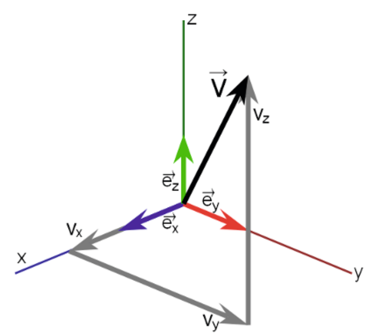
Part V
Appendices

A

Vector calculus

It is often useful in physics to describe the position of some object using three numbers $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. This is what we call a vector $\vec{v}$ and denote by a little arrow above the letter. The three numbers are the components of the vector along the three coordinate axes. The first number tells us how far the vector in question goes in the x-direction, the second how far in the y-direction and the third how far in the z-direction. For example, $\vec{w} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$ is a vector that points exclusively in the y-direction.



Vectors can be added

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad \vec{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} \quad \rightarrow \quad \vec{v} + \vec{w} = \begin{pmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_y \end{pmatrix} \tag{A.1}$$

or multiplied

$$\vec{v} \cdot \vec{w} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \cdot \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = v_x w_x + v_y w_y + v_z w_z. \tag{A.2}$$

The result of this multiplication is not a vector, but a number (= a scalar), hence the name: **Scalar product**. The scalar product of a vector with itself is directly related to its length:

$$\text{length}(\vec{v}) = \sqrt{\vec{v} \cdot \vec{v}}. \tag{A.3}$$

Take note that we can't simply write three quantities below each other between two brackets and expect it to be a vector. For example,

© Springer International Publishing Switzerland 2015
J. Schwichtenberg, *Physics from Symmetry*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19201-7

let's say we put the temperature T, the pressure P and the humidity H of a room between two brackets:

$$\begin{pmatrix} T \\ P \\ H \end{pmatrix} . \tag{A.4}$$

Nothing prevents us from doing so, but the result would be rather pointless and definitely not a vector, because there is no linear connection between these quantities that could lead to the mixing of these quantities. In contrast, the three position coordinates transform into each other, for example if we look at the vector from a different perspective[1]. Therefore writing the coordinates below each other between two big brackets is useful. Another example would be the momentum of some object. Again, the components mix if we look at the object from a different perspective and therefore writing it like the position vector is useful.

[1] This will be made explicit in a moment.

For the moment let's say a vector is a quantity that transforms exactly like the position vector $\vec{v}$. This means, if under some transformation we have $\vec{v} \to \vec{v}' = M\vec{v}$ any quantity that transforms like $\vec{w} \to \vec{w}' = M\vec{w}$ is a vector. Examples are the momentum or acceleration of some object.

We will encounter this idea quite often in physics. If we write quantities below each other between two brackets, they aren't necessarily vectors, but the quantities can transform into each other through some linear operation. This is often expressed by multiplication with a matrix.

A.1 Basis Vectors

We can make the idea of components along the coordinate axes more general by introducing basis vectors. Basis vectors are linearly independent[2] vectors of length one. In three dimensions we need three basis vectors and we can write every vector in terms of these basis vectors. An obvious choice is:

[2] A set of vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ is called linearly independent if the equation $c_1\vec{a} + c_2\vec{b} + c_3\vec{c} = 0$ is only true for $c_1 = c_2 = c_3 = 0$. This means that no vector can be written as a linear combination of the other vectors, because if we have $c_1\vec{a} + c_2\vec{b} + c_3\vec{c} = 0$ for numbers different than zero, we can write $c_1\vec{a} + c_2\vec{b} = -c_3\vec{c}$.

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{A.5}$$

and an arbitrary three-dimensional vector $\vec{v}$ can be expressed in terms of these basis vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \quad \text{(A.6)}$$

The numbers v_1, v_2, v_3 are called the components of $\vec{v}$. Take note that these components depend on the basis vectors.

The vector[3] $\vec{w}$ we introduced above can therefore be written as $\vec{w} = 0\vec{e}_1 + 4\vec{e}_2 + 0\vec{e}_3$. An equally good choice for the basis vectors would be

$$\tilde{\vec{e}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \tilde{\vec{e}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad , \quad \tilde{\vec{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \quad \text{(A.7)}$$

In this basis the vector $\vec{w}$ looks quite different:

$$\vec{w} = 2\sqrt{2}\tilde{\vec{e}}_1 - 2\sqrt{2}\tilde{\vec{e}}_2 + 0\tilde{\vec{e}}_3 = 2\sqrt{2}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 2\sqrt{2}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} . \quad \text{(A.8)}$$

Therefore we can write $\vec{w}$ in terms of components with respect to this new basis as

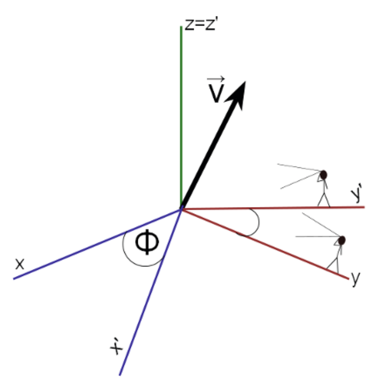
$$\tilde{\vec{w}} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix} .$$

This is not a different vector just a different description! To be precise, $\tilde{\vec{w}}$ is the description of the vector $\vec{w}$ in a coordinate system that is rotated relative to the coordinate system we used in the first place.

A.2 Change of Coordinate Systems

The connection between different coordinate systems can be made precise through the use of matrices. Two different coordinate systems can mean that we have two different observers that look at our experiment from different perspectives or this can simply mean that **one** observer decides to use a different set of basis vectors. How are those descriptions related? To avoid complications, let's assume that the origin of the two coordinate systems coincide and both coordinate systems have the same z-axes. Therefore only the x and y coordinates

[3] $\vec{w} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$



are different. Let's assume further that the position of something important in the experiment is described by the vector $\vec{v}$.

If the first observer sees the vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, we can compute how the same vector looks like in the coordinate system of the second observer $\vec{v} = \begin{pmatrix} v_{x'} \\ v_{y'} \\ v_{z'} \end{pmatrix}$ by using the usual trigonometric functions $\sin(\phi)$,$\cos(\phi)$ and $\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)}$, as illustrated in Fig. A.1.

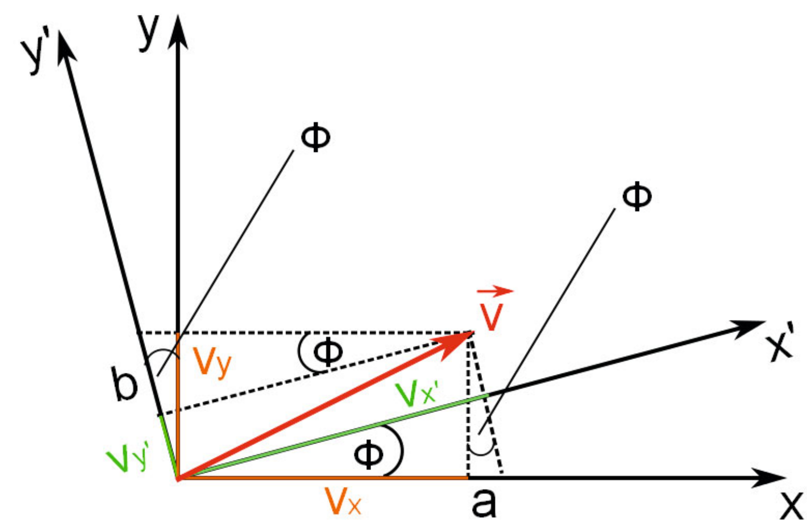


Fig. A.1: Illustration of the components of a vector in two different coordinate systems. Details can be found in the text.

The relationship between v_x and $v_{x'}$ can be computed using

$$\cos(\phi) = \frac{v_{x'}}{v_x + a} \to v_{x'} = (v_x + a)\cos(\phi)$$

and

$$\tan(\phi) = \frac{a}{v_y} \to a = v_y \tan(\phi).$$

This yields

$$v_{x'} = \big(v_x + v_y \tan(\phi)\big)\cos(\phi) = \left(v_x + v_y \frac{\sin(\phi)}{\cos(\phi)}\right)\cos(\phi)$$

$$= v_x \cos(\phi) + v_y \sin(\phi).$$

Analogously we can use

$$\cos(\phi) = \frac{v_y}{v_{y'} + b} \rightarrow v_{y'} = v_y \frac{1}{\cos(\phi)} - b$$

and

$$\tan(\phi) = \frac{b}{v_{x'}} \rightarrow b = v_{x'} \tan(\phi),$$

which yields using $\sin^2(\phi) + \cos^2(\phi) = 1$

$$v_{y'} = v_y \frac{1}{\cos(\phi)} - v_{x'} \tan(\phi) = v_y \frac{1}{\cos(\phi)} - (v_x \cos(\phi) + v_y \sin(\phi)) \frac{\sin(\phi)}{\cos(\phi)}$$

$$= v_y \frac{\sin^2(\phi) + \cos^2(\phi)}{\cos(\phi)} - v_x \sin(\phi) - v_y \frac{\sin^2(\phi}{\cos(\phi)} = v_y \cos(\phi) - v_x \sin(\phi)$$

Therefore $v_{y'} = -v_x \sin(\phi) + v_y \cos(\phi)$.

We can write this using a rotation matrix:

$$\begin{pmatrix} v_{x'} \\ v_{y'} \\ v_{z'} \end{pmatrix} = R_z(\phi)\vec{v} = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi)v_x + \sin(\phi)v_y \\ -\sin(\phi)v_x + \cos(\phi)v_y \\ v_z \end{pmatrix}. \quad \text{(A.9)}$$

We multiply each row of the matrix with the unrotated vector to compute the rotated vector. As already noted above, the component along the z-axis v_3 is the same for both observers. The matrix $R_z(\phi)$ describes a rotation by the angle ϕ about the z-axis.

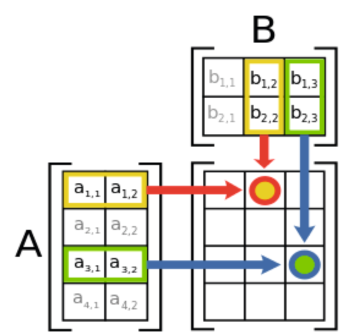


Fig. A.2: Schematic matrix multiplication. The important thing to keep in mind is **row times column**. The first index denotes the row number, the second the column number. In the example, the red element of the product matrix is $c_{1,2} = a_{1,1}b_{1,2} + a_{1,2}b_{2,2}$ and the blue element is $c_{3,3} = a_{3,1}b_{1,3} + a_{3,2}b_{2,3}$. In general $c_{i,j} = a_{i,k}b_{k,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots$. Figure by Olivier Perrin (Bilou Wikimedia Commons) released under a CC BY-SA 3.0 licence: `http://creativecommons.org/licenses/by-sa/3.0/deed.en`. URL: `http://commons.wikimedia.org/wiki/File:Matrix_multiplication_diagram_2.svg`, Accessed: 28.1.2015

A.3 Matrix Multiplication

Computations like this are tremendously simplified through the use of matrices. The rule for Matrix multiplication is always row times column. We can see the scalar product introduced above as a special case of this, if we interpret a vector as a matrix with one column and three rows (a 3×1 matrix). The scalar product of two vectors is then

$$\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w} = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix} \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = v_x w_x + v_y w_y + v_z w_z, \quad \text{(A.10)}$$

where the T denotes transposing, which means that every columns becomes a row and every row a column. Therefore, $\vec{v}^T$ is a matrix with 1 row and 3 columns. Written in this way the scalar product is once more a matrix product with row times columns.

Analogously, we get the matrix product of two matrices from the multiplication of each row of the matrix to the left with a column of the matrix to the right. This is explained in Fig. A.2. An explicit example for the multiplication of two matrices is

$$M = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 1 \\ 4 & 8 \end{pmatrix}$$

$$MN = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 2\cdot 0 + 3\cdot 4 & 2\cdot 1 + 3\cdot 8 \\ 1\cdot 0 + 0\cdot 4 & 1\cdot 1 + 0\cdot 8 \end{pmatrix} = \begin{pmatrix} 12 & 26 \\ 0 & 1 \end{pmatrix} \tag{A.11}$$

The rule to keep in mind is **row times column**. Take note that the multiplication of two matrices is not commutative, which means in general $MN \neq NM$.

A.4 Scalars

An important thing to notice is that the scalar product of two vectors has the same value for all observers. This can be seen as the definition of a scalar: A scalar is the same for all observers. This does not simply mean that every number is a scalar, because each component of a vector is a number, but as we have seen above a different number for different observers. In contrast the scalar product of two vectors must be the same for all observers. This follows from the fact that the scalar product of a vector with itself is directly related to the length of the vector. Changing the perspective or the location we choose to look at our experiment may not change the length of anything. The length of a vector is called an invariant for rotations, because it stays the same no matter how we rotate our system.

A.5 Right-handed and Left-handed Coordinate Systems

When we talked above about two observers, we implicitly assumed they agree in terms of the definition of their coordinate system. In fact, there are two possible choices, which are again related by matrix multiplication, but not by rotations. One observer may choose what we call a right-handed coordinate system and another observer what we call a left-handed coordinate system.

There is no way to rotate a left-handed into a right-handed coordinate system. Instead, such coordinate systems are related through a

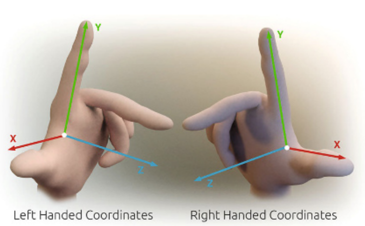


Fig. A.3: Right-handed and left-handed coordinate system. Figure by Primalshell (Wikimedia Commons) released under a CC-BY-SA-3.0 licence: `http://creativecommons.org/licenses/by-sa/3.0/deed.en`. URL: `http://commons.wikimedia.org/wiki/File:3D_Cartesian_Coodinate_Handedness.jpg` , Accessed: 1.12.2014 reflection in a mirror. This means the descriptions in a right-handed and a left-handed coordinate system are related by a transformation of the form

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \rightarrow \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix}, \tag{A.12}$$

which means we flip the sign of all spatial coordinates. The conventional name for this kind of transformation is **parity transformation**. We can describe a parity transformation by

$$\vec{v} \rightarrow \vec{v}' = P\vec{v} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix}. \tag{A.13}$$

B

Calculus

B.1 Product Rule

The product rule

$$\frac{d\Big(f(x)g(x)\Big)}{dx} = \Big(\frac{df(x)}{dx}\Big)g(x) + f(x)\Big(\frac{dg(x)}{dx}\Big) \equiv f'g + fg' \qquad \text{(B.1)}$$

follows directly from the definition of derivatives

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)g(x+h) - f(x+h)g(x)] + [f(x+h)g(x) - f(x)g(x)]}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

B.2 Integration by Parts

> A (likely apocryphal) story goes: when Peter Lax was awarded the National Medal of Science, the other recipients (presumably non-mathematicians) asked him what he did to deserve the Medal. Lax responded: " I integrated by parts."
>
> **- Willie Wong[1]**

[1] Told on www.math.stackexchange.com

An important rule for integrals follows directly from the product rule. Integrating the product rule[2]

[2] See Eq. B.1 and we then use for the first term the fundamental theorem of calculus $\int_a^b dx\, h'(x) = h(b) - h(a)$.

© Springer International Publishing Switzerland 2015
J. Schwichtenberg, *Physics from Symmetry*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19201-7

$$\underbrace{\int_a^b dx \frac{d\Big(f(x)g(x)\Big)}{dx}}_{=f(x)g(x)\Big|_a^b} = \int_a^b dx \Big(\frac{df(x)}{dx}\Big) g(x) + \int_a^b dx f(x) \Big(\frac{dg(x)}{dx}\Big) \quad \text{(B.2)}$$

and rearranging the terms yields

$$\int_a^b dx \Big(\frac{df(x)}{dx}\Big) g(x) = f(x)g(x)\Big|_a^b - \int_a^b dx f(x) \Big(\frac{dg(x)}{dx}\Big). \quad \text{(B.3)}$$

This rule is particularly useful in physics when working with fields, because if we integrate over all space, i.e. $a = -\infty, b = \infty$, the boundary term vanishes $f(x)g(x)\Big|_{a=-\infty}^{b=\infty} = 0$, because all fields must vanish at infinity[3].

[3] We discover in Sec. 2.4 that nothing can move faster than the speed of light. Therefore the field configuration infinitely far away mustn't have any influence on physics at finite x.

B.3 The Taylor Series

The Taylor series is a formula that enables us to write any infinitely differentiable function in terms of a power series

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \dots \quad \text{(B.4)}$$

- On the one hand we can use it if we want to know how we can write some function in terms of a series. This can be used for example to show that $e^{ix} = \cos(x) + i\sin(x)$.

- On the other hand we can use the Taylor series to get approximations for a function about a point. This is useful when we can neglect for some reasons higher order terms and don't need to consider infinitely many terms. If we want to evaluate a function $f(x)$ in some neighbouring point of a point y, say $y + \Delta y$, we can write[4]

[4] Don't let yourself get confused by the names of our variables here. In the formula above we want to evaluate the function at x by doing computations at a. Here we want to know something about f at $y + \Delta y$, by using information at y. To make the connection precise: $x = y + \Delta y$ and $a = y$.

$$f(y+\Delta y) = f(y) + (y + \Delta y - y)f'(y) + \dots = f(y) + \Delta y f'(y) + \dots. \quad \text{(B.5)}$$

This means we get an approximation for the function value at $y + \Delta y$, by evaluating the function at y. In the extreme case of an infinitesimal neighbourhood $\Delta y \rightarrow \epsilon$, the change of the function can be written by one (the linear) term of the Taylor series.

$$\Delta f = f(y+\epsilon) - f(y) = f(y) + \epsilon f'(y) + \dots - f(y) = \epsilon f'(y) + \underbrace{\dots}_{\approx 0 \text{ for } \epsilon^2 \approx 0} \quad \text{(B.6)}$$

This formula is one of the most useful mathematical tools and we can derive it using the fundamental theorem of calculus and integration by parts. The fundamental theorem tells us

$$\int_a^x dt\, f'(t) = f(x) - f(a) \quad \to \quad f(x) = f(a) + \int_a^x dt\, f'(t). \tag{B.7}$$

We can rewrite the second term by integrating by parts, because we have of course an implicit 1 in front of $f'(t) = 1f'(t)$, which we can use as a second function in Eq. B.3: $g'(t) = 1$. The rule for integration by parts tells us that we can rewrite an integral $\int_a^b v'u = vu|_a^b + \int vu'$ by integrating one term and differentiating the other. Here we integrate $g'(t) = 1$ and differentiate $f'(t)$, i.e. in the formula $g' = v'$ and $f' = u$. This yields

$$f(x) = f(a) + \int_a^x dt\, f'(t) = f(a) + g(t)f(t)\Big|_a^x - \int_a^x dt\, g(t)f''(t). \tag{B.8}$$

Now we need to know what $g(t)$ is. At this point the only information we have is $g'(t) = 1$, but there are infinitely many functions with this derivative: For any constant c the function $g = t + c$ satisfies $g'(t) = 1$. Our formula becomes particularly useful[5] for $g = t - x$, i.e. we use minus the upper integration boundary $-x$ as our constant c. Then we have for the second term in the equation above

[5] The equation holds for arbitrary c and of course you're free to choose something different, but you won't get our formula. We choose the constant c such that we get a useful formula for $f(x)$. Otherwise $f(x)$ would appear on the left- and right-hand side.

$$g(t)f(t)\Big|_a^x = (t-x)f(t)\Big|_a^x = \underbrace{(x-x)}_{=0}f(x) - (a-x)f(a) = (x-a)f(a) \tag{B.9}$$

and the formula now reads

$$f(x) = f(a) + (x-a)f'(a) + \int_a^x dt\, (x-t)f''(t). \tag{B.10}$$

We can then evaluate the last term once more using integration by parts, now with[6] $v' = (x-t)$ and $u = f''(t)$:

[6] Take note that integrating $v' = (x-t)$ yields a minus sign: $\to v = -\frac{1}{2}(x-t)^2 + d$, because our variable here is t and with some constant d we choose to be zero.

$$\to \int_a^x dt\, \underbrace{(x-t)}_{=v'}\underbrace{f''(t)}_{=u} = \underbrace{-\frac{1}{2}(x-t)^2}_{=v}\underbrace{f''(t)}_{=u}\Big|_a^x - \int_a^x dt\, \underbrace{\Big(-\frac{1}{2}(x-t)^2\Big)}_{=v}\underbrace{f'''(t)}_{=u'}$$

where the boundary term is again simple

$$-\frac{1}{2}(x-t)^2 f''(t)\Big|_a^x = -\frac{1}{2}\underbrace{(x-x)^2}_{=0}f''(x) + \frac{1}{2}(x-a)^2 f''(a).$$

This gives us the formula

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \int_a^x dt\, \frac{1}{2}(x-t)^2 f'''(t). \tag{B.11}$$

We could go on and integrate the last term by parts, but the pattern should be visible by now. The Taylor series can be written in a more compact form using the mathematical sign for a sum $\sum$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \\ &= \frac{f^{(0)}(a)(x-a)^0}{0!} + \frac{f^{(1)}(a)(x-a)^1}{1!} + \frac{f^{(2)}(a)(x-a)^2}{2!} \\ &\quad + \frac{f^{(3)}(a)(x-a)^3}{3!} + \dots, \end{aligned} \tag{B.12}$$

where $f^{(n)}$ denotes the n-th derivative of f, e.g. $f^{(2)} = f''$ and $n!$ is the factorial of n, i.e. $n! = 1 \cdot 2 \cdot 3 \dots n$. For example for $n = 5$ we have $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$, $2! = 2 \cdot 1 = 2$ and by definition $0! = 1$. Series are the topic of the next section.

B.4 Series

In the last section we stumbled upon a very important formula that includes an infinite sum. In this section some basic tricks for sum manipulation and some very important series will be introduced.

B.4.1 Important Series

In the last section we learned that we can write every infinitely differentiable function as a series. Let's start with maybe the most important function: The exponential function e^x. The Taylor series for the exponential function can be written right away

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{B.13}$$

by using the defining feature of the exponential function that the derivative is the exponential function itself: $(e^x)' = e^x$, evaluating the Taylor series about $a = 0$ and using $e^0 = 1$. This yields the Taylor series (Eq. B.12) for the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{e^0 (x-0)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{B.14}$$

This series can be seen as a definition of e^x.

Two other important, infinitely differentiable functions are $\sin(x)$ and $\cos(x)$. We can compute the Taylor series for these functions, by using $(\sin(x))' = \cos(x)$, $(\cos(x))' = -\sin(x)$, $\cos(0) = 1$ and

$\sin(0) = 0$.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)(x-0)^n}{n!}$$

Because of $\sin(0) = 0$ every term with uneven n vanishes, which we can use if we split the sum. Observe that

$$\sum_{n=0}^{\infty} n = \sum_{n=0}^{\infty} (2n+1) + \sum_{n=0}^{\infty} (2n)$$
$$1+2+3+4+5+6\ldots = 1+3+5+\ldots \quad +2+4+6+\ldots \tag{B.15}$$

Splitting the sum for $\sin(x)$ yields

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{\sin^{(2n+1)}(0)(x-0)^{2n+1}}{(2n+1)!} + \underbrace{\sum_{n=0}^{\infty} \frac{\sin^{(2n)}(0)(x-0)^{2n}}{(2n)!}}_{=0} \\ &= \sum_{n=0}^{\infty} \frac{\sin^{(2n+1)}(0)(x-0)^{2n+1}}{(2n+1)!}. \end{aligned} \tag{B.16}$$

Every even derivative of $\sin(x)$, i.e. $\sin^{(2n)}$ is again $\sin(x)$ (with possibly a minus sign in front of it) and therefore the second term vanishes because of $\sin(0) = 0$. Every uneven derivative of $\sin(x)$ is $\cos(x)$, with possibly a minus sign in front of it. We have

$$\begin{aligned} \sin(x)^{(1)} &= \cos(x) \\ \sin(x)^{(2)} &= \cos'(x) = -\sin(x) \\ \sin(x)^{(3)} &= -\sin'(x) = -\cos(x) \\ \sin(x)^{(4)} &= -\cos'(x) = \sin(x) \\ \sin(x)^{(5)} &= \sin'(x) = \cos(x) \end{aligned} \tag{B.17}$$

The pattern is therefore $\sin^{(2n+1)}(x) = (-1)^n \cos(x)$, as you can check by putting some integer values for n into the formula[7]. We can therefore rewrite Eq. B.16 as

[7] $\sin^{(1)}(x) = \sin^{(2\cdot 0+1)}(x) = (-1)^{\cdot 0}\cos(x) = \cos(x)$, $\sin^{(3)}(x) = \sin^{(2\cdot 1+1)}(x) = (-1)^1\cos(x) = -\cos(x)$

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{\sin^{(2n+1)}(0)(x-0)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \cos(0)(x-0)^{2n+1}}{(2n+1)!} \\ &\underbrace{=}_{\cos(0)=1} \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} \end{aligned} \tag{B.18}$$

This is the Taylor series for $\sin(x)$, which again can be seen as a definition of $\sin(x)$. Analogous we can derive

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}, \tag{B.19}$$

because this time uneven derivatives are proportional to $\sin(0) = 0$.

B.4.2 Splitting Sums

In the last section we used a trick that is quite useful in many computations. There we used the example

$$\begin{aligned} \sum_{n=0}^{\infty} n &= \sum_{n=0}^{\infty} (2n+1) + \sum_{n=0}^{\infty} (2n) \\ 1+2+3+4+5+6\ldots &= 1+3+5+\ldots \quad +2+4+6+\ldots, \end{aligned} \tag{B.20}$$

to motivate how we can split any sum in terms of even and uneven integers. $2n$ is always an even integer, whereas $2n+1$ is always an uneven integer. We already saw in the last section that this can be useful, but let's look at another example. What happens if we split the exponential series with complex argument ix?

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} \end{aligned} \tag{B.21}$$

This can be rewritten using that $(ix)^{2n} = i^{2n}x^{2n}$ and $i^{2n} = (i^2)^n = (-1)^n$. In addition we have of course $i^{2n+1} = i \cdot i^{2n} = i(-1)^n$. Then we have

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} \\ &= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}}_{=\cos(x) \text{ see Eq. B.19}} + i \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}}_{=\sin(x) \text{ see Eq. B.18}} \\ &= \cos(x) + i\sin(x) \end{aligned} \tag{B.22}$$

B.4.3 Einstein's Sum Convention

Sums are very common in physics and writing the big sum sign $\sum$ all the time can be quite cumbersome. For this reason a clever man introduced a convention, called Einstein sum convention. According to this convention every time an index appears twice in some term, like n in the sums above, an implicit sum is understood. This means

$$a_n b_n \equiv \sum_n a_n b_n. \tag{B.23}$$

Other examples are

$$a_n b_n c_m \equiv \sum_n a_n b_n c_m. \tag{B.24}$$

$$a_m b_n c_m \equiv \sum_m a_m b_n c_m. \tag{B.25}$$

but

$$a_n b_m \neq \sum_n a_n b_m, \tag{B.26}$$

because in general $m \neq n$. An index without a partner is called a **free index**, an index with a partner a **dummy index**, for reasons that will be explained in the next section.

For example in the sum $a_n b_n c_m \equiv \sum_n a_n b_n c_m$, the index n is a dummy index, but m is a free index. Equivalently, in $a_m b_n c_m \equiv \sum_m a_m b_n c_m$, the index m is a dummy an n is free.

B.5 Index Notation

B.5.1 Dummy Indices

It is important to take note that the name of indices with a partner plays absolutely no role. Renaming $n \to k$, changes absolutely nothing[8], as long as n is contracted

$$a_n b_n c_m = a_k b_k c_m \equiv \sum_n a_n b_n c_m \equiv \sum_k a_k b_k c_m. \tag{B.27}$$

[8] Of course we can't change an index into another **type** of index. For example, we can change $i \to j$ but not $i \to \mu$, because Greek indices like μ are always summed from 0 to 3 and Roman indices, like i from 1 to 3.

On the other hand free indices can not be renamed freely. For example, $m \to q$ can make quite a difference because there must be some term on the other side of the equation with the same free index. This means when we look at a term like $a_n b_n c_m$ isolated, we must always take into account that there might be other terms with the same free index m that must be renamed, too. Let's look at an example

$$F_i = \epsilon_{ijk} a_j b_k. \tag{B.28}$$

A new thing that appears here is that some object, here ϵ_{ijk}, is allowed to carry more than one index, but don't let that bother you, because we will come back to this in a moment. Therefore, if we look at $\epsilon_{ijk}a_j b_k$ we can change the names of j and k as we like, because these indices are contracted. For example $j \to u$, $k \to z$, which yields $\epsilon_{iuz}a_u b_z$ is really the same. On the other hand i is not a dummy index and we can't rename it $i \to m$: $\epsilon_{muz}a_u b_z$, because then our equation would read

$$F_i = \epsilon_{muz}a_u b_z. \tag{B.29}$$

This may seem pedantic at this point, because it is clear that we need to rename i at F_i, too in order to get something sensible, but more often than not will we look at isolated terms and it is important to know what we are allowed to do without changing anything.

B.5.2 Objects with more than One Index

Now, let's talk about objects with more than one index. Objects with two indices are simply matrices. The first index tells us which row and the second index which column we should pick our value from. For example

$$M_{ij} \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{B.30}$$

This means for example that M_{12} is the object in the first row in the second column.

We can use this to write matrix multiplication using indices. The product of two matrices is

$$MN \equiv (MN)_{ij} = M_{ik}N_{kj}. \tag{B.31}$$

On the left hand side we have the element in row i in column j of the product matrix (MN), which we get from multiplying the i-th row of M with the j-th column of N. The index k appears twice and therefore an implicit sum is assumed. One can give names to objects with three or more indices (tensors). For the purpose of this book two are enough and we will discuss only one exception, which is the topic of one of the next sections.

B.5.3 Symmetric and Antisymmetric Indices

A matrix is said to be symmetric if $M_{ij} = M_{ji}$. This means in our two dimensional example $M_{12} = M_{21}$ and an example for a symmetric matrix is

$$\begin{pmatrix} 9 & 3 \\ 3 & 17 \end{pmatrix} \tag{B.32}$$

A matrix is called **totally antisymmetric** if $M_{ij} = -M_{ji}$ for **all** i, j holds. An example would be.

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \tag{B.33}$$

Take note that the diagonal elements must vanish here, because $M_{11} = -M_{11}$, which is only satisfied for $M_{11} = 0$ and analogously for M_{22}.

B.5.4 Antisymmetric × Symmetric Sums

An important observation is that every time we have a sum over something symmetric in its indices multiplied with something antisymmetric in the same indices, the result is zero:

$$\sum_{ij} a_{ij} b_{ij} = 0 \tag{B.34}$$

if $a_{ij} = -a_{ji}$ and $b_{ij} = b_{ji}$ holds for all i, j. We can see this by writing

$$\sum_{ij} a_{ij} b_{ij} = \frac{1}{2}\Big(\sum_{ij} a_{ij} b_{ij} + \sum_{ij} a_{ij} b_{ij}\Big) \tag{B.35}$$

As explained earlier we are free to rename our dummy indices $i \to j$ and $j \to i$, which we use in the second term

$$\to \sum_{ij} a_{ij} b_{ij} = \frac{1}{2}\Big(\sum_{ij} a_{ij} b_{ij} + \sum_{ij} a_{ji} b_{ji}\Big) \tag{B.36}$$

Then we use the symmetry of b_{ij} and antisymmetry of a_{ij}, to switch the indices in the second term, which yields[9]

$$\begin{aligned} \to \sum_{ij} a_{ij} b_{ij} &= \frac{1}{2}\Big(\sum_{ij} a_{ij} b_{ij} + \sum_{ij} \underbrace{a_{ji}}_{=-a_{ij}} \underbrace{b_{ji}}_{=b_{ij}}\Big) \\ &= \frac{1}{2}\Big(\sum_{ij} a_{ij} b_{ij} - \sum_{ij} a_{ij} b_{ij}\Big) = 0 \end{aligned} \tag{B.37}$$

[9] If this looks like a cheap trick to you, compute some explicit examples to see that this is really true.

B.5.5 Two Important Symbols

One of the most important matrices is of course the unit matrix. In two dimensions we have

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.38}$$

In index notation the unit matrix is called the Kronecker symbol, denoted δ_{ij}, which is then defined for arbitrary dimensions by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{B.39}$$

The Kronecker symbol is symmetric because $\delta_{ij} = \delta_{ji}$.

Equally important is the Levi-Civita symbol ϵ_{ijk}, which is defined in two dimensions by

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } (i,j) = \{(1,2)\} \\ 0 & \text{if } i = j \\ -1 & \text{if } (i,j) = \{(2,1)\} \end{cases} \tag{B.40}$$

In matrix form

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{B.41}$$

In three dimensions the Levi-Civita symbol is

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = \{(1,2,3),(2,3,1),(3,1,2)\} \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \\ -1 & \text{if } (i,j,k) = \{(1,3,2),(3,2,1),(2,1,3)\} \end{cases} \tag{B.42}$$

and in four dimensions

$$\epsilon_{ijkl} = \begin{cases} 1 & \text{if } (i,j,k,l) \text{ is an even permutation of} \{(1,2,3,4)\} \\ -1 & \text{if } (i,j,k,l) \text{ is an uneven permutation of} \{(1,2,3,4)\} \\ 0 & \text{otherwise} \end{cases} \tag{B.43}$$

For example $(1,2,4,3)$ is an uneven (because we make one change) and $(2,1,4,3)$ is an even permutation (because we make two changes) of $(1,2,3,4)$.

The Levi-Civita symbol is totally anti-symmetric because if we change two indices, we always get, by definition, a minus sign: $\epsilon_{ijk} = -\epsilon_{jik}$, $\epsilon_{ijk} = -\epsilon_{ikj}$ etc. or in two dimensions $\epsilon_{ij} = -\epsilon_{ji}$.

C

Linear Algebra

Many computations can be simplified by using matrices and tricks from the linear algebra toolbox. Therefore, let's look at some basic transformations.

C.1 Basic Transformations

The **complex conjugate of a matrix** is defined by

$$M_{ij}^{\star} = \begin{pmatrix} M_{11}^{\star} & M_{12}^{\star} \\ M_{21}^{\star} & M_{22}^{\star} \end{pmatrix}, \tag{C.1}$$

which means we simply take the complex conjugate of each element[1].

The **transpose of a matrix** is defined by $M_{ij}^T = M_{ji}$, in matrix form

$$M_{ij} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \rightarrow \quad M_{ij}^T = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}, \tag{C.2}$$

which means we swap columns and rows of the matrix. An important consequence of this definition and the definition of the product of two matrices is that we have $(MN)^T \neq M^T N^T$. Instead $(MN)^T = N^T M^T$, which means by transposing we switch the position of two matrices in a product. We can see this directly in index notation

$$MN \equiv (MN)_{ij} = M_{ik} N_{kj}$$

$$(MN)^T \equiv ((MN)_{ij})^T = (MN)_{ji} = (M_{ik} N_{kj})^T$$

$$(M_{ik} N_{kj})^T = M_{ik}^T N_{kj}^T = M_{ki} N_{jk} = N_{jk} M_{ki} \equiv N^T M^T, \tag{C.3}$$

[1] Recall that the complex conjugate of a complex number $z = a + ib$, where a is the real part and b the imaginary part, is simply $z^{\star} = a - ib$.

© Springer International Publishing Switzerland 2015
J. Schwichtenberg, *Physics from Symmetry*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19201-7

where in the last step we use the general rule that in matrix notation we always multiply rows of the left matrix with columns of the right matrix. To write this in matrix notation, we change the position of the two terms to $N_{jk}M_{ki}$, which is rows of the left matrix times columns of the right matrix, as it should be and we can write in matrix notation $N^T M^T$.

Take note that in index notation we can always change the position of the objects in question freely, because for example M_{ki} and N_{jk} are just individual elements of the matrices, i.e. ordinary numbers.

C.2 Matrix Exponential Function

We already derived how the exponential function looks as a series, and therefore we can define what we mean when we put a matrix into the exponential function. e^M, with an arbitrary matrix M, is defined by this series

$$\mathrm{e}^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}. \tag{C.4}$$

It is important to take note that in general $\mathrm{e}^M \mathrm{e}^N \neq \mathrm{e}^{M+N}$. The identity $\mathrm{e}^M \mathrm{e}^N = \mathrm{e}^{M+N}$ is only correct if $MN = NM$.

C.3 Determinants

The determinant of a matrix is a rather unintuitive, but immensely useful notion. For example, if the determinant of some matrix is non-zero, we automatically know that the matrix is invertible[2]. Unfortunately proving this lies beyond the scope of this text and the interested reader is referred to the standard texts about linear algebra.

[2] A matrix M is invertible, if we can find an inverse matrix, denoted by M^{-1}, with $M^{-1}M = 1$.

The determinant of a 3×3 matrix can be defined using the Levi-Civita symbol

$$\det(A) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} A_{1i} A_{2j} A_{3k} \tag{C.5}$$

and analogously for n-dimensions

$$\det(A) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \dots \sum_{i_n=1}^{n} \epsilon_{i_1 i_2 \dots i_n} A_{1i_1} A_{2i_2} \dots A_{ni_n}. \tag{C.6}$$

It is instructive to look at an explicit example in two dimensions:

$$\det(A) = \det \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = (3 \cdot 2) - (5 \cdot 1) = 1$$

Or for a general three dimensional matrix

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \tag{C.7}$$

C.4 Eigenvalues and Eigenvectors

Two very important notions from linear algebra that are used all over physics are eigenvalues and eigenvectors. The eigenvectors $\vec{v}$ and eigenvalues λ are defined for each matrix M by the equation

$$M\vec{v} = \lambda\vec{v}. \tag{C.8}$$

The important thing is that we have on both sides of the equation the same vector $\vec{v}$. In words this equation means that the vector $\vec{v}$ remains, up to a constant λ, unchanged if multiplied with the matrix M. To each eigenvector we have a corresponding eigenvalue. There are quite sophisticated computational schemes for finding the eigenvectors and eigenvalues of a matrix and the details can be found in any book about linear algebra.

To get a feeling for the importance of these notions think about rotations. We can describe rotations by matrices and the eigenvector of a rotation matrix defines the rotational axis.

C.5 Diagonalization

Eigenvectors and eigenvalues can be used to bring matrices into diagonal form, which can be quite useful for computations and physical interpretations. It can be shown that any diagonalizable matrix M can be rewritten in the form[3]

$$M = N^{-1}DN, \tag{C.9}$$

where the matrix N consists of the eigenvectors as its column and D is diagonal with the eigenvalues of M on its diagonal:

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = N^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} N = \begin{pmatrix} \vec{v}_1, \vec{v}_2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \vec{v}_1, \vec{v}_2 \end{pmatrix} \tag{C.10}$$

[3] In general, a transformation of the form $M' = N^{-1}MN$ refers to a basis change. M' is the matrix M in another coordinate system. Therefore, the result of this section is that we can find a basis where M is particularly simple, i.e. diagonal.

D

Additional Mathematical Notions

D.1 Fourier Transform

The idea of the Fourier transform is similar to the idea that we can express any vector $\vec{v}$ in terms of basis vectors[1] ($\vec{e}_1, \vec{e}_2, \vec{e}_3$). In ordinary Euclidean space the most common choice is

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{D.1}$$

and an arbitrary three-dimensional vector $\vec{v}$ can be expressed in terms of these basis vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{D.2}$$

The idea of the Fourier transform is that we can do the same thing with functions[2]. For periodic functions such a basis is given by $\sin(kx)$ and $\cos(kx)$. This means we can write every periodic function $f(x)$ as

$$f(x) = \sum_{k=0}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \tag{D.3}$$

with constant coefficients a_k and b_k.

An arbitrary (not necessarily periodic) function can be written in terms of the basis e^{ikx} and e^{-ikx}, but this time with an integral instead of a sum[3]

[1] This is explained in more detail in appendix A.1.

[2] In a more abstract sense, functions are abstract vectors. This is meant in the sense that functions are elements of some vector space. For different kinds of functions a different vector space. Such abstract vector spaces are defined similar to the usual Euclidean vector space, where our ordinary position vectors live (those with the little arrow $\vec{}$). For example, take note that we can add two functions, just as we can add two vectors, and get another function. In addition, it's possible to define a scalar product.

[3] Recall that an integral is just the limit of a sum, where the discrete k in $\sum_k$ becomes a continuous variable in $\int dk$.

© Springer International Publishing Switzerland 2015

J. Schwichtenberg, *Physics from Symmetry*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19201-7

$$f(x)=\int_0^\infty dk\Big(a_k e^{ikx}+b_k e^{-ikx}\Big), \qquad \text{(D.4)}$$

which we can also write as

$$f(x)=\int_{-\infty}^\infty dk f_k e^{-ikx}. \qquad \text{(D.5)}$$

The expansion coefficients f_k are often denoted $\tilde{f}(k)$, which is then called the Fourier transform of $f(x)$.

D.2 Delta Distribution

In some sense, the delta distribution is to integrals what the Kronecker delta[4] is to sums. We can use the Kronecker delta δ_{ij} to pick one specific term of any sum. For example, consider

[4] The Kronecker delta is defined in appendix B.5.5.

$$\sum_{i=1}^{3} a_i b_j = a_1 b_j + a_2 b_j + a_3 b_j \qquad \text{(D.6)}$$

and let's say we want to pick the second term of the sum. We can do this using the Kronecker delta δ_{2i}, because then

$$\sum_{i=1}^{3} \delta_{2i} a_i b_j = \underbrace{\delta_{21}}_{=0} a_1 b_j + \underbrace{\delta_{22}}_{=1} a_2 b_j + \underbrace{\delta_{23}}_{=0} a_3 b_j = a_2 b_j. \qquad \text{(D.7)}$$

Or more general

$$\sum_{i=1}^{3} \delta_{ik} a_i b_j = a_k b_j. \qquad \text{(D.8)}$$

The delta distribution $\delta(x-y)$ is defined by

$$\int dx f(x)\delta(x-y) = f(y). \qquad \text{(D.9)}$$

Completely analogous to the Kronecker delta, the delta distribution picks one term[5] from the integral. In addition, we can use this analogy to motivate from the equality

[5] The term where $x=y$. For example, $\int dx f(x)\delta(x-2)=f(2)$.

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad \text{(D.10)}$$

the equality

$$\frac{\partial f(x_i)}{\partial f(x_j)} = \delta(x_i - x_j). \qquad \text{(D.11)}$$

This is of course by no means a proof, but this equality can be shown in a rigorous way, too. There is a lot more one can say about this object, but for the purpose of this book it is enough to understand what the delta distribution does. In fact, this is how the delta distribution was introduced in the first place by Dirac.

Bibliography

Ian J.R. Aitchison and Anthony J.G. Hey. *Gauge Theories in Particle Physics*. CRC Press, 4th edition, 12 2012. ISBN 9781466513174.

John C. Baez and Javier P. Muniain. *Gauge Fields, Knots, and Gravity*. World Scientific Pub Co Inc, 1st edition, 9 1994. ISBN 9789810220341.

Alessandro Bettini. *Introduction to Elementary Particle Physics*. Cambridge University Press, 2nd edition, 4 2014. ISBN 9781107050402.

Ta-Pei Cheng. *Relativity, Gravitation and Cosmology: A Basic Introduction*. Oxford University Press, 2nd edition, 1 2010. ISBN 9780199573646.

Paul A. M. Dirac and Physics. *Lectures on Quantum Mechanics*. Dover Publications, 1st edition, 3 2001. ISBN 9780486417134.

Albert Einstein. The foundation of the general theory of relativity. 1916.

Albert Einstein and Francis A. Davis. *The Principle of Relativity*. Dover Publications, reprint edition, 6 1952. ISBN 9780486600819.

Richard P. Feynman and Albert R. Hibbs. *Quantum Mechanics and Path Integrals: Emended Edition*. Dover Publications, emended editon edition, 7 2010. ISBN 9780486477220.

Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics, Volume 3*. Addison Wesley, 1st edition, 1 1971. ISBN 9780201021189.

Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics: Volume 2*. Addison-Wesley, 1st edition, 2 1977. ISBN 9780201021172.

Daniel Fleisch. *A Student's Guide to Vectors and Tensors*. Cambridge University Press, 1st edition, 11 2011. ISBN 9780521171908.

© Springer International Publishing Switzerland 2015

J. Schwichtenberg, *Physics from Symmetry*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19201-7

Jon Fripp, Deborah Fripp, and Michael Fripp. *Speaking of Science*. Newnes, 1st edition, 4 2000. ISBN 9781878707512.

William Fulton and Joe Harris. *Representation Theory: A First Course*. Springer, 1st corrected edition, 8 1999. ISBN 9780387974958.

Herbert Goldstein, Charles P. Poole Jr., and John L. Safko. *Classical Mechanics*. Addison-Wesley, 3rd edition, 6 2001. ISBN 9780201657029.

David J. Griffiths. *Introduction to Quantum Mechanics*. Pearson Prentice Hall, 2nd edition, 4 2004. ISBN 9780131118928.

David J. Griffiths. *Introduction to Electrodynamics*. Addison-Wesley, 4th edition, 10 2012. ISBN 9780321856562.

Francis Halzen and Alan D. Martin. *Quarks and Leptons: An Introductory Course in Modern Particle Physics*. Wiley, 1st edition, 1 1984. ISBN 9780471887416.

Nadir Jeevanjee. *An Introduction to Tensors and Group Theory for Physicists*. Birkhaeuser, 1st edition, August 2011. ISBN 978-0817647148.

Robert D. Klauber. *Student Friendly Quantum Field Theory*. Sandtrove Press, 2nd edition, 12 2013. ISBN 9780984513956.

Cornelius Lanczos. *The Variational Principles of Mechanics*. Dover Publications, 4th edition, 3 1986. ISBN 9780486650678.

Michele Maggiore. *A Modern Introduction to Quantum Field Theory*. Oxford University Press, 1st edition, 2 2005. ISBN 9780198520740.

Franz Mandl and Graham Shaw. *Quantum Field Theory*. Wiley, 2nd edition, 5 2010. ISBN 9780471496847.

Rollo May. *The Courage to Create*. W. W. Norton and Company, reprint edition, 3 1994. ISBN 9780393311068.

Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation*. W. H. Freeman, 1st edition, 9 1973. ISBN 9780716703440.

Matthew Robinson. *Symmetry and the Standard Model*. Springer, 1st edition, August 2011. ISBN 978-1-4419-8267-4.

Robert S. Root-Bernstein and Michele M. Root-Bernstein. *Sparks of Genius*. Mariner Books, 1st edition, 8 2001. ISBN 9780618127450.

Lewis H. Ryder. *Quantum Field Theory*. Cambridge University Press, 2nd edition, 6 1996. ISBN 9780521478144.

J. J. Sakurai. *Modern Quantum Mechanics*. Addison Wesley, 1st edition, 9 1993. ISBN 9780201539295.

Michael Spivak. *A Comprehensive Introduction to Differential Geometry, Vol. 1, 3rd Edition*. Publish or Perish, 3rd edition, 1 1999. ISBN 9780914098706.

Andrew M. Steane. An introduction to spinors. *ArXiv e-prints*, December 2013.

John Stillwell. *Naive Lie Theory*. Springer, 1st edition, August 2008a. ISBN 978-0387782140.

John Stillwell. *Naive Lie Theory*. Springer, 1st edition, 8 2008b. ISBN 9780387782140.

Edwin F. Taylor and John Archibald Wheeler. *Spacetime Physics*. W. H. Freeman, 2nd edition, 3 1992. ISBN 9780716723271.

Frank Wilczek. Riemann-einstein structure from volume and gauge symmetry. *Phys. Rev. Lett.*, 80:4851–4854, Jun 1998. DOI: 10.1103/PhysRevLett.80.4851.

Harry Woolf, editor. *Some Strangeness in the Proportion*. Addison-Wesley, 1st edition, 2 1981. ISBN 9780201099249.

Anthony Zee. *Quantum Field Theory in a Nutshell*. Princeton University Press, 1st edition, 3 2003. ISBN 9780691010199.

Anthony Zee. *Einstein Gravity in a Nutshell*. Princeton University Press, 1st edition, 5 2013. ISBN 9780691145587.

Index

© Springer International Publishing Switzerland 2015
J. Schwichtenberg, *Physics from Symmetry*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19201-7