Appendix
On the History of Web Geometry

It seems impossible to grasp the ins and outs of a mathematical field without setting it back in its historical context. An attempt for web geometry, certainly incomplete and biased, is made in the next few pages.

Origins

If the *birth* of web geometry can be ascribed to the middle of the 1930s in Hamburg (see below), some precedents can be found as early as the middle of the nineteenth century. The concepts and problems of web geometry spring from two different fields of the nineteenth century mathematics: projective differential geometry and nomography.

Web geometry comes mainly from the first. At that time, projective differential geometry mainly consisted in the study of projective properties of curves and surfaces in the ordinary space $\mathbb{R}^3$, that is of their differential properties that are invariant up to homographies.

Gaussian geometry, which had appeared before, studied the properties of curves and surfaces in ordinary euclidean space that are invariant up to isometric transformations. Gauss and other mathematicians pointed out how useful the first and second fundamental forms are for the study of surfaces. They also brought to light the relevance of derived concepts, such as the principal, asymptotic, and conjugated directions. When considering the integral curves of these tangent direction fields, the mathematicians of the time were considering what they called 2-nets of '*lines*' on surfaces, that is the data of two families of curves, or in more modern terms, 2-webs. When they endeavored to generalize these constructions to projective differential geometry, some *3-nets* projectively attached to surfaces in $\mathbb{R}^3$ quite naturally made their appearance (for instance, Darboux introduced a 3-web called after him in [41]; see also Sect. 1.4.4 in this book).

J.V. Pereira, L. Pirio, *An Invitation to Web Geometry*, IMPA Monographs 2,
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These webs were useful back then because they encoded properties of the surfaces under study. Thomsen's paper [129] is a good illustration of this fact. In this article, Thomsen shows that a surface in $\mathbb{R}^3$ is isothermally asymptotic[1] if and only if its Darboux 3-web is hexagonal.[2] At that time, the study of 3-webs on surfaces from the point of view of projective differential geometry was on the agenda.

A particular feature of Thomsen's result is his characterization of the geometric-differential property of being isothermally asymptotic by a closedness property of more topological nature that is (or not) verified by a configuration traced on the surface itself. It is this feature which struck some mathematicians and led to the study of webs at the beginning of the 1930s.

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The second source of web geometry is nomography. This discipline, nowadays practically extinct, belonged to the field of applied mathematics in the 1900s. It was established as an autonomous mathematical discipline by M. d'Ocagne. It consisted in a method of 'graphical calculus' which allowed engineers to calculate rather fast. To explain its principle (which to-day appears rather naïve), let

$$L(V_1, V_2, V_3) = 0$$

be a mathematical law linking three physical variables V_1, V_2, and V_3. Is there a quick and accurate way to determine one variable, say V_i, from the other two: V_j and V_k? To solve this problem, people used nomograms. A nomogram is a graphic which represents curves according to values of the variables V_1, V_2, and V_3. For instance, to find the value of v_1 in function of values v_2 and v_3 of the variables V_2 and V_3 (respectively), one has to find the intersection point of the curves $V_2 = v_2$ and $V_3 = v_3$. Through (or near) this point goes a curve $V_1 = v_1$, and v_1 is the sought value (Fig. A.1).

What nowadays seems to be far from actual mathematics was once an important part of the mathematical culture. It was probably after considering some results of nomography that Hilbert formulated the 13th of the famous 23 problems that he stated at the International Congress of Mathematics of 1900.

The main disadvantage of nomography was the problem of its readability. Of course, the nomograms where the curves coincided with (pieces of) lines were easier to use. Hence the problem to know whether it is possible to linearize the curves of a given nomogram. Or equivalently, whether it is possible to linearize a 3-web of curves on the plane. For more precisions on the links between nomography and web geometry, the interested reader can consult [2].

[1]Geometers of the nineteenth century had established a very rich "bestiary" of surfaces in $\mathbb{R}^3$. The *isothermally asymptotic surfaces* (or "*F-surfaces*") formed one of the classes in their classification (see [51] for a modern definition.)

[2]Thomsen's result applies to real surfaces in $\mathbb{R}^3$, thus his statement is different depending on one takes place at a neighborhood of an elliptic point or a hyperbolic point of the considered surface.

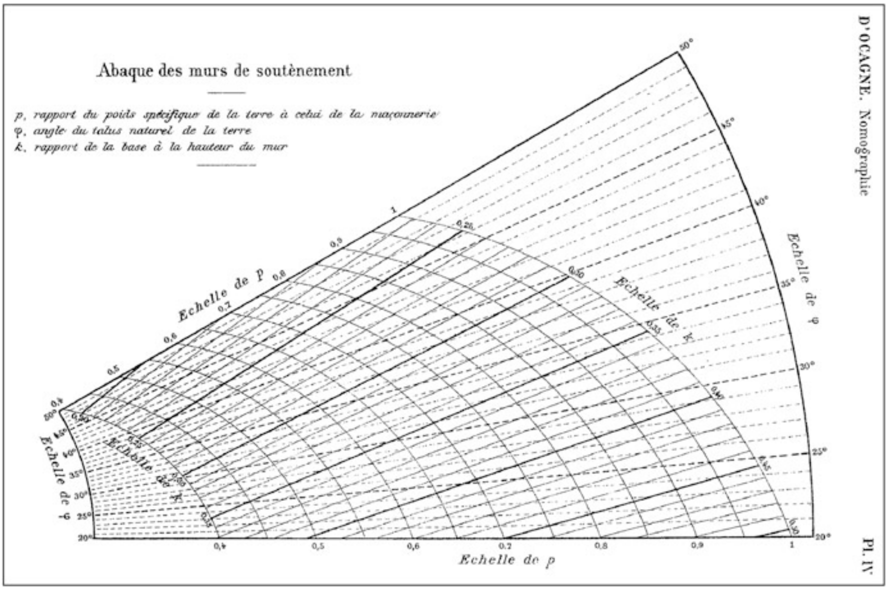

Fig. A.1 A nomogram from the book [43] by D'Ocagne

Birth of Web Geometry: Spring of 1927 in Naples

Thomsen's paper [129] is considered as marking the birth of web geometry. According to Blaschke (see the beginning of the foreword in [17]) this paper is the result of his Spring walks with Thomsen on Posillipo hill, in the vicinity of Naples, in 1927. Even if it concerns the study of some surfaces in $\mathbb{R}^3$, it clearly shows that a plane configuration made of three families of curves (i.e., a 3-web) admits local analytic invariants. It seems that the equivalence between the vanishing of the curvature of a 3-web (which is a condition of analytic nature) and the hexagonality condition (which is a property of topological nature)[3] struck these two mathematicians and led them (with others) to study the matter (Fig. A.2).

Early Developments: Hamburg School (1927–1938)

A short time after Thomsen's paper [129] was published, a group led by Blaschke was set up in Hamburg to do research on webs. Blaschke and his coworkers[4] found many results which established web geometry as a discipline. It is a remarkable fact

[3] See Theorem 1.2.4 in this book.

[4] Bol, Chern, Mayrhoffer, Podehl, Walberer were active members of this group. Kähler, Zariski, Reidemester, and others also worked on this subject but in an occasional way.

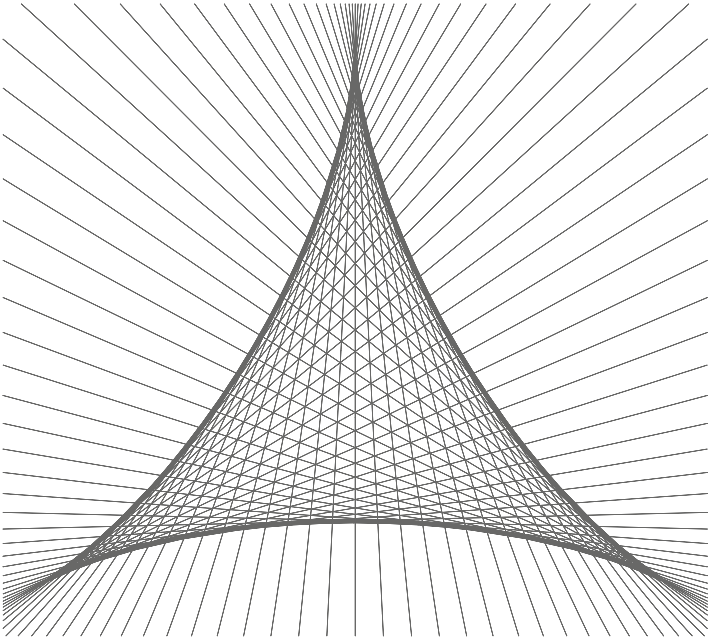

Fig. A.2 A 3-web with vanishing curvature formed by the tangents of a hypocycloid with 3 cusps

that a series of more than 60 papers were published in a variety of journals between 1927 and 1938, mainly by members of the Hamburg school of web geometry, under the common label of "Topologie Fragen der Differentialgeometrie."[5]

Their work developed along three main directions:

- The study of webs from the differential geometry viewpoint, through the analytical invariants which can be associated with them;
- The study of the relations between webs and abstract geometric configurations linked to the algebraic theory of (quasi-)groups;
- The interpretation of web geometry as a relative of projective algebraic geometry, notably via the notion of abelian relation.

Due to a lack of competence, we will not expand on the two former issues but we will focus on the latter direction which is also the main subject of this book. It deals with the links between webs and algebraic geometry, which have their origins in results obtained by Blaschke, Bol, and Howe.

At the beginning, these results were mainly about planar webs. Firstly, Blaschke came up with an interpretation of a theorem by Graf and Sauer [61] in the framework of web geometry. This theorem says that a linear 3-web carrying an abelian relation

[5]In English, "Topological questions of differential geometry".

is constituted by the tangents to a plane algebraic curve of class 3.[6] As soon as 1932, Blaschke and Howe [18] generalized this theorem to the case of linear k-webs carrying at least one complete abelian relation, thus bringing to light the usefulness of the notion of abelian relation. Bol's result giving the explicit bound $\frac{1}{2}(k-1)(k-2)$ on the dimension of the space of abelian relations of a planar k-web appeared shortly afterwards in [19] and allowed to define the rank of a web. Using this formalism, Howe noticed that Lie's result about the double-translation surfaces can be understood in the framework of web geometry as the striking fact that a planar 4-web of rank 3 is algebraizable. The relationship between the planar webs of maximal rank and Abel's Theorem was reported the following year by Blaschke in [15], which brought up the final definition of the notion of algebraic web. In the same paper, Blaschke expounded the generalization of Lie's Theorem to the 5-webs of maximal rank, a result which was later proved to be incorrect by Bol. Surprisingly, Blaschke also exhibited Bol's 5-web $\mathcal{B}_5$ as an example of non algebraizable 5-web of rank 5, while it is of maximal rank 6 (see below).

In 1933, Blaschke set about studying webs in dimension three. In [14], he established a bound $\pi(3,k)$ on the rank of a k-web of hypersurfaces in $\mathbb{C}^3$. One year later, Bol gave in [20] one of the most important results obtained at the time: *for $k \geq 6$, a k-web of hypersurfaces on $\mathbb{C}^3$ of maximal rank $\pi(3,k)$ is algebraizable*. This success certainly played a role in Blaschke's attempt in [15, 16] to obtain algebraization results for planar webs of maximal rank. Only in 1936 it was made clear that the result he was looking for was unattainable. In [21], Bol realized that $\mathcal{B}_5$ carries one more abelian relation, related to Abel's five terms equation for the dilogarithm; hence, it is an instance of a 5-web of rank 6 which is not algebraizable.

In this same year 1936, Chern defended his PhD dissertation on webs, written under Blaschke's direction. He then published two papers. The 60th issue of the "Topologische Fragen der Differentialgeometrie" series [30] is of special interest here. Generalizing Blaschke's result, he obtains a bound on the rank of a web of codimension one in arbitrary dimension[7] which now bears his name.

Thus, in 1936, most of the notions studied in this book had been brought to light. A general survey of the state the art at the time can be found in the third part of the book [17], to which the reader is asked to refer.

Finally, in this very year Blaschke shifted his interest from web geometry to integral geometry. Few members of the Hamburg school worked again on webs, with the notable exceptions of Blaschke and Chern, but this time in a different way (see below).

[6] By definition, the **class** of a reduced plane algebraic curve C is the number of tangent lines to C passing through a generic point of $\mathbb{P}^2$. Via projective duality, it corresponds to the degree of the dual curve $\check{C} \subset \check{\mathbb{P}}^2$.

[7] See Theorem 2.3.8 in this book.

Web Geometry in Mid Twentieth Century (1938–1960)

Blaschke strongly supported exchanges between mathematicians. From 1927 to 1960 he travelled a lot and had the opportunity to give lectures about web geometry in numerous countries (for instance, in Romania, Greece, Spain, Italy, the USA, India, Japan), thus inspiring people of various nationalities and backgrounds to do research on web geometry.[8]

It seems that Blaschke went to Italy many times during this period. As a by-product, an Italian school of web geometry developed at that time. Bompiani, Terracini, and Buzano were its most prominent contributors. Their work was chiefly about the links between the geometry of planar webs and the projective differential geometry of surfaces. In the 1950s and 1960s, a second Italian school of web geometry appeared, probably thanks to Bompiani's influence. He, Vaona, and Villa (among others) published papers on the projective deformation of planar 3-webs, but with no major outcome.

The work of the Romanian mathematicians Pantazi and Mihăileanu must be mentioned as well. During the 1930s and 1940s, they obtained interesting results on how to determine the rank of planar webs. These results were published as short notes in Romanian journals (see [87, 101]) and were then forgotten.

The war and later Blaschke's political stance during the war (see [123, p. 423]) put an end to his influence for some time. When things came back to normal, he gave lectures again, on webs among other things. Although he didn't obtain new results, these lectures induced new researches once more, for instance by Dou in Barcelona [44–46] and by Ozkän in Turkey [100].

Russian School (From 1965 Onwards)

More than at the Hamburg school, it is at the Moscow school of differential geometry that the Russian school of web geometry, led first by Akivis, and then by Akivis and Goldberg, seems to have its origin. Under the influence of the work of Élie Cartan, a Russian school of differential geometry developed in USSR at the instigation of Finikov from the 1940s onwards. Projective differential geometry was studied in full generality and involved the study of some nets (which could be called webs but only in a weak sense) projectively attached to (analytic) projective subvarieties. It is probably this fact which led to the study of webs for their own sake in arbitrary dimension and/or codimension, from the 1960s onwards. Akivis was joined by Goldberg quite early. They explored several directions in web geometry, published many papers and had many students.

[8]For instance, it is in attending some conferences given by Blaschke at Pekin in 1933 that Chern became interested in web geometry and decided to go and study at Hamburg.

The work of this school chiefly dealt with the differential geometry of webs and with the interactions between webs and the theory of quasigroups. The links with algebraic geometry were not their major concern. Their results had little influence in the West for two main reasons: (1) their papers were in Russian, hence they were not distributed in the West; (2) the method they used was the *Cartan–Laptev method*,[9] which non-specialists do not understand easily.

The reader who wishes to get an outline of the methods and results of this school may consult the books [3] and [59].

Chern's and Griffiths' Work (1977–1980)

Throughout his professional life Chern kept being interested in webs, particularly in the notion of web of maximal rank, as shown in [31, 32] and [33]. This point can be illustrated by quoting the last lines of [33]:

> *Due to my background I like algebraic manipulation, as Griffiths once observed. Local differential geometry calls for such works. But good local theorems are difficult to come by. The problem on maximal rank webs discussed above*[10] *is clearly an important problem, and will receive my attention.*
> *My mathematical education goes on.*

In 1978, he resumed working on webs of maximal rank jointly with Griffiths. In the long paper [34], they set about demonstrating that a k-web of codimension one and of maximal rank $\pi(n, k)$ is algebraizable when $n > 2$ and $k \geq 2n$. Their proof is not complete (cf. [36]) and it is necessary to make an extra non-natural assumption to ensure the validity of the result. They also got a sharp bound for the rank of webs of codimension two in [35].

Griffiths' interest in the subject probably came from the links between web geometry and algebraization results like the converse of Abel's Theorem discussed in Chap. 4. He discussed web geometry at the opening talk he gave at Abel's bicentennial conference held at Oslo in 2002, see [64].

Although it contains a non-trivial mistake, the paper [34] has been quite influential in web geometry. It has popularized the subject and led the Russian school to pay attention to the notions of abelian relations and rank. It is probably from [34] that Trépreau has taken up Bol's method to obtain a proof of the result originally aimed at by Chern and Griffiths. The present book would not exist if [34] had not been written. The reader should read it, as it contains a masterfully written introductory part putting things in perspective and offers different proofs of many of the results included here.

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[9] Cartan–Laptev method is a reinterpretation/generalization of the methods of the mobile frame and of equivalence of É. Cartan by the Russian geometer G. Laptev.

[10] Chern is referring to the classification of webs of maximal rank.

Modern Developments (1980–2000)

A number of new interesting results in web geometry have been obtained in the last 20 years of the twentieth century. The only results mentioned here are those relating to rank, abelian relations and webs of maximal rank.

The abelian relations of Bol's web all come (after analytic prolongation) from its dilogarithmic abelian relation, which thus appears more fundamental than the other relations. In 1982, in [54], Gelfand and MacPherson found a beautiful geometric interpretation of this relation. In it Bol's web appears defined on the space of projective configurations of five points of $\mathbb{RP}^2$. In [40], Damiano considers, for $n \geq 2$ a curvilinear $(n+3)$-web D_n naturally defined on the space of projective configurations of $n+3$ points in $\mathbb{RP}^n$. He shows that this web is of maximal rank and gives a geometric interpretation of the "main abelian relation" of D_n, thus obtaining a family of exceptional webs which generalizes Bol's web.

During the 1980s, Little studied some algebro-geometrical consequences of the converse to Abel's Theorem. He worked in particular on double-translation hypersurfaces (see for instance [85]) and published one paper on webs [86]. In it, he used some results of Mumford and Roitman on the space of 0-cycles on a projective surface of positive genus to construct examples of non-algebraizable two-codimensional webs of maximal rank from K3 surfaces embedded in projective space.

From 1980 to 2000, Goldberg studied the webs of codimension strictly bigger than one from the point of view of their rank. He obtained many results, most of which are expounded in [59]. Among his results, one finds the construction of several non-algebraizable webs of maximal rank.

Motivated by some questions in the geometric study of differential equations, Nakai began to study webs starting from the end of the 1980s using methods coming from differential topology. He worked mainly on webs in a real setting in relation with PDEs but he also produced papers on complex and real analytic webs such as [92, 93]. He also produced several expository papers [94–96] that certainly helped to make the subject more popular.

At the beginning of the 1990s, Hénaut started studying webs in the complex analytical realm. He published about 15 papers on the matter. His research is mainly about rank and abelian relations, and is concerned with webs of arbitrary codimension as well as planar webs (see [75] for an outline of the results he obtained before 2000). The papers [72, 73] have to be mentioned, as related with the topic of this book. At the time when he started working, the field attracted little attention. Without any doubt his tenacity played a major role to popularize web geometry in France, and in other countries as well.

At the occasion of the visit of Nakai, some *Journées sur les tissus* were organized at Toulouse in 1996. These 'journées' gathered researchers from different backgrounds, as testified by the content of the book [66].

To finish this very quick overview of the works on webs during the 1980s and the 1990s, let's mention some papers by researchers working on foliations and dynamical systems such as [28, 56].

Recent Developments (Since 2000)

The activity on webs during the two decades before 2000 announced an even greater vitality in this area for the years to follow. In particular, major results concerning the problem of algebraizing webs of maximal rank have been obtained since 2001. We mention below those that we consider the most important.

In 2001, the second author [107, 109] and Robert [119] independently showed that the Spence-Kummer 9-web associated with the trilogarithm is an example of exceptional planar web. This was the second example of such a web, the other one having being discovered by Bol more than 60 years before.

A notable event for the field was the conference *Géométrie des tissus et équations différentielles* co-organized by Hénaut and Nakai. This conference, held at the CIRM in 2003, was attended by researchers from all over the world. For some mathematicians it was their first opportunity to meet web geometry. In the 10 years following this conference, a myriad of new examples of planar exceptional webs were brought to light [91, 106, 108, 113] showing that the classification problem for non-algebraic planar webs of maximal rank is more difficult than suggested by Chern and Griffiths, but also more interesting due to the variety and distinct natures of known webs of this kind.

In 2005, Trépreau provided a proof of the result which Chern and Griffiths aimed at in [34], i.e. the algebraization of one-codimensional k-webs on $(\mathbb{C}^n, 0)$ of maximal rank, when $k \geq 2n$ and $n > 2$. This result being one of the main subjects of this book, we will not comment it any longer here.

The results hitherto mentioned were the sign that the old problematic of the algebraization of webs, that goes back to the Hamburg school of web geometry, was undergoing a revival, as is testified by the Bourbaki seminar [105] devoted to this topic.

From 2005, Trépreau and the second author of this book started working on the algebraization of maximal rank webs of codimension strictly greater than 1. If Poincaré's approach generalizes and is still fruitful to study these webs, it leads to a geometrical problem that does not really hold for webs of codimension 1. Indeed, using constructions similar to the ones presented in this book, it is possible to reduce the question of the algebraization of maximal rank webs of codimension bigger than 1 to the classification of some extremal projective varieties carrying a 'big' family of rational normal curves.

This question has been completely solved in [114] in all but one particular case, for which some unexpected exceptions arise. Using the classifications obtained in [114], the authors were able to obtain a general algebraization result for maximal rank webs of codimension greater than 2 in [115]. The result is the expected one (namely '*maximal rank webs are algebraizable*') except for one unexpected case corresponding to the geometric exceptions mentioned just above.

Finally, this particular case has been investigated in [111] where the author constructed '*algebraic exceptional webs*', that is webs of maximal rank that are algebraic in a generalized sense, but not in the classical one.

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So, as for maximal rank webs of codimension 1, for which exceptional webs exist in dimension 2, there are algebraic exceptional webs of codimension bigger than 2. All these exceptional webs, planar or not, are far from being well understood. Their number and their still mysterious geometry make them exciting mathematical objects worth to be studied further. We hope that they will.

To conclude, it is worth mentioning that although major results about the algebraization of webs have been obtained in the last decade, the field must not be reduced to the study of exceptional webs. Many other natural questions remain widely open. Among them, let us mention the one about the algebraization of webs of codimension $c \geq 2$ with maximal q-rank when $q < c$ (see [76] for details). Another one is linked to the nice but seemingly forgotten result by Blaschke and Walberer [17, §37] about the algebraization of maximal rank webs whose codimension does not divide the dimension of the ambient space.

There is still a lot of space for future research in web geometry.

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Index

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