Appendix
Operations with Matrices and Vectors

This Appendix contains the minimum information necessary to understand the material in this monograph involving the matrices and vectors.

A.1 Definitions

Throughout this monograph, matrices are understood as rectangular tables, composed of scalar elements consisting, in general, of m rows and n columns:

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \ddots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \tag{A.1}$$

and referred to as $m \times n$ matrices. Vectors are, essentially, $n \times 1$ matrices (column $n-$ vectors, or just $n-$vectors) or $1 \times n$ matrices (row $n-$vectors):

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} \tag{A.2}$$

The scalars can be considered as 1×1 matrices, or $1-$vectors.

The transpose $\mathbf{A}^T$ of matrix $\mathbf{A}$, Eq. (A.1), is an $n \times m$ matrix with elements

$$\left(\mathbf{A}^T\right)_{ij} = A_{ji}\,(i = 1, \dots m, j = 1, \dots n) \tag{A.3}$$

E.A. Ustinov, *Sensitivity Analysis in Remote Sensing*,
SpringerBriefs in Earth Sciences, DOI 10.1007/978-3-319-15841-9

In particular, vectors **a** and **b** in Eq. (A.2) are mutually transpose to each other if $a_i = b_i$ $(i = 1, \ldots n)$:

$$\mathbf{b} = \mathbf{a}^T,\ \mathbf{a} = \mathbf{b}^T \tag{A.4}$$

The square matrix **A** is called a symmetric matrix if $\mathbf{A}^T = \mathbf{A}$.

Block matrices are defined as matrices, elements of which are matrices themselves (or vectors and scalars, in particular). These elements should satisfy two requirements: (1) matrix elements of each row of the block matrix should have equal numbers of rows; (2) matrix elements of each column of the block matrix should have equal numbers of columns. In particular, an $m \times n$ matrix can be considered both as a column $m-$vector consisting of m row $n-$vectors, and as a row $n-$vector consisting of n column $m-$vectors.

A.2 Algebra of Matrices and Vectors

Matrices with matching numbers of rows and columns are added to, and subtracted from each other just element by element:

$$(\mathbf{A} \pm \mathbf{B})_{ij} = (\mathbf{A})_{ij} \pm (\mathbf{B})_{ij} \tag{A.5}$$

The elements of the product of a matrix **A** with m columns and a matrix **B** with m rows are defined as a sum of m terms:

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{m} (\mathbf{A})_{ik} (\mathbf{B})_{kj} \tag{A.6}$$

The resulting matrix **AB** has the number of rows corresponding to that of the matrix **A** and number of columns corresponding to that of the matrix **B**. The number of rows of the matrix **A** and number of columns of the matrix **B** can be arbitrary. In general, the matrix product is not commutative, and $\mathbf{AB} \neq \mathbf{BA}$.

By direct substitution it can be seen that

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T,\ (\mathbf{ABC})^T = \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T, \text{ etc.} \tag{A.7}$$

Important particular cases are presented below.

The product **ab** of the row $n-$vector **a** and the column $n-$vector **b** is a scalar:

$$\mathbf{ab} = \sum_{k=1}^{n} (\mathbf{a})_k (\mathbf{b})_k \tag{A.8}$$

The product **ba** of these vectors is an $n \times n$ matrix with elements:

$$(\mathbf{ba})_{ij} = (\mathbf{b})_k(\mathbf{a})_k,\ (i,j = 1,\ldots n) \tag{A.9}$$

The product $a\mathbf{b}$ of the scalar a and the row $n-$vector **b** is a row $n-$vector with elements:

$$(a\mathbf{b})_j = a(\mathbf{b})_j \tag{A.10}$$

The product $\mathbf{a}b$ of the column $n-$vector **a** and the scalar b is a column $n-$vector with elements:

$$(\mathbf{a}b)_j = (\mathbf{a})_j b \tag{A.11}$$

In the remainder of this Appendix, speaking of vectors we mean column vectors.

The scalar, or inner, product $(\mathbf{a}, \mathbf{b}) \equiv \mathbf{a} \cdot \mathbf{b}$ of two $n-$vectors **a** and **b** is defined as a scalar:

$$c = \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b} = \sum_{k=1}^{n} (\mathbf{a})_k(\mathbf{b})_k \tag{A.12}$$

The tensor, or outer, product $\mathbf{a} \otimes \mathbf{b}$ of an $m-$vector **a** and $n-$vector **b** is defined as an $m \times n$ matrix

$$\mathbf{C} = \mathbf{a} \otimes \mathbf{b} = \mathbf{ab}^T \tag{A.13}$$

with elements

$$(\mathbf{C})_{ij} = (\mathbf{a})_i(\mathbf{b})_j \tag{A.14}$$

The identity matrix **I** is a square matrix with diagonal elements $(\mathbf{I})_{ii} \equiv 1$ and non-diagonal elements $(\mathbf{I})_{i \neq j} \equiv 0$. It commutes with any square matrix **A** of matching size:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \tag{A.15}$$

The inverse matrix $\mathbf{A}^{-1}$ of the square matrix **A** is defined as a square matrix satisfying the equality

$$\mathbf{AA}^{-1} = \mathbf{I} \tag{A.16}$$

By definition $\mathbf{A}^{-1}$ commutes with **A**.

The transpose and inversion operations are commutative:

$$\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T \tag{A.17}$$

The simple proof of this consists in reduction of Eq. (A.17) to an identity $\mathbf{I} = \mathbf{I}$ by identical transformations: multiplication of both sides of Eq. (A.17) by $\mathbf{A}^T$, application of Eq. (A.7) to $\mathbf{A}$ and $\mathbf{A}^{-1}$, and application of Eq. (A.16) to pairs $\mathbf{A}, \mathbf{A}^{-1}$ and $\mathbf{A}^T, \left(\mathbf{A}^T\right)^{-1}$ in both sides of the resulting equality.

A.3 Differential Operations

Derivatives of matrices and vectors with respect to a single scalar argument are carried out element by element. In the general case of an $m \times n$ matrix, Eq. (A.1) the derivative with respect to an argument x is an $m \times n$ matrix with elements:

$$\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\right)_{ij} = \frac{\mathrm{d}(\mathbf{A})_{ij}}{\mathrm{d}x} \tag{A.18}$$

Derivatives with respect to spatial coordinates $\mathbf{r}$ in 2D and 3D space, which are combined, correspondingly, in a 2− or 3− vector

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \text{ or } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{A.19}$$

are represented by the nabla operator ∇, which is, correspondingly, a 2− or 3− vector

$$\nabla = \frac{\partial}{\partial \mathbf{r}} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix}, \text{ or } \nabla = \frac{\partial}{\partial \mathbf{r}} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \tag{A.20}$$

The nabla operator can be applied both to scalars and vectors following the multiplication rules presented above. If $a(\mathbf{r})$ is a scalar function, then

$$\nabla a = \frac{\partial a}{\partial \mathbf{r}} = \begin{pmatrix} \partial a/\partial x \\ \partial a/\partial y \end{pmatrix}, \text{ or } \nabla a = \frac{\partial a}{\partial \mathbf{r}} = \begin{pmatrix} \partial a/\partial x \\ \partial a/\partial y \\ \partial a/\partial z \end{pmatrix} \tag{A.21}$$

is, correspondingly, a 2− or 3− vector. The result is a gradient of a: $\nabla a = \text{grad } a$. If $\mathbf{a}(\mathbf{r})$ is a 2− or 3− vector function, then

$$\nabla^T \mathbf{a} = \left(\frac{\partial}{\partial \mathbf{r}}\right) \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y}, \text{ or } \nabla^T \mathbf{a} = \left(\frac{\partial}{\partial \mathbf{r}}\right) \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \quad (A.22)$$

The result is a divergence of $\mathbf{a}$: $\nabla^T \mathbf{a} = \text{div } \mathbf{a}$.

Derivatives of products of matrices and vectors are, in suitable notations, the straight forward generalizations of the case of scalar functions of a single scalar argument:

$$(uv)' = u'v + uv' \quad (A.23)$$

For the general case of the product of a matrix $\mathbf{A}$ with m columns and a matrix $\mathbf{B}$ with m rows, by direct substitution it can be seen that the elements of the derivative with respect to a single scalar argument have the form [cf. Eq. (A.6)]:

$$(\mathbf{AB})' = \mathbf{A}'\mathbf{B} + \mathbf{AB}' \quad (A.24)$$

A.4 Integral Operations

Integration of matrices and vectors with respect to any scalar argument are carried out element by element. In the general case of an $m \times n$ matrix $\mathbf{A}$, Eq. (A.1) the integral over an argument x is an $m \times n$ matrix with elements:

$$\left(\int \mathbf{A} \, \mathrm{d}x\right)_{ij} = \int (\mathbf{A})_{ij} \, \mathrm{d}x \quad (A.25)$$

Integration over a multidimensional domain D_x of arguments x is reduced to a corresponding multiple integration:

$$\left(\int\limits_{D_x} \mathbf{A} \, \mathrm{d}x\right)_{ij} = \int\limits_{D(x)} (\mathbf{A})_{ij} \, \mathrm{d}x \quad (A.26)$$

By analogy with the inner product (a, b) of two scalar functions $a(x)$ and $b(x)$

$$(a, b) = \int\limits_{D_x} a(x)b(x) \, \mathrm{d}x \quad (A.27)$$

the inner product of two vector functions $\mathbf{a}(x)$ and $\mathbf{b}(x)$ has the form:

$$(\mathbf{a}, \mathbf{b}) = \int_{D_x} \mathbf{a}^T(x)\mathbf{b}(x)\ \mathrm{d}x \tag{A.28}$$

Similarly, the inner product of two matrices $\mathbf{A}$ and $\mathbf{B}$ with a matching number of rows has the form:

$$(\mathbf{A}, \mathbf{B}) = \int_{D_x} \mathbf{A}^T(x)\mathbf{B}(x)\ \mathrm{d}x \tag{A.29}$$

In particular, the inner product of an $n \times m$ matrix $\mathbf{A}$ and an $n-$ vector $\mathbf{b}$ has the form:

$$(\mathbf{A}, \mathbf{b}) = \int_{D_x} \mathbf{A}^T(x)\mathbf{b}(x)\ \mathrm{d}x \tag{A.30}$$

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E.A. Ustinov, *Sensitivity Analysis in Remote Sensing*,
SpringerBriefs in Earth Sciences, DOI 10.1007/978-3-319-15841-9