha-x)}d\alpha. \quad (8)$$

All now reduces to determining the limit to which this integral continually approaches when n increases indefinitely. For this we divide it into two parts, the one taken from $-\pi$ to x, the other from x to π. If one replaces α in the first one by $x - 2\beta$, and in the second α by $x + 2\beta$, β being a new variable the two integrals change into these, abstraction being made of the factor $\frac{1}{\pi}$:

$$\int_{0}^{(\pi+x)/2}\frac{\sin(2n+1)\beta}{\sin\beta}\phi(x-2\beta)d\beta,$$

$$\int_{0}^{(\pi-x)/2}\frac{\sin(2n+1)\beta}{\sin\beta}\phi(x+2\beta)d\beta. \quad (9)$$

Let us consider the second of these two integrals. The quantity x being less than or equal to π, abstraction being made of sign, $(\pi - x)/2$ cannot fall outside the limits 0 and π. If $(\pi - x)/2 = 0$, which happens when $x = \pi$, the integral is null whatever n is; in all other cases it will converge for increasing values of n to a limit that we are going to determine. Let us suppose first of all that $(\pi - x)/2$ is less than or at most equal to $\frac{\pi}{2}$, and let us note that the function $\phi(x + 2\beta)$ can have several breaks in continuity from $\beta = 0$ to $\beta = (\pi - x)/2$, and that it can also have several maxima and minima in this same interval. Let us denote by $\ell, \ell', \ell'', \ldots, \ell^{(\nu)}$, ranged in order of size, the different values of β which arise at one or other of those circumstances, and let us decompose our integral into several others taken between the limits:

$$0 \text{ and } \ell,\ \ell \text{ and } \ell',\ \ell' \text{ and } \ell'', \ldots, \ell^{(\nu)} \text{ and } \frac{1}{2}(\pi - x).$$

All these integrals occur in the case of statement (6). They all therefore converge to the limit zero as n increases, with the exception of the first which converges to the limit $\frac{1}{2}\phi(x + \varepsilon)$, ε being an infinitely small positive number.

If $\frac{1}{2}(\pi - x)$ is greater than $\frac{\pi}{2}$, which will happen when x has a negative value, one divides the integral into two others, one taken from $\beta = 0$ to $\beta = \frac{\pi}{2}$, the other from $\beta = \frac{\pi}{2}$ to $\beta = \frac{1}{2}(\pi - x)$. The first of these two integrals is one that we have already considered, it will therefore converge to the limit $\frac{\pi}{2}\phi(x+\varepsilon)$. As for the second, it can be changed into this, on replacing β by $\pi-\gamma$, γ being a new variable:

$$\int_{(\pi+x)/2}^{\frac{\pi}{2}} \phi(x+2\pi-2\gamma)\frac{\sin(2n+1)(\pi-\gamma)}{\sin(\pi-\gamma)}d\gamma,$$

or, which comes to the same thing, n being an integer:

$$\int_{(\pi+x)/2}^{\frac{\pi}{2}} \phi(x+2\pi-2\gamma)\frac{\sin(2n+1)(\gamma)}{\sin(\gamma)}d\gamma.$$

It thus has a form analogous to the preceding, and decomposing it like the preceding into several others, one sees that it will converge to the limit zero, the sole case excepted where $\frac{1}{2}(\pi+x)$ has the value null, that is to say when $x=-\pi$; in this case it continually approaches the limit $\left[\frac{\pi}{2}\right]\phi(\pi-\varepsilon)$, ε always having the same signification.[5] In summarising all that has gone before, we find that the second integral of (9) is zero when $x=\pi$, that it converges to the limit $\frac{\pi}{2}[\phi(\pi-\varepsilon)+\phi(-\pi+\varepsilon)]$ when $x=-\pi$, and that in all other cases it continually approaches the limit $\frac{\pi}{2}\phi(x+\varepsilon)$.

The first of the integrals (9) is entirely analogous to the second; on applying similar considerations to it, one finds that it is null when $x=-\pi$, that it converges to the limit $\frac{\pi}{2}[\phi(\pi-\varepsilon)+\phi(-\pi+\varepsilon)]$ when $x=\pi$ and that in all other cases it has the limit $\frac{\pi}{2}\phi(x-\varepsilon)$. Knowing in this way the limits of each of the integrals (9), it is easy to find the limit that integral (8) continually approaches when n becomes greater and greater. It is sufficient for this to recall that each integral is equal to the sum of the integrals (9) divided by π. Now, integral (8) being equivalent to the sum of the first $2n+1$ terms of the series (7), the convergence of this series is proved, and one finds by means of the preceding results that it is equal to

$$\frac{1}{2}[\phi(x-\varepsilon)+\phi(x+\varepsilon)]$$

for every value of x between $-\pi$ and π, and that for each of the extreme values *pi* and π it is equal to

$$\frac{1}{2}[\phi(\pi-\varepsilon)+\phi(-\pi+\varepsilon)].$$

The preceding exposition embraces all cases; it simplifies when the value of x is not one of those that presents a break in continuity. In fact, the quantities $\phi(x+\varepsilon)$ and $\phi(x-\varepsilon)$ then each being equivalent to $\phi(x)$, one sees that the series has the value $\phi(x)$.

The preceding considerations prove in a rigorous manner that, if the function $\phi(x)$, whose values are supposed finite and definite, has only a finite number of breaks of continuity between the limits $-\pi$ and π, and if moreover it has only a definite number

[5]The factor $\left[\frac{\pi}{2}\right]$ was omitted by Dirichlet.

of maxima and minima between these same limits, then series (7), whose coefficients are definite integrals depending on the function $\phi(x)$, is convergent and generally has the value expressed by

$$\frac{1}{2}[\phi(x-\varepsilon)+\phi(x+\varepsilon)],$$

where ε denotes an infinitely small number. It remains to consider the case where the assumptions that we have made about the number of breaks in continuity and on the natural number of maxima and minima cease to hold. These singular cases can be reduced to those that we have considered. It is only necessary for series (8) to make sense when the breaks in continuity are infinite in number, that the function $\phi(x)$ satisfies the following condition.

It is then necessary that the function $\phi(x)$ be such that, if one denotes by a and b two arbitrary quantities lying between $-\pi$ and π, one can always find between a and b two quantities r and s so close that the function is continuous in the interval from r to s. One easily appreciates the necessity of this restriction on considering that the different terms of the series are definite integrals and going back to the fundamental notion of integrals. One will then see that the integral of a function only signifies something when the function satisfies the condition previously stated. One will have an example of a function that does not satisfy this condition, if one supposes that $\phi(x)$ is equal to a definite constant c when the variable x has a rational value, and is equal to a different constant d when this variable is irrational. The function thus defined has finite and definite values for each value of x, however, one cannot substitute it in the series because the different integrals which enter this series lose all signification in this case. The restriction which I have made precise, and that of not becoming infinite, are the only ones to which the function $\phi(x)$ must be subjected, and all the cases that we have not excluded can be reduced to those we have considered in the preceding. But this matter, to be done with all the clarity that one desires, requires some details related to the fundamental principles of analysis, and will be expounded in another note, where I will also consider some other quite remarkable properties of series (7).

A.3 Riemann on Elementary Complex Function Theory

G.B.F. Riemann, *Foundations for a general theory of functions of a variable complex quantity*.[6] Inauguraldissertation, Göttingen, 1851, second unaltered publication, Göttingen, 1867.

§ 1.

[6]In making this translation, I consulted the one in *A Source Book in Classical Analysis*, G. Birkhoff and U. Merzbach (eds.) 1973, pp. 48–50, and some phrases of that translation appear here.

If one thinks of z as a variable that successively takes all possible real values, then if to each value of z there corresponds a single value of w, w is called a function of z. If, when z varies continuously on an interval, w also varies continuously, this function is called continuous on this interval.

This definition evidently does not assume any fixed law describing the function since, once this function has been defined on a given interval, it may be extended arbitrarily outside it.

The dependence of w on z may be given by a mathematical law which determines from each value of z the corresponding value of w. Formerly only certain kinds of functions (functiones continuae in Euler's terminology) were considered capable of satisfying the same law of dependence for all values of z in a given interval. Recent researches have shown, however, that there exist analytic expressions by which any continuous function may be represented on a given interval. It is therefore all the same whether the function is defined arbitrarily or by a formula. Because of the theorem recalled above, the two concepts are equivalent.

However, it is otherwise when z is not restricted to real values, and complex numbers of the form $z = x + iy$ (where $i = \sqrt{-1}$) are included.

Let $x + yi$ and $x + yi + dx + dyi$ be two values of the quantity z differing infmitesimally, to which correspond the values $u + iv$ and $u + vi + du + dvi$ of the quantity w. Then if the dependence of the quantity w on z is taken to be arbitrary … the ratio $\frac{du+dvi}{dx+dyi}$ will in general vary with dx and dy. For, if one sets $dx+dyi = \varepsilon e^{\phi i}$, then

$$\begin{aligned}\frac{du+dvi}{dx+dyi} &= \frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial}{\partial y}\right)\\ &\quad+\frac{1}{2}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)i\right]\\ \frac{dx-dyi}{dx+dyi} &= \frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)i\\ &\quad+\frac{1}{2}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)i\right]e^{-2\phi i}.\end{aligned}$$

But no matter how w is determined as a function of z by tying together simple algebraic operations the value of the derivative [or differentialquotient] $\frac{dw}{dz}$ must always be independent of the particular value of dz.[7] Evidently, not every dependence of the complex quantity w on the complex quantity can be expressed in this way.

This characteristic property of all functions defined by explicit operations, will be taken as basic in what follows, where a function will be considered independent of its expression, and we, without proving the general validity and permissibility of

[7]This requirement is evidently satisfied in all cases where from the expression for w in terms of z an expression for $\frac{dw}{dz}$ in terms of z is obtained by means of the rules for differentiation; the rigorous general validity of this remains for now undiscussed.

expressing every dependence through operations on quantity, shall proceed from the following definition:

A variable complex quantity w is called a function of another variable complex quantity z when its variation is such that the value of the derivative $\frac{dw}{dz}$ is independent of the value of the differential dz.

§ 2.

Let the quantities z and w be complex be taken as variables that can take each take complex values. The presentation of their variation on a connected two-dimensional domain is made essentially easier by a connection to spatial intuition. One thinks of each value $x + iy$ of the quantity z as represented by the point O of the plane A with rectangular coordinates (x, y), and each value $u + vi$ by the point Q of the plane B with rectangular coordinates (u, v). Every dependence of the quantity w on z will then be represented by the dependence of the position of the point q on that of the point O. If to each value of z there corresponds a value of w, which varies continuously with z, in other words, if u and v are continuous functions of x and y—then to each point of the plane A will correspond a point of the plane B, every curve generally to a curve, and every connected piece of surface to a connected piece of surface. One can think of this dependence of the quantity w on the quantity z as a mapping of the plane A onto the plane B.

§ 3.

We now investigate what properties this mapping has when w is a function of the complex quantity z, that is when $\frac{dw}{dz}$ is independent of dz. We denote by o be an arbitrary point of the plane A in the neighbourhood of O, and by q its image in the plane B, also, let $x + yi + dx + dyi$ and $u + vi + du + dvi$ be the values of z and w at these points. Then dx, dy and du, dv can be regarded as rectangular coordinates of the points o and q relative to the points O and Q considered as origins, and if one writes $dx + dy = \varepsilon e^{\phi i}$ and $du + dvi = \eta e^{\psi i}$ then the quantities $\varepsilon, \phi, \eta, \psi$ will be the polar coordinates of these points for the same origins. Now if o' and o'' are any two specified locations of the point o infinitely near to O, and if one denotes the other variables by the corresponding indices, then we have

$$\frac{du' + dv'i}{dx' + dy'i} = \frac{du'' + dv''i}{dx'' + dy''i}$$

and consequently

$$\frac{du' + dv'i}{du'' + dv''i} = \frac{\eta'}{\eta''}e^{(\psi' - \psi'')i} = \frac{dx' + dy'i}{dx'' + dy''i} = \frac{\varepsilon'}{\varepsilon''}e^{(\phi' - \phi'')i},$$

whence $\frac{\eta'}{\eta''} = \frac{\varepsilon'}{\varepsilon''}$ and $\psi' - \psi'' = \phi' - \phi''$, that is to say, in the triangles $o'Oo''$ and $q'Qq''$ the angles $o'Oo''$ and $q'Qq''$ are equal and the sides enclosing them proportional.

Thus there exists equality in the smallest parts between two corresponding infinitesimal triangles and consequently between the plane A and the lane B in general. An exception occurs only in the special case when the corresponding variations in the quantities z and w does not have a finite ratio, which is here tacitly assumed.

§ 4.

If one writes the derivative $\frac{du+dvi}{dx+dyi}$ in the form

$$\frac{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i\right) dx + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} i\right) dyi}{dx + dyi}$$

it is clear that its value will be independent of dx and dy if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These conditions are therefore necessary and sufficient for $w = u + vi$ to be a function of $z = x + yi$. Consequently the individual parts of this function satisfy the following

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2},$$

which will be taken as fundamental for the study of the properties of the individual parts of such a function, considered separately.

A.4 Riemann's Definition of the Integral

From Riemann *On the representability of a function by a trigonometric series*: The concept of a definite integral and the scope of its validity.

§ 4

The indeterminancy which still prevails on a number of fundamental points of the theory of definite integrals compels us to make some preliminary remarks about the concept of a definite integral and the scope of its validity.

So first: what have we to understand by $\int_a^b f(x)dx$?

To determine this, we take a sequence of values $x_1, x_2, \ldots, x_{n-1}$ increasing in size between a and b, and for brevity denote $x_1 - a$ by δ_1, $x_2 - x_1$ by $\delta_2, \ldots, b - x_{n-1}$ by δ_n, and by ε a positive real fraction. Then the value of the sum

$$S = \delta_1 f(a + \varepsilon_1 \delta_1) + \delta_2 f(x_1 + \varepsilon_2 \delta_2) + \delta_3 f(x_2 + \varepsilon_3 \delta_3) + \ldots + \delta_1 f(x_{n-1} + \varepsilon_n \delta_n)$$

depends on the choice of intervals δ and the quantities ε. If this now has the property that however δ and ε may be chosen, as soon as all δ's become infinitely small, it approaches infinitely close to a fixed limit A, then this value is called $\int_a^b f(x)dx$.

If it does not have this property, then $\int_a^b f(x)dx$ has no meaning.

However, in several cases, attempts have been made to attribute a meaning to this symbol, and among these expansions of the concept of a definite integral there is one that is accepted by all mathematicians. To wit, if the function $f(x)$ becomes infinitely large as the argument approaches a particular value c in the interval (a, b), then obviously the sum S, whatever degree of smallness we attribute to the δ's, can obtain any value whatsoever. Thus it has no limiting value, and $\int_a^b f(x)dx$, as above, would have no meaning. But, should

$$\int_a^{c-\alpha_1} f(x)dx + \int_{c+\alpha_2}^b f(x)dx,$$

when α_1 and α_2 become infinitely small, approach a fixed limit, $\int_a^b f(x)dx$ is understood to be this limit value.

Other statements by Cauchy about the concept of the definite integral in cases where, under the basic concept, one does not exist may be useful for certain classes of investigations. However, they were not introduced at a general level and, given their great arbitrariness, are scarcely appropriate.

§ 5

Second, let us now look at the scope of validity of this concept, or at the question: in which cases is a function integrable and in which not?

We shall first look at the concept of integral in the strict sense, i.e., we assume that the sum S converges when all the δ's become infinitely small. If we call the greatest variation of the function between a and x_1, i.e., the difference between its largest and smallest values in this interval, D_1, that between x_1 and x_2, $D_2, \ldots,$ that between x_{n-1} and b, D_n, then

$$\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$$

must become infinitely small with the quantities δ. We further assume that, as long as all δ's remain smaller than d, the greatest value that this sum can have is Δ; Δ will then be a function of d that is always decreasing along with d and becomes infinitely small with this quantity. Now if the total size of the intervals in which the variations are greater than σ is $= s$, the contribution these intervals make to the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ is evidently $\geq \sigma s$. One therefore has

$$\sigma s \leq \delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n \leq \Delta,$$

consequently $s \leq \frac{\Delta}{\sigma}$.

Now $\frac{\Delta}{\sigma}$, when σ is given, can always be made infinitely small by a suitable choice of d; hence the same is true of s, and this results in:

For the sum S to converge as all δ's become infinitely small, not only does the function $f(x)$ need to be finite, but also the total size of the intervals in which the variations are $> \sigma$ (whatever σ may be) must be able to be made arbitrarily small by a suitable choice of d.

This theorem can also be inverted:

If the function $f(x)$ is always finite and, as all the quantities δ decrease the total size s of those intervals where the variations of function $f(x)$ are larger than a given value σ continually becomes infinitely small, then the sum S will converge as all δ's become infinitely small.

For, those intervals where the variations are $> \sigma$ contribute an amount smaller than s to the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ multiplied by the greatest variation of the function between a and b, which (by hypothesis) is finite. The remaining intervals contribute an amount $< \sigma(b - a)$. Evidently one can first assume σ to be arbitrarily small and then always (by hypothesis) so determine the size of the intervals that s also becomes arbitrarily small, whence the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ can be given any arbitrary smallness one wishes, and consequently the sum S can be contained within arbitrarily narrow limits.

Thus we have found conditions that are necessary and sufficient for the sum S to converge as the values of δ infinitely decrease and so we can speak in the strict sense of an integral of the function $f(x)$ between a and b.

If the concept of the integral is now expanded as above, it is evident that for integration always to be possible, the latter of the two conditions we found is still necessary; but in place of the condition that the function always be finite comes the condition that the function only becomes infinite as the argument approaches particular values, and that a determinate limiting value emerge as the integration limits are approach infinitely close to this value.

A.5 Schwarz on Squaring the Circle

From H.A. Schwarz, Ueber einige Abbildungsaufgaben, *Journal für die reine und angewandte Mathematik* vol. 70 (1869), pp. 105–120; in *Gesammelte Mathematische Abhandlung* vol. 2 (1890), pp. 65–83, this extract pp. 66–70.[8]

[8]This translation is adapted from the one in *A Source Book in Classical Analysis*, G. Birkhoff and U. Merzbach (eds.) 1973, pp. 56–59.

[Schwarz first commented on the striking contrast between Riemann's general mapping theorem and the absence of specific formulas for mapping triangles (say) on the unit circle. He then began to solve that problem as follows.]

The following fruitful theorem leads to the solution of this and many other mapping problems: If to a continuous sequence of real values of the complex argument of an analytic function there corresponds a continuous sequence of real values of the function, then to any two conjugate values of the argument there correspond conjugate values of the function. In the u-plane, whose points represent the values of a complex variable u geometrically, let U' be a bounded, simply connected domain whose boundary contains a finite segment ℓ of the real axis.

Let a single-valued analytic function t of the complex variable u, $t = f(u)$ behave like an entire function for all values of u in the interior of U'. That is, if u_0 denotes an arbitrary value of u in the interior of the domain U' then the function $f(u)$ can be expanded in a power series in $u - u_0$ for the values of u lying in a neighbourhood of u_0 that converges for all sufficiently small absolute values of $u - u_0$. We shall assume that t remains bounded as u approaches the boundary ℓ and is real for all points of the line ℓ, and that for all values of the argument u in the interior and the boundary of the domain U' the function $t = f(u)$ varies continuously with the argument u.

To the domain U' there corresponds a domain U'' whose points lie symmetrically with the points of U' with respect to the real axis.

At all points of the domain U'' an analytic function t is defined that in the domains U' and U'' takes conjugate values at conjugate values of u in the domains. If one thinks of the two domains U' and U'' as joined along the segment ℓ, then one obtains a simply connected domain $U' + U''$. For all values of the argument u in the interior of $U' + U''$, the value of t is uniquely defined; and indeed in the interior of U' and the interior of U'' as an analytic function of this argument which behaves like an entire function. On crossing the line ℓ, and along that line, the value of t varies continuously. It follows from this that the function t determined for the domain U'' is an analytic continuation of the function determined for the domain U', and is indeed a continuation extending beyond the line ℓ. The proof of the validity of this statement can be carried out as follows if, as may be assumed, the domain $U' + U''$ everywhere covers the u-plane only simply.

If u_0 denotes a value of u in the interior of U' then, by a theorem of Cauchy, the integral

$$\frac{1}{2\pi i} \int \frac{f(u)}{u - u_0} du$$

taken in a positive sense around the boundary of the domain U' or that of U'', has the value $f(u_0)$ in the first case and the value 0 in the second. On adding these two integrals, the integrations along the line ℓ, carried out twice and in opposite senses, cancel out; and the following equation holds

$$f(u_0) = \frac{1}{2\pi i} \int \frac{f(u)}{u - u_0} du,$$

where the integral is taken along the boundary of the domain $U' + U''$, and for all values of the quantity u_0 that are among the points represented geometrically in the interior of these domains, represents a continuous function of this argument whose values coincide everywhere with the values of the function $t = f(u)$.

It follows that the function so determined also behaves like an entire function for all values in the interior of the segment ℓ. Thus, under the assumptions made, conjugate values of the argument correspond to conjugate values of the function, or, expressed geometrically, the conformal mapping of the (u)-plane onto the (t)-plane, whose points represent the values of the complex quantity t, is symmetric with respect to the real axis for both planes: to symmetric points there correspond symmetric images.

If one analytically continues the function $t = f(u)$, symmetrically on both sides of the real axis in the (u)-plane, one is led to the result that singular points of some kind lie either singly on the real axis or in pairwise symmetry on both sides of it. This theorem can immediately be extended to an analytic function which maps a straight line segment in the domain of the argument or on its boundary onto a straight line segment in the plane whose points represent the value of the analytic function geometrically.

For the special problem of mapping the surface of a square onto the surface of a disk, one easily guesses that, to make the centre of the square correspond to the centre of the disk, the four straight lines that are symmetry axes of the square should in any case be mapped onto straight lines. This comparison locates the four singular points on the circumference of the circle that correspond to the four corners of the square under this special assumption.

Now the solution of the given problem could evidently be simplified by replacing the disk by a half- plane, which can be achieved by the transformation by reciprocal radii [inversion, JJG]. Indeed, the resulting simplification lies in the circumstance that the boundaries of the two regions to be mapped onto each other are now straight lines. By the general law given above, the mapping function can be continued analytically outside of the square in which it had been originally defined. If the centre of the transformation is taken to be one of the singular points on the circumference of the circle, it follows that the points $t = \infty, t = -1, t = 0, t = +1$ on the real axis can be taken as singular points, while the half-plane lying on the positive side of the real axis is a conformal image of the disk.

If the position of a point inside the given square is determined by the complex number u then the problem requires that for all [such] z, the complex variable t be an analytic function for all values of the argument u that correspond to points lying in the interior of the given square, with the property that all values corresponding to points u on the periphery of the square shall have real values. The domain of the argument u can now be extended by the above principle, first to the interior of four squares symmetrically placed adjacent to the given square, then by repeated application to an arbitrarily large domain of the (u)-plane.

It follows that the [extended] function t must, for all finite values of its argument u in the extension of its domain, be a single-valued function and indeed a doubly periodic function of u, the ratio of whose fundamental periods is $\sqrt{-1}$. This already indicates the lemniscatic function.

The boundary of the square has singular points at its four corners. Under the requirement that the perimeter of the square is to be mapped onto the perimeter of the circle or onto the perimeter of the half-plane so that they are equal in the smallest parts [infinitesimally conformal], these points must be excepted; otherwise the given problem would contain an impossible condition.

Each piece of the surface of the square near a corner, being a right-angled sector near the vertex, must be mapped onto a straight angle by the function describing the given mapping.

This leads to the problem of finding the most general function that maps the sector subtending an angle $\alpha\pi$ lying near the vertex $u = 0$ in the (u)-plane,

$$u = re^{\phi i}; \quad 0 \leq \phi \leq \alpha\pi; \quad 0 < r < r_0$$

conformally onto the half-plane

$$t = \rho e^{\psi i}, \quad 0 \leq \psi \leq \pi$$

so that inside the given boundaries each point $u = re^{\phi i}$ corresponds continuously to a point $t = \rho e^{\psi i}$, while the values

$$r = 0, \rho = 0; \phi = 0, \psi = 0; \phi = \alpha\pi, \psi = \pi$$

correspond to each other. The simplest function providing such a mapping is the function $v = u^{1/\alpha}$. Every other function t of the argument u that also provides a mapping with the stated properties considered as a function of the variable v has, by the above theorem, the character of an entire function for the value $v = 0$ and in the vicinity of this value.

Likewise conversely the quantity v can be expressed as an analytic function of the argument t, which behaves like an entire function for all values of the complex quantity t in the vicinity of and including $t = 0$. Hence one obtains the following analytic representations, valid in the vicinities of the values $v = 0$ and $t = 0$:

$$v = u^{1/\alpha}; \quad t = Cv(1 + a_1 v + a_2 v^2 + \cdots); v = \frac{1}{C} t(1 + b_1 t + b_2 t^2 + \cdots);$$

$$u = v^{\alpha}; \quad u = \frac{1}{C^{\alpha}} t^{\alpha}(1 + c_1 t + c_2 t^2 + \cdots).$$

The constant C is positive and different from 0; the coefficients a, b, c all have real values; the latter follows from the fact that to all sufficiently small positive values of u with respect to the quantity v positive values of t must again correspond.

[Schwarz continued and eventually obtained what became known as the 'Schwarz–Christoffel' transformation

$$u' = \int_0^t \frac{dt}{\sqrt{4t(1-t^2)}}$$

which is a lemniscatic integral that represents the interior of two half-planes divided by the real axis onto the interior of a square with sides $\int_0^1 \frac{dt}{\sqrt{4t(1-t^2)}}$. The substitution $s = \frac{t-i}{t+i}$ takes one from the half-plane lying on the positive side of the real axis in the (t)-plane to the surface of the circle lying in the (s)-plane with radius 1 and centre the point $s = 0$.]

Appendix B
Series of Functions

Very often, functions are defined as convergent infinite sums of other functions, and when this is done we can ask if the properties of the individual terms of the sum (integrable, continuous, differentiable, ...) also belong to the limit function. The over-simplified general philosophy is that they do if the convergence is uniform, otherwise not. The troublesome case is differentiability: the terms may be differentiable and the convergence uniform, without the limit function being differentiable. When the relevant property is assured, the question of term by term integration or differentiation arises.

Two classes of infinite series are generally encountered: power series and Fourier series. A power series has a radius of convergence, and on any smaller interval (or, in the complex case, disc) convergence is uniform, so with that restriction on the domain of definition, the theorems stated below do apply.

Fourier series are much more complicated. The 'standard hypotheses' on a function are that it is periodic (say with period 2π), regulated on $[-\pi, \pi]$, and has the 'right' Fourier coefficients:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

A regulated function is defined as the uniform limit of step functions. Among the regulated functions are functions continuous on a closed interval, say $[a, b]$, and they, like all regulated functions, are integrable. A regulated function, f and its indefinite integral, $F(t) := \int_a^t f(x)dx$, satisfy the fundamental theorem of the calculus:

$$F'(t) = f(t).$$

Even when a function f satisfies the standard hypotheses, sufficient hypotheses for the Fourier series of the function f to convergence pointwise to f are that the function f be differentiable, and sufficient hypotheses for a Fourier series of a function f to

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converge uniformly to the function are that the function f be C^2 (twice continuously differentiable).

Uniform Convergence and Continuity

A sequence of functions $f_n(x)$ defined on a domain D tends to a function $f(x)$ on that domain if for each $x \in D$ and every $\varepsilon > 0$ there is an $N = N(\varepsilon, x)$ such that $n > N$ implies that $|f_n(x) - f(x)| < \varepsilon$. If N can be chosen for all $x \in D$ the convergence is said to be uniform, otherwise it is said to be pointwise.

If a sequence of functions $\{f_n\}$, each continuous at a point $x_0 \in (a, b)$, tends uniformly to a function f then the function f is continuous at the point x_0.

If a series of functions $\sum_n f_n$, each continuous at a point x_0, tends uniformly to a function f then the function f is continuous at the point x_0.

Uniform Convergence and Integrability

If a series of functions $\sum_n f_n$, each continuous and Riemann-integrable on an interval (a, b), tends uniformly to a function f then the function f is Riemann-integrable and the integral of the sum is sum of the integrals; i.e. term by term integration is valid.

Uniform Convergence and Differentiability

Here, uniformity alone is not enough. Indeed, it is possible for a sequence of differentiable functions $\{f_n\}$ to converge uniformly on an open interval and for the sequence $\{f_n'\}$ to fail to converge pointwise on that interval, as we shall see below. However, the following more delicate results hold.

If $\{f_n\}$ is a sequence of functions, each with a finite derivative at every point of the open interval (a, b), and:
If there is at least one point x_0 in (a, b) such that the sequence $\{f_n(x_0)\}$ converges, and
If there is a function g on (a, b) such that the sequence of derived functions $\{f_n'\}$ converges uniformly to the function g on (a, b),
Then there is a function f such that the sequence $\{f_n\}$ converges uniformly to f on (a, b), and for each x_0 in (a, b) the derivative $f'(x)$ exists and $f'(x) = g(x)$.

If $\sum_n f_n$ is a series of functions, each with a finite derivative at every point of the open interval (a, b), and:
If there is at least one point x_0 in (a, b) such that the series $\sum_n f_n(x_0)$ converges, and
If there is a function g on (a, b) such that the series of derived functions $\sum_n f_n'$ converges uniformly to the function g on (a, b),
Then there is a function f such that the series $\sum_n f_n$ converges uniformly to f on (a, b), and for each x_0 in (a, b) the derivative $f'(x)$ exists and $\sum_n f'(x) = g(x)$.

Non-uniform Convergence of Smooth Functions

Define $f(x)$ by:

$$f(x) = \begin{cases} -1 & x \le -\frac{1}{n} \\ \sin\left(\frac{n\pi x}{2}\right) & -\frac{1}{n} \le x \le \frac{1}{n} \\ 1 & x \ge \frac{1}{n} \end{cases}$$

All these functions are differentiable, but the limit is -1 when $x < 0$, 0 when $x = 0$, and +1 when $x > 0$.

Indeed, worse can happen. For example, a sequence of functions, (f_n), can tend uniformly to a smooth function, f, and yet the sequence of derived functions (f_n') can fail to converge even pointwise, and a fortiori therefore there is no question of the sequence converging to some limit f'. This is illustrated by the sequence $\frac{\sin(nx)}{\sqrt{n}}$.

Suppose, however, that a sequence of functions, (f_n) defined on the interval (a, b) tends pointwise (not necessarily uniformly) to a smooth function, f, that each function f_n is differentiable and has a bounded derivative on (a, b) and the sequence of derived functions (f_n') converges uniformly to some limit g. Then the conditions of the above theorem apply, and we can deduce that the sequence of derived functions (f_n') converges uniformly to some limit f', the derivative of f—in other words, term by term differentiation is permitted.

The subtleties are well illustrated by Abel's series. Recall that $f_n(x) := \sum_{k=1}^{n}(-1)^{k-1}\frac{\sin(kx)}{k}$. In the interval $(-1, 1)$, say, these functions do a remarkable job of approximating the function $x/2$ (recall Fig. 4.3). Indeed, they do so well that you might imagine that the derived functions

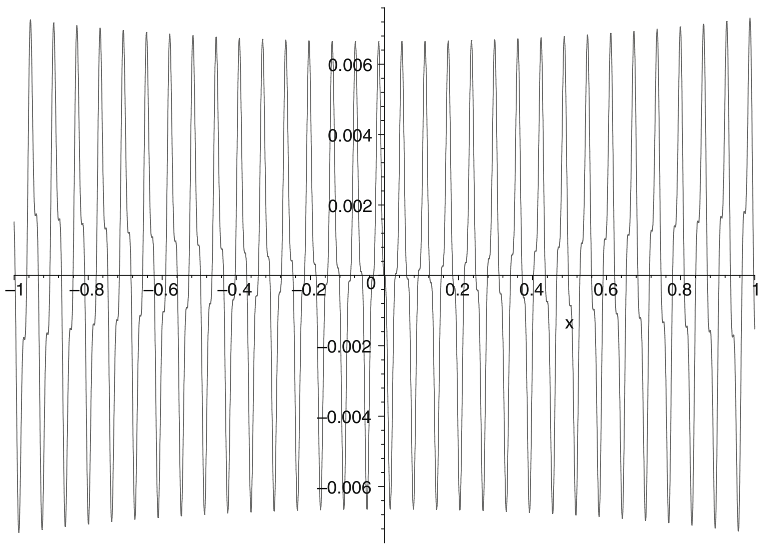

Fig. B.1 Much of the tail of Abel's series

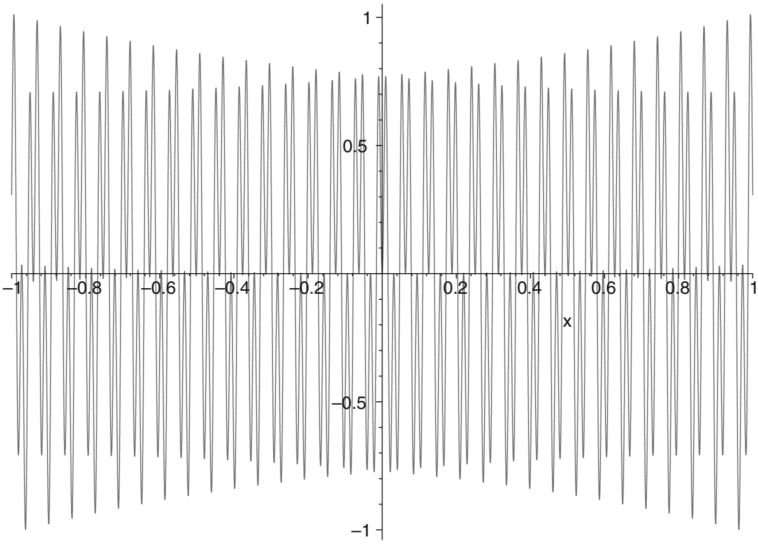

Fig. B.2 Much of the tail of the derivative of Abel's series

$$f_n'(x) := \Sigma_k^n (-1)^{k-1} \cos(kx)$$

approximate the constant function $1/2$ equally well—but they do not (recall Fig. 4.4). Figure B.1 shows the contribution of terms 101 to 300 to $f_n(x) - x/2$ and Fig. B.2 shows their contribution to $f_n'(x) - 1/2$; they make very visible the fact that the convergence of the $f_n(x)$ to $x/2$ is uniform, but that the derived functions do not even converge for all x. You need to look carefully to see that in the first case the uniform bound is already down to 0.007, whereas in the second case it is still 1. This suggests, but does not prove the stated results.

Note that pointwise convergence of the sequence (f_n) would have been enough for the theorem, but even uniform convergence, as here, is not enough to control the behaviour of the sequence of derived functions.

This illustrates a profound moral, that two functions may differ very little and yet their derivatives may differ a lot, as is the case with $f_n(x) = \Sigma_k^n (-1)^{k-1} \frac{\sin(kx)}{k}$ and $x/2$.

Appendix C
Potential Theory: A Mathematical Summary

The next two subsections revise the mathematics of what was discussed historically in Chap. 13. The first covers vector methods (**div**, **grad**, and **curl**). The second looks ahead to the connection between potential theory and the theory of harmonic functions, which became very important for Riemann's pioneering development of complex function theory.

Vector Methods

The calculus of vector fields is concerned with three operators on functions or vector fields: **grad**, **div**, and **curl**. To define them we need some notation. We write a function u of n variables $x_1, x_2, \ldots, x_n$ as $u(x_1, x_2, \ldots, x_n)$, and the ith partial derivative of u as either $\frac{\partial u}{\partial x_i}$, or as u_{x_i}. We write a vector field as $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, where each component v_i is a function of the variables $x_1, x_2, \ldots, x_n$.

First, **grad** applies to a function u and produces a vector:

$$\mathbf{grad}\; u = \nabla u := \left(u_{x_1}, u_{x_2}, \ldots, u_{x_n}\right).$$

Next, **div** applied to a vector field $\mathbf{v}$ produces a scalar:

$$\mathbf{div}(\mathbf{v}) = \nabla \cdot \mathbf{v} = v_{x_1} + v_{x_2} + \cdots + v_{x_n}.$$

Finally, **curl** applied to a vector field produces a vector field, but the definition we give here (the usual one) only makes sense when there are 3 variables ($n = 3$)—be careful. In this setting, the ith component of the vector field $\mathbf{curl}\;\mathbf{v} = \nabla \times \mathbf{v}$ is $\frac{\partial v_j}{\partial x_k} - \frac{\partial v_k}{\partial x_j}$, where the variables are taken in cyclic order i, j, k.

A major feature of the study of vector fields in the plane or in space is the use of line and surface integrals. The main theorems about them, Gauss's divergence theorem and Stokes's theorem, generalise the fundamental theorem of the one variable calculus. Both concern an arbitrary vector field $\mathbf{v}$ in space.

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Gauss's divergence theorem concerns a region U of $\mathbb{R}^3$ inside a closed surface S in space, and relates the normal component of **v** on the boundary S to the integral of the divergence of the vector field inside U:

$$\int_{surf} \mathbf{v}.\mathbf{n} = \int_{vol} \nabla.\mathbf{v}.$$

It is about surface and volume integrals.

Stokes's theorem concerns a surface in S in $\mathbb{R}^3$ bounded by a closed curve γ, and relates the integral of the tangential component of **v** along the boundary to the integral of the normal component of the curl of the vector field over the surface S:

$$\int_{curve} \mathbf{v}.ds = \int_{surf} \nabla \times \mathbf{v}.\mathbf{n}.$$

It is about line and surface integrals.

A **potential function** associated to a vector field **v** is a function u such that $\nabla u = \mathbf{v}$. It is an immediate consequence of Stokes's theorem that in many circumstances the line integral of a potential functions depends only on the end points, not the path. It is interesting to note that Stokes first stated his theorem as a problem in the Smith's Prize examination at Cambridge in 1854. The Smith's Prize was an annual event designed to test the best recent Cambridge graduates and to give then a chance to develop an interest in original research. Technical mastery of the material, which is all the proof requires, was regarded a necessary pre-requisite for any kind of further study.

The Gauss–Green Theorem

A good way to think of the mathematics involved in Gauss's and Green's theorems is to realise that it all derives from one theorem, which I shall call the Gauss–Green theorem. This theorem refers to a region U of $\mathbb{R}^n$ with a boundary ∂U on which there is a unit outward normal $^\circ = (\nu^1, \nu^2, \ldots, \nu^n)$. Write the gradient of a function u with continuous first derivatives on U as $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$, and the (outward) normal derivative as $\frac{\partial u}{\partial \nu} = {}^\circ.\nabla u$

The Gauss–Green theorem states that

$$\int_U u_{x_i} dx = \int_{\partial U} u\nu^i dS, \quad i = 1 \ldots n.$$

It immediately implies, on integrating uv by parts, the corollary that

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS, \quad i = 1 \ldots n.$$

We now upgrade the functions u, v to have continuous second derivatives, and obtain **Green's formulas**:

$$1. \quad \int_U \triangle u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$

$$2. \quad \int_U \nabla v . \nabla u dx = -\int_U u \triangle v + \int_{\partial U} u \frac{\partial v}{\partial \nu} dS$$

$$3. \quad \int_U (u \triangle v - v \triangle u)\, dx = \int_{\partial U} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS.$$

Formula 1 follows from the Corollary on replacing u with u_{x_i} and v with 1.
Formula 2 follows from the Corollary on replacing v with v_{x_i}.
Formula 3 follows from Formula 2 by interchanging u and v and subtracting.

Potential and Harmonic Functions

There is a close connection between potential and harmonic functions, as the next result indicates. The function $f(r) = \frac{1}{r}$ is the one appropriate to problems involving mass, magnetism or electricity. Write this as

$$f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$$

and set $R = (x^2 + y^2 + z^2)^{-1/2}$. We note that $\frac{\partial R}{\partial x} = -xR^3$ with similar formulae for the other partial derivatives, so

$$\nabla f = (xR^3, yR^3, zR^3).$$

We now compute the Laplacian of f, $\Delta f = \nabla \cdot \nabla f$. We differentiate the first component, xR^3 with respect to x and find

$$\frac{\partial}{\partial x}(xR^3) = R^3 - 3x^2 R^5.$$

So the Laplacian turns out to be

$$\begin{aligned} &R^3 - 3x^2R^5 + R^3 - 3y^2R^5 + R^3 - 3z^2R^5 \\ &= 3R^3 - 3(x^2 + y^2 + z^2)R^5 = 3R^3 - 3R^3 = 0. \end{aligned}$$

This computation is valid in any region in which $f \neq \infty$, that is, in any region in which there is no mass, and we deduce that in any region containing no mass the potential function $f(r) = \frac{1}{r}$ is harmonic—it satisfies Laplace's equation.

This turns out to be generally true: typically a potential function in a region where it is never infinite is harmonic. This makes for a very close connection between potential theory and the theory of harmonic functions.

In fact, although it is a matter of vector algebra that $\nabla \cdot \nabla \times \mathbf{F} = 0$ for any smooth vector field $\mathbf{F}$, and that $\nabla \times \nabla f = 0$ for any smooth function f, both these results have non-trivial consequences. In each case, we consider only nice regions of $\mathbb{R}^3$—for example star-shaped regions—in which nothing becomes infinite. In physical terms there are no sources present. In the first case, we can say that $\nabla \cdot \mathbf{F} = 0$ if and only if there is a smooth vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$. In the second case, we can say that $\nabla \times \mathbf{F} = 0$ if and only if there is a smooth function f such that $\mathbf{F} = \nabla f$. We will prove these claims in a moment.

Using these results, consider a vector field $\mathbf{F}$ which admits a potential function f, so $\mathbf{F} = \nabla f$. Now we calculate the Laplacian of the function f, and we find

$$\Delta f = \nabla \cdot \nabla f = \nabla \cdot \mathbf{F},$$

and so by the first result $\Delta f = 0$ if and only if f is the potential function of a vector field whose divergence vanishes, which happens if and only if the vector field is a curl, that is, it is of the form $\nabla \times G$ for some other vector field $\mathbf{G}$.

We deduce that every potential function, in regions where it is not infinite (no sources are present) is a harmonic function.

We shall now see that in any simply connected domain, if a vector field $\mathbf{G} = 0$ then there is vector field $\mathbf{F}$ such that $\mathbf{G} = \nabla \times \mathbf{F}$. Let us first fix some notation: $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{G} = (G_1, G_2, G_3)$.

Now we use the fact that for any smooth function f, $\nabla \times \nabla f = 0$ to simplify the problem. We find a function f such that $\frac{\partial f}{\partial z} = -F_3$, which is easy: define f by $f(\mathbf{r}) = -\int_0^{\mathbf{r}} f_3 dz +$ any arbitrary function $g(x, y)$. The integration is done treating f as a function of z alone. The arbitrary function g can be set to zero or used to keep the expressions symmetric when working on a specific problem.

We are reduced to solving the problem of finding a vector field $\mathbf{F}$ of the form $\mathbf{F} = (F_1, F_2, 0)$. The equations we have to solve are therefore these three

$$\begin{aligned} F_{2_z} &= G_1, \\ -F_{1_z} &= G_2, \\ F_{1_y} - F_{2_x} &= G_3. \end{aligned}$$

This is three equations for two unknowns, so the system is said to be over-determined and perhaps cannot be solved. Notice that they imply that $G_{1_x} + G_{2_y} + G_{3_z} = 0$, which is just the condition that $\nabla \cdot G = 0$, so the given condition is necessary; we are about to see that it is sufficient.

We integrate the first equation with respect to z and find

$$F_2(\mathbf{r}) = \int_0^{\mathbf{r}} G_1 dz + h(x, y),$$

where $h(x, y)$ is an arbitrary function of x and y. We substitute this into the third for equation, and get these two equations for the remaining component F_1 of the vector field $\mathbf{F}$:

$$F_{1_y}(\mathbf{r}) = \int_0^{\mathbf{r}} G_1 dz + G_3 + h_x(x, y),$$

$$F_{1_z} = -G_2.$$

This is two equations for F_1. We solve the second one by a simple integration, to get

$$F_1(\mathbf{r}) = -\int_0^{\mathbf{r}} G_2 dz + k(x, y),$$

where $k(x, y)$ is another arbitrary function of x and y. Then we differentiate this expression for F_1 with respect to y and substitute that in the other equation, to get

$$F_{1_y}(\mathbf{r}) = -\int_0^{\mathbf{r}} G_{2_y} dz + k_y(x, y) = \int_0^{\mathbf{r}} G_1 dz + G_3 + h_x(x, y).$$

This rearranges to become

$$k_y(x, y) - h_x(x, y) = \int_0^{\mathbf{r}} \left(G_{2_y} - G_1\right) - G_3.$$

To finish off we can set one of h or k to zero and integrate. In any given problem it is likely that one or other of these methods will make the integration easy or the expression for F_1 attractive. But in any case we have shown that if a vector field $\mathbf{G} = 0$ then there is vector field $\mathbf{F}$ such that $\mathbf{G} = \nabla \times \mathbf{F}$. The only condition is that the path integrals make sense, and this is guaranteed in $\mathbb{R}^3$ or any other simply connected space.

Next, let us be more precise about when a vector field $\mathbf{F}$ is the gradient of a (potential) function. Notice that this is the case if and only if the line integral of the field is independent of the path. 'Only if' goes this way: we have $\int_\gamma \mathbf{F}.d\mathbf{r} = \int_\gamma \nabla f.d\mathbf{r}$ and this works out, by the chain rule, to be the integral of df along the path, which is plainly the difference in the values of f at the endpoints whatever path is chosen. The 'if' part is little more than writing down $f(\mathbf{x}) := \int_{\mathbf{p}}^{\mathbf{x}} \mathbf{F}.d\mathbf{r}$ for an arbitrary $\mathbf{p}$ and checking that it makes sense—which it does because the integral is assumed to be independent of the path.

So a vector field is the gradient of a function if and only if integrals along it are independent of the path, which is the same—by Stokes's theorem—as saying if and only if integrals around closed loops always enclose a surface within which the curl $\nabla \times \mathbf{F}$ vanishes. And we know that the vanishing of the curl of $\mathbf{F}$ is a necessary condition for $\mathbf{F}$ to be given by a potential function. So a vector field is given by a potential function if and only if any closed loop in the regions outside the sources of the field contains a surface. This is usually true in questions about gravitation, but not in electro-magnetic theory where circuits flow along closed loops of wire or, it is sometimes supposed, along infinitely long wires. Later mathematicians (and indeed Gauss) were to see this as the start of the study of the topology of spaces of the form $\mathbb{R}^3 - stuff$.

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