

# References

Numbers in angle brackets at the end of each listing show pages of these notes on which the work is referred to. “MR” refers to the review of the work in *Mathematical Reviews*, readable online at <http://www.ams.org/mathscinet/>.

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# List of Exercises

I hope the telegraphic descriptions given below will help you recall in most cases roughly what the exercises are. When they don't, you can always look back to the page in question (shown at the left). If you find any of the descriptions incorrect, or think of a more effective wording to briefly describe some exercise, let me know.

- 1 **Chapter 1: About the Course, and These Notes.**
- 11 **Chapter 2: Making Some Things Precise.**
  - " 2.1. Generalities.
  - " 2.2. What is a group?.
  - 12 2.2:1 precise def. of homomorphism
  - 13 2.2:2  $\delta(x, y) = xy^{-1}$
  - " 2.2:3  $\sigma(x, y) = xy^{-1}x$
  - 14 2.3. Indexed sets.
  - 15 2.4. Arity.
  - " 2.5. Group-theoretic terms.
  - 17 2.5:1  $T \leftrightarrow T'$  (terms)
  - " 2.5:2 (a), (b)  $\implies$  (c) in def. of  $T$ ?
  - 18 2.5:3 " $(s \cdot t)$ " etc. are OK
  - " 2.5:4 are " $\mu s t$ " etc. OK?
  - " 2.5:5 is  $\text{symb}(x) = x$  OK?
  - 19 2.6. Evaluation.
  - " 2.6:1 construct "eval":  $T \rightarrow |G|$
  - 20 2.7. Terms in other families of operations.
  - 21 2.7:1 derived op.s of majority  $M_3$
  - " 2.7:2  $\text{Aut}(\mathbb{C}, +, \cdot, \beta) = ?$
  - 22 2.7:3 testing whether  $\beta$  is derived op
  - " 2.7:4 Can  $s_1 \succ s_2 \succ \dots$ ?
- 25 **Chapter 3: Free Groups.**
  - " 3.1. Motivation.
  - 26 3.1:1 if  $a, b, c$  don't generate  $F \dots$
  - " 3.1:2 free  $\implies$  gen'd by  $\{a, b, c\}$
  - 28 3.1:3 no "free finite groups"
  - " 3.2. The logician's approach: construction from group-theoretic terms.
  - 31 3.2:1  $(xy)^{-1} \sim y^{-1}x^{-1}$
  - " 3.2:2 (3.2.1)–(3.2.3), (3.2.6)–(3.2.8)  $\implies$  above?
  - 32 3.2:3 (3.2.1)–(3.2.3), then (3.2.4)–(3.2.5), then (3.2.6)–(3.2.8)?
- " 3.3. Free groups as subgroups of big enough direct products.
- 35 3.3:1 universal prop. of  $J \subseteq (S_3)^{216}$ ?
- " 3.3:2 structure of  $J \subseteq (S_3)^{6^n}$ ?
- 36 3.3:3 no "free fields"
- " 3.3:4 Is  $\mathbb{Q}(X)$  free in some sense?
- " 3.3:5 "free skew fields"?
- 37 3.3:6 functorial operations on groups
- " 3.3:7 " " on finite groups
- " 3.4. The classical construction: free groups as groups of words.
- 42 3.4:1 some calculations in free group
- " 3.4:2 the Hall-Witt identity
- 44 3.4:3  $s_v \neq t_v$  in some finite  $G$ ?
- " 3.4:4 two variants of above
- " 3.4:5 a free group of  $2 \times 2$  matrices
- 45 **Chapter 4: A Cook's Tour of Other Universal Constructions.**
  - 46 4.1. The subgroup and normal subgroup of  $G$  generated by  $S \subseteq |G|$ .
  - 47 4.1:1  $\text{norm-sgp}(S) = \langle gsg^{-1} \rangle$
  - " 4.1:2  $\text{norm-sgp}(\{x^n, y\})$
  - " 4.2. Imposing relations on a group. Quotient groups.
  - 48 4.2:1 motivating normality
  - 49 4.2:2 universal prop. of  $(G/H, [e])$
  - " 4.3. Groups presented by generators and relations.
  - 52 4.3:1 one group, three descriptions
  - 53 4.3:2 endomorphisms of above group
  - " 4.3:3 norm. form in  $\langle a, b \mid ab = b^2a \rangle$
  - " 4.3:4 norm. form in  $\langle a, b \mid ab = b^2a^2 \rangle$
  - " 4.3:5  $\langle w, x, y \mid w = f(x, y), x = g(y) \rangle$
  - " 4.3:6 group of symmetries of  $\mathbb{Z} \times \mathbb{Z}$
  - " 4.3:7 # of gen.s and rel.s for fin. gps
  - 54 4.3:8 using finite subset of rel-set
  - " 4.3:9 fin. gen'd but not fin. pres'd

- " 4.3:10  $\langle x, y \mid \dots \rangle$  trivial?
- " 4.3:11 more of the same
- 55 4.3:12  $y^{-1}x^2y = x^{-2}$ ,  $x^{-1}y^2x = y^{-2}$
- 56 **4.4. Abelian groups, free abelian groups, and abelianizations.**
- 58 4.4:1  $\mathbb{Z}^X$  not free abelian
- 59 4.4:2 abelianize 4.3:1, ..., 4.3:11
- " 4.4:3  $f^{\text{ab}} : G^{\text{ab}} \rightarrow H^{\text{ab}}$
- " 4.4:4 in 4.3:2, is  $\text{Aut}G \rightarrow \text{Aut}(G^{\text{ab}})$  1-1?
- " 4.4:5 if  $H \subseteq G$ , what of  $( )^{\text{ab}}$ ?
- " 4.4:6  $\text{GL}(n, K)^{\text{ab}}$
- " 4.4:7  $\exists G^{\text{solv}}$ ? if fin.? Free solvable?
- " 4.4:8  $\mathbb{Z}^n$  needs  $n$  gens,  $n(n-1)/2$  rels
- 60 **4.5. The Burnside problem.**
- 61 4.5:1 implications re Burnside prob
- " 4.5:2 residual finiteness and " "
- " 4.5:3 " " " Ex. 3.4:3;  $\exists G^{\text{res.fin.}}$ ?
- 62 **4.6. Products and coproducts of groups.**
- 66 4.6:1  $\coprod_X Z$  is free on  $X$
- 67 4.6:2  $(F * G) * H \cong F * G * H$
- " 4.6:3  $G \times H \cong H \times G$ , ...
- " 4.6:4 el'ts of finite order in  $G * H$
- 68 4.6:5  $m : G * H \rightarrow G \times H$
- " 4.6:6  $G \rightarrow G \times G$  and dual
- 69 4.6:7  $\langle X \mid \text{some pairs commute} \rangle$
- " **4.7. Products and coproducts of abelian groups.**
- 71 4.7:1 abelianizing  $\prod, \coprod$
- " **4.8. Right and left universal properties.**
- 73 4.8:1 right and left universal  $G$ -sets
- 74 4.8:2  $G$ -set constr'ns using  $G \rightarrow \{e\}$
- " 4.8:3 no duals to "free", "ab"
- " 4.8:4 universal subset?
- " 4.8:5 universal  $X$  with  $A \times X \rightarrow B$ ?
- " 4.8:6 right and left univ.  $R$ -modules
- 75 **4.9. Tensor products.**
- " 4.9:1 (linear  $\wedge$  bilinear)  $\implies 0$
- 76 4.9:2  $F(X) \otimes F(Y) \cong F(X \times Y)$
- 77 4.9:3 univ.  $(A, ?) \rightarrow C$ ,  $(?, ?) \rightarrow C$ ?
- " 4.9:4 bilinearity and nonabelian gps
- " 4.9:5 range of a bilinear map
- 78 4.9:6  $\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$
- " 4.9:7 description of  $- \otimes \mathbb{Z}_n$
- " 4.9:8 when is  $A \otimes B = \{0\}$ ?
- " 4.9:9 identities of bilinear maps
- 79 **4.10. Monoids.**
- 81 4.10:1 constr. of cong., vs. §3.2
- 82 4.10:2 inf, sup, etc. of congruences
- " 4.10:3 largest congruence  $\subseteq X$ ?
- 83 4.10:4 describe  $\langle a, b \mid ab = e \rangle$
- " 4.10:5  $ab = ac = dc = e$
- " 4.10:6  $ab = ac, ba = bc, ca = cb$
- " 4.10:7  $ab = b^2a$
- " 4.10:8  $abba = baab; abbab = baabb$
- 85 4.10:9  $\text{Cong } f \rightrightarrows S$  univ? ... CoCong?
- " 4.10:10 which  $H \subseteq G$  are equalizers?
- " **4.11. Groups to monoids and back again.**
- 86 4.11:1 embeds in  $\text{gp} \iff S \rightarrow S^{\text{gp}}$  1-1
- " 4.11:2  $S^{\text{gp}}$  for Ex's 4.10:4-4.10:7
- " 4.11:3 if  $S \subseteq G$ , is  $S^{\text{gp}} \rightarrow G$  1-1?
- 87 4.11:4  $xyz = zyx = e; S, \dots, U(S^{\text{ab}})$
- " 4.11:5 semigroups  $\rightrightarrows$  monoids
- 88 **4.12. Associative and commutative rings.**
- 92 4.12:1 symmetric el'ts in  $\mathbb{Z}\langle x, y \rangle$
- 94 4.12:2 univ.  $R \rightarrow$  integral domain?
- " 4.12:3  $yx - xy = 1$  (Weyl algebra)
- 95 4.12:4 subrings of  $\mathbb{Z} \times \mathbb{Z} (\times \mathbb{Z})$
- " 4.12:5  $\langle x \mid x^2 = x \rangle \cong \mathbb{Z} \times \mathbb{Z}$
- " 4.12:6 gens and rels for  $R \times S$
- " 4.12:7  $2x = 1$  vs.  $(4x = 2, 2x^2 = x)$
- " 4.12:8 when must  $R$  be fin. pres'd?
- 96 4.12:9 free on  $\emptyset$  etc.;  $\mathbb{Z}_n$
- " 4.12:10 similarly, but without  $e$ , 1
- " 4.12:11 if 1 not preserved by homs
- " **4.13. Coproducts and tensor products of rings.**
- 97 4.13:1  $\mathbb{Z}_m \otimes \mathbb{Z}_n$ , two ways
- 98 4.13:2  $\mathbb{Z}[i] \otimes \mathbb{Z}_p$
- " 4.13:3  $\otimes$  and field composita
- " 4.13:4  $\mathbb{C} \otimes \mathbb{C}, \mathbb{Q}(2^{1/3}) \otimes \mathbb{Q}(2^{1/3})$
- 99 4.13:5 univ. prop. of  $\otimes$  of rings
- " 4.13:6  $\mathbb{Z}S \otimes \mathbb{Z}T$
- " 4.13:7 other ring structures on  $\otimes$ ?
- 100 4.13:8 v.d.W. for  $R * S$  (assoc rings)
- " 4.13:9 centers of coproduct rings
- " 4.13:10 coproducts involving  $\mathbb{Q}$
- " **4.14. Boolean algebras and Boolean rings.**
- 102 4.14:1 normal form in free Bool. ring
- " 4.14:2 ident's in Bool. ring vs.  $\mathbf{P}(S)$
- " 4.14:3 {idemp'ts} in a comm. ring
- " 4.14:4 univ. rep'n of a Boolean ring
- 103 4.14:5 finite Boolean ring is  $\cong 2^S$
- " 4.14:6 {finite & cofinite s'sets of  $\mathbb{Z}$ }
- " 4.14:7 coproducts of Boolean rings
- 104 **4.15. Sets.**
- " 4.15:1 laws of  $\prod$  and  $\coprod$  for sets
- 105 **4.16. Some algebraic structures we have not looked at.**
- " **4.17. The Stone-Ćech compactification of a topological space.**
- 107 4.17:1  $X$  is dense in  $S\text{-}\check{C}$  compactf'n
- 109 4.17:2 embeddability in compact

- " 4.17:3 leaving out "Hausdorff"
- " 4.17:4  $C - \mathbb{R}$  has two connected cpts
- " 4.17:5 curves of finite length
- 110 4.17:6 variants of above two
- 111 4.17:7  $c \in C$  gives ring hom.
- 112 4.17:8 maximal ideals of subrings
- " 4.17:9 when  $S-\tilde{C}$  just adds one pt
- " 4.17:10 group compactifications
- 113 **4.18. Universal covering spaces.**
- 119 **Chapter 5: Ordered Sets, Induction, and the Axiom of Choice.**
- " **5.1. Partially ordered sets.**
- 120 5.1:1 isotone bijection  $\neq$  isom'm
- 121 5.1:2 " $\leq$ " vs. " $<$ "
- " 5.1:3  $<$ -respecting maps
- 122 5.1:4  $\prod$ , presentations,  $\coprod$ , etc.
- 123 5.1:5 universal subset-repr'ns
- " 5.1:6 "least"  $\neq$  "unique minimal"
- 124 5.1:7 minimal presentations
- 125 5.1:8 {maximal el'ts} and cofinality
- " 5.1:9 chains and antichains
- 126 5.1:10 maximal partial orderings
- " 5.1:11 Fredman:  $1/3 \leq N_{x,y}/N \leq 2/3$ ?
- 127 5.1:12 metric on {orderings}
- 128 5.1:13 reconstruction problem
- " **5.2. Digression: preorders.**
- 129 5.2:1 preorder  $\longleftrightarrow$  ( $\approx$ ,  $\leq$ )
- " 5.2:2 preorder on  $\mathbb{N}^{\mathbb{N}}$
- 130 5.2:3 growth rates of monoids
- " 5.2:4 growth rates of free monoids
- 131 5.2:5 Gel'fand-Kirillov dimension
- 132 5.2:6  $GK(S_1 \times S_2)$
- " 5.2:7  $GK(S) \notin (1, 2)$
- " 5.2:8 grwth rates:  $k$ -algs vs monoids
- " 5.2:9 GK and transcendence degree
- 133 5.2:10 finite topologies as preorders
- " **5.3. Induction, recursion, and chain conditions.**
- 135 5.3:1 factoring into irreducibles
- 136 5.3:2 no recursion on  $[0, 1]$
- 137 5.3:3 stronger version of preceding
- 139 5.3:4 lexicographic order
- 140 5.3:5 symmetric polynomials
- " 5.3:6 recursively defined  $n_{i,j}$
- 141 5.3:7 in top.sp., (all cpct)  $\iff$  ACC
- " 5.3:8 factoriz'n into irred's  $\not\iff$  ACC
- " **5.4. The axioms of set theory.**
- 146 5.4:1 closing a set under " $\epsilon$ "
- " **5.5. Well-ordered sets and ordinals.**
- 147 5.5:1 examples of well-ordered sets
- 148 5.5:2 finish proof of Lemma 5.5.1
- 151 5.5:3  $\alpha$  embeds in  $\beta \iff \alpha \leq \beta$
- " 5.5:4 height fn on p.o.set with DCC
- " 5.5:5 indep'ce of cdns def'ing ordinal
- 152 5.5:6 lim. ords are l.u.b.s of members
- 153 5.5:7 characterizing op.s on ordinals
- 154 5.5:8  $1 + \omega \neq \omega + 1$ , etc.
- " 5.5:9 ordinal written as disj. union
- " 5.5:10 ordinal arith and lex ordering
- 155 5.5:11 {chains in  $P$ }  $\not\leftrightarrow$   $P$
- 157 5.5:12 inf ord (and card) products
- 159 5.5:13 cofinality and cofinal subsets
- " 5.5:14 regular cardinals
- " 5.5:15 regular ordinals
- 160 5.5:16 cofinal s'sets of  $\prod$ 's of chains
- " 5.5:17 natural topology on ordinals
- " **5.6. Zorn's Lemma.**
- 163 5.6:1 "weakening" Zorn's Lemma
- 164 5.6:2 set-rep. of Boolean rings
- " 5.6:3 minimal prime ideals
- 165 5.6:4 extending partial to total order
- " 5.6:5 well-ordered cofinal subsets
- " 5.6:6 disjoint cofinal subsets
- " 5.6:7 cofinal subsets and subchains
- " 5.6:8 extending p.o. w DCC to w.o.
- 166 5.6:9 chains in  $\mathbf{P}(S)$
- " 5.6:10 inf. p.o.s.  $\supseteq \omega$ ,  $\omega^{\text{op}}$  or antichn
- " 5.6:11 partial well-orderings
- 167 5.6:12 e.g. w card  $>$  height  $\cdot$  width
- " 5.6:13 countable chains embed in  $\mathbb{Q}$
- " 5.6:14 Tukey equivalence of p.o.sets
- 168 **5.7. Some thoughts on set theory.**
- 173 **Chapter 6: Lattices, Closure Operators, and Galois Connections.**
- " **6.1. Semilattices and lattices.**
- 174 6.1:1 axioms  $\implies \vee$  arises as l.u.b.
- 176 6.1:2 natural non-lattices
- " 6.1:3 almost-lattices
- 177 6.1:4 Compatibility  $\implies$  Id'potence
- " 6.1:5 in a lattice, maximal = greatest
- " 6.1:6 non-homs of lattices
- " 6.1:7 affine and proj. geometries
- 178 6.1:8 universal (semi)lattices
- " 6.1:9 free semilattices
- 179 6.1:10 size of free lattices
- " 6.1:11 univ. set-rep's of (semi)lats
- " 6.1:12 disjoint cofinal subsemilats
- 180 6.1:13 gen'n by 2 proper sublats, sgps
- 181 6.1:14 modularity: cond'ns & eg.s
- " 6.1:15 distributivity: cond'ns & eg.s
- " 6.1:16 distrib. lat.s & vector-sp. bases
- 182 6.1:17 infinite chains and " " "

- 183 6.1:18 nec. & suf. cdns re preceding?  
 " **6.2.** 0, 1, and completeness.
- 185 6.2:1 when is  $\{< \beta$  or  $\text{co-}< \gamma\}$  cplt?  
 186 6.2:2 complete lattice of open sets  
 " 6.2:3 which ordinals are cplt lats?  
 " 6.2:4 fixed point theorem  
 " 6.2:5 cp. lat. w/o chains  $\cong$   $\text{card} > \omega$
- 187 6.2:6 representations in  $\mathbf{P}(S)$   
 " 6.2:7 l.u.b.s of chains  $\cong$  cardinals  
 188 6.2:8 completeness and interpolation  
 " 6.2:9 nine kinds of interpolation  
 189 6.2:10 fin. interpol'n for polynomials  
 " 6.2:11 gen. numbers of a lattice  
 " 6.2:12 maps  $\omega^X \rightarrow$  semilattice
- 190 6.2:13 identities for complete lattices  
 191 6.2:14  $\wedge$  and  $\vee$  of compact elements  
 " 6.2:15  $\text{ACC} \iff$  all el'ts compact  
 192 6.2:16 representing a lattice in  $\mathbf{E}(X)$   
 193 6.2:17 getting path-sufficiency
- 194 **6.3. Closure operators.**  
 " 6.3:1 verify corresp.:  $\{\text{closed}\} \leftrightarrow \text{cl}$   
 195 6.3:2 every "cl" comes from a "G"  
 " 6.3:3  $G$  for generating equiv relns  
 " 6.3:4 building up  $\text{cl}(X)$
- 196 6.3:5 verify embedding lemma  
 197 6.3:6 alt proof of Lemma 6.3.5(ii)  
 " 6.3:7 univ. prop.s of above constrns?  
 " 6.3:8 prin. ideals and completeness  
 " 6.3:9 infinite 3-generator lattice
- 198 6.3:10 " " " with no infinite chain?  
 " 6.3:11 inf chain in f.g. lat of ab gps?  
 " 6.3:12 proof of Lemma 6.3.6  
 " 6.3:13 finitary cl ops &  $\alpha$ -gen'd ssets  
 " 6.3:14 finitary cl ops & compact elts
- 199 6.3:15 compact vs. finitely generated  
 " 6.3:16 find 8 closure operators  
 " 6.3:17 which cl ops are topologies?  
 " 6.3:18 cardinality of  $\text{cl}(X)$   
 " 6.3:19  $\text{cl}_{\text{sys}}^S$  determines closure ops
- 200 6.3:20 Clofam, Clop, and Closys  
 " 6.3:21  $\{\text{finitary}\}$  within Clop  
 " 6.3:22 Péter Frankl's question
- 201 **6.4. Digression: a pattern of threes.**
- 205 **6.5. Galois connections.**  
 206 6.5:1 all Gal cncnts arise from  $R$ 's  
 " 6.5:2 simpler def of Gal connection  
 207 6.5:3 all cl ops arise from Gal cncnts  
 209 6.5:4 Gal cncntn from " $\leq$ " on  $\mathbb{Q} \times \mathbb{Q}$   
 " 6.5:5 Gal cncntn from "nonempty  $\cap$ "  
 " 6.5:6 topology on dual vector space  
 " 6.5:7 commutants, bicommutants
- 210 6.5:8 centralizer subgroups
- 213 **Chapter 7: Categories and Functors.**
- " **7.1. What is a category?**  
 220 7.1:1 Cayley's Thm for categories  
 " **7.2. Examples for categories.**  
 223 7.2:1 which categories are p.o. sets?  
 226 7.2:2 groups up to finite index  
 227 7.2:3 bimodules as morphisms  
 " 7.2:4 sets modulo finite sets  
 " 7.2:5 categories from  $S$ -sets
- 228 **7.3. Other notations and viewpoints.**  
 230 7.3:1 categories without objects  
 232 **7.4. Universes.**  
 235 7.4:1 which maps, gps, cats are small?  
 " 7.4:2  $\{\text{universes}\}$  is a proper class  
 " 7.4:3 rank, her.card, and universes  
 236 7.4:4  $\{\text{universes}\}$  is well-ordered  
 " 7.4:5 Ax. of Univ's w/o Ax. of  $\infty$   
 " 7.4:6  $\text{card}(\mathbb{U})$  is regular limit card
- 238 **7.5. Functors.**  
 242 7.5:1  $G \mapsto \{\text{order } 2\}$  is not a functor  
 " 7.5:2  $G \mapsto$  center is not a functor  
 " 7.5:3  $G \mapsto \text{Aut}(G)$  is not a functor  
 " 7.5:4 algebraic closure not a functor  
 " 7.5:5 functors  $\mathbf{FSet} \rightarrow \mathbf{FSet}$
- 244 7.5:6  $G \mapsto G^{\text{ab}}$  is not full or faithful  
 " 7.5:7 monoid  $\mapsto \{\text{invertible}\}$  as funct  
 " 7.5:8  $P \mapsto P_{\text{cat}}$  as functor  
 " 7.5:9 full vs "onto on morphisms"
- 245 7.5:10 proof of Cayley for categories  
 " 7.5:11 concretizability questions
- 247 **7.6. Contravariant functors, and functors of several variables.**  
 249 7.6:1  ${}^{\text{op}}$  as a functor  
 " 7.6:2 co- and contravariant  $\mathbf{P}(-)$   
 " 7.6:3  $X \mapsto \{\text{equivalence relations}\}$   
 250 7.6:4 subgroups; normal subgroups  
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# Symbol Index

Boldface numbers indicate pages where definitions are given. If a symbol is defined in one place and used again without explanation more than a page or so away, I show the page(s) where it is defined, and often some of the pages where it is used or where the entity it symbolizes is discussed; but I do not attempt to show all significant occurrences of each subject. For that, the Word Index, with its headings and subheadings, is more useful.

Symbols are ordered alphabetically. For Greek letters, I use their Latin spelling, e.g., “Omega” for  $\Omega$ . Other symbols are alphabetized in various ways; e.g.,  $\vee$  and  $\wedge$  are alphabetized as “vee” and “wedge” based on their L<sup>A</sup>T<sub>E</sub>X names; the symbol  $=$ , and related symbols such as  $\cong$ , are alphabetized, in an arbitrary order, under “equals”; and  $\mapsto$  and  $\downarrow$  are similarly alphabetized under “arrow”. Fortunately, you do not have to know all the details; some symbols will require more search than others, but this index is only a few pages long.

Font-differences, capitalization, and “punctuation” such as brackets, do not affect ordering unless everything else is equal. Operator-symbols are often shown in combination with letters with which they are commonly used, e.g.,  $\langle X | Y \rangle$  is alphabetized under  $XY$ .

Category theory, introduced in Chapter 6, brings with it a proliferation of symbols for categories. I do not record below cases where the meaning is obvious, like **Group**, nor cases discussed only briefly, like **GermAnal** (germs of analytic functions, which can be found in the Word Index), but only category names used in more than one place, for which some aspect of the definition (e.g., the associativity assumption in **Ring**<sup>1</sup>) or the abbreviation (as with **Ab**) is not obvious.

$ A $	underlying set (etc.) of object $A$ , <b>12, 380, 462</b> .
$A^*$	set determined by $A$ under a Galois connection, <b>205–211</b> .
$A/\sim$ ,	
$A/(s_i = t_i)_{i \in I}$	quotient-set or factor-algebra of $A$ , <b>30, 48, 82</b> .
<b>Ab</b>	category of abelian groups; prefix for “abelian” in categories such as <b>AbMonoid</b> , <b>221, 525</b> .
$\aleph_0, \aleph_\alpha$	least (respectively, $\alpha$ -th) infinite cardinal, <b>35, 158, 159</b> .
$\forall$	“for all”, <b>11, 211</b> .

- $\text{Ar}(\mathbf{C})$  the class of morphisms ('arrows') of the category  $\mathbf{C}$ , **217**.
- $\text{ari}_\Omega$ ,  $\text{ari}$  arity function (cf.  $\Omega$  below), 20, **379**.  
 $\downarrow$  see " $(S \downarrow T)$ ,  $(S \downarrow \mathbf{C})$ ,  $(\mathbf{C} \downarrow T)$ " below.  
 $\mapsto$  action of a function on elements, **11**.
- $\text{Aut}(X)$  automorphism group of  $X$ , **59**, 213, 241–242.
- $\beta: (A, B) \rightarrow C$  bilinear map (temporary notation), **75–79**.
- Binar**, **Binar**<sup>e</sup> variety of sets with a binary operation, resp., a binary operation and neutral element, **505**, **514**, 525.
- $\mathbb{C}$  the complex numbers.
- $\mathbf{C}^{\mathbf{D}}$  category of all functors  $\mathbf{D} \rightarrow \mathbf{C}$ , **278**, 287–290.
- $\mathbf{C}^{\text{pt}}$  category of pointed objects of  $\mathbf{C}$ , **264**, 275, **506**, 524.
- $\mathbf{C}(X, Y)$  set (or in Chapter 10, algebra) of morphisms  $X \rightarrow Y$  in the category  $\mathbf{C}$ , **218**, **462**, **466**.
- $\text{card}(X)$  cardinality of the set  $X$ , **156**.
- Cat**, **Cat** <sub>$\mathbb{U}$</sub>  category of all  $\mathbb{U}$ -small categories, **243**.
- $\text{cl}$  general symbol for a closure operator, **194**.
- Cl**( $\mathbf{V}$ ),  $\text{Cl}_n(\mathbf{V})$  clonal theory of variety  $\mathbf{V}$ , and its  $n$ th object, **441**.
- Clone**,  
**Clone**<sup>( $\gamma$ )</sup> category of all covariant ( $\leq \gamma$ -)clonal categories, **440–441–453**, 528.
- CommRing**<sup>1</sup> variety of all commutative associative rings with unity, **221**, 314, 457–461, 468, 472.
- $\text{deg}(x)$  in §10.6, degree of element  $x$  in a comonoid object of **Monoid**, **481–485**.
- $\Delta$  diagonal functor  $\mathbf{C} \rightarrow \mathbf{C}^{\mathbf{D}}$ , **313**, **337**, 340, 346, 349.
- $\mathbf{E}(X)$  lattice of equivalence relations on  $X$ , **191–193**, 299.
- $\text{End}(X)$  monoid of endomorphisms of  $X$ , **59**, 214.
- $\approx$  equivalence of categories, **285**.
- $\cong$  isomorphism (of algebras, categories, etc.).
- $\eta$ ,  $\varepsilon$  unit and counit of an adjunction, **309**.
- $\exists$  "there exists", **11**, 211.
- $\exists!$  "there exists a unique", **47**.
- $F_\Omega$ ,  $F_{\mathbf{V}}$  free-algebra functors (see mainly "free" in Word Index), **393**, **408**, 416.
- $f|X$  restriction of the function  $f$  to the set  $X$ , 40, **135**.

$G_{\text{cat}}$	see “ $S_{\text{cat}}, G_{\text{cat}}, P_{\text{cat}}$ ” below.
$G^{\text{ab}}$	see “group: abelianization of” in Word Index.
$\gamma_0, \gamma_1, \gamma$	in Chapters 9 (for $\gamma_0, \gamma_1$ ) and 10 (for $\gamma$ ): infinite cardinals bounding arities in a type, <b>401, 461</b> .
$\text{GL}(n, K)$	general linear group, 59, 60, 373, 477, 510.
$G_{\text{md}}$	“underlying” monoid of the group $G$ , <b>85–87, 222, 240</b> .
$\mathbf{H}(C)$	in § 9.6, set of all homomorphic images of algebras in the set $C$ , <b>416, 514</b> .
$h_Y, h^Y$	covariant and contravariant hom functors, $\mathbf{C}(Y, -)$ and $\mathbf{C}(-, Y)$ , <b>246, 250, 253</b> .
$\text{Hom}(X, Y)$	set of homomorphisms from $X$ to $Y$ ; mostly superseded by $\mathbf{C}(X, Y)$ after §7.1.
<b>HtpTop</b>	category of topological spaces, with homotopy classes of maps for morphisms, <b>225, 241, 244, 251</b> .
<b>HtpTop</b> <sup>(pt)</sup>	as above, but for pointed topological spaces, <b>469, 527</b> .
$\text{Id}_{\mathbf{C}}$	identity functor of the category $\mathbf{C}$ , <b>240, 281, 285, 290</b> .
$\text{id}_X$	identity morphism of the object $X$ , <b>218</b> .
$K_f$	congruence determined by the map $f$ , <b>81, 84</b> .
$K[t]$ ,	
$\mathbb{Z}[x_1, \dots, x_n]$	polynomial algebra or ring, 36, 89.
$K\langle t \rangle$ ,	
$\mathbb{Z}\langle x_1, \dots, x_n \rangle$	free $K$ -algebra or ring (“noncommutative polynomial ring”), 89–91.
$KS, \mathbb{Z}S$	monoid algebra or group algebra, <b>93</b> .
$\lambda, \mu, \rho$	indices in coproducts of copies of $R$ (e.g., $R^\lambda \amalg R^\mu \amalg R^\rho$ ) and their elements (e.g., $x^\lambda$ ), <b>481–488, 495, 507</b> .
$\leq, <, \geq$ ,	
$\preceq, \prec$ , etc.	symbols for partial orderings and preorderings, <b>119–141</b> .
$\varprojlim, \varinjlim$	limits and colimits (including inverse and direct limits); see mainly Word Index, <b>325–335–368</b> .
$\limsup$	limit superior, <b>22, 131</b> .
$M_3$	ternary majority vote operation (but see also next), <b>21, 519</b> .

- $M_3, N_5$  5-element “forbidden sublattices” in characterizations of distributivity and modularity, **180**, 187.
- $\mu, \mu_G$ , etc. explicit notation for composition operation of a group, monoid, or category, **11**, **218**.
- N** natural numbers as subcategory of **Set**, **439**.
- $\mathbb{N}$  natural numbers as monoid, ordered set, etc., **439**.
- $\text{Ob}(\mathbf{C})$  object-set of the category **C**, **217**.
- $\Omega$  algebra type (family of operation-symbols with specified arities), 20, 22, **379**–414.
- $\Omega\text{-Alg}$  variety of all algebras of type  $\Omega$ , **381**–387, 393–397, 406.
- $\omega, \omega_\alpha$  least infinite ordinal, respectively least ordinal of cardinality  $\aleph_\alpha$ , **148**–160, 166–167.
- 1,  $1_L$  multiplicative neutral element of ring; greatest element in a lattice, 88, **183**, 506.
- $( )^{\text{op}}$  opposite (of a partially ordered set, semigroup, monoid, ring, or category), **121**, **240**, **248**–249, 255.
- $\mathbf{P}(C)$  in §9.6, set of all products of algebras in the set  $C$ ; otherwise, see  $\mathbf{P}(S)$  below, **416**, 514.
- $P_{\text{cat}}$  see “ $S_{\text{cat}}, G_{\text{cat}}, P_{\text{cat}}$ ” below.
- $\mathbf{P}(S)$  power set of  $S$ , **100**–103, 120, 123, 144, 166, 175, 299.
- $\pi_1(X, x_0)$  fundamental group of the pointed topological space  $(X, x_0)$ , 213, 216, 225, 246, 287, 470, 527.
- POSet**,
- POSet** $_<$  categories of partially ordered sets with isotone, respectively, strict isotone, maps, **221**.
- $\prod_{i \in I} X_i$ ,
- $\coprod_{i \in I} X_i$  product, coproduct of a family of objects, **266**–268.
- ${}_R B_S$  notation indicating that  $B$  is an  $(R, S)$ -bimodule, **498**.
- $R^\lambda, x^\rho$ , etc. see “ $\lambda, \mu, \rho$ ”.
- R-Mod-S** variety of all  $(R, S)$ -bimodules, **498**, 526.
- R-Mod**,
- Mod-R** categories of left, respectively right  $R$ -modules, **221**, 496.

- $|R|^\sigma$  in § 10.6, set of elements in coproduct monoid  $R^\lambda \amalg R^\mu \amalg R^\rho$  associated with index-string  $\sigma$ , **481–482**.
- RelSet** category with sets for objects, relations for morphisms, **225**, 238, 242, 251, 259.
- Rep(C, V)** category of all representable functors from **C** to **V**, **469–529**.
- Ring**<sup>1</sup> variety of associative rings with unity, **221**, 496.
- $(S \downarrow T)$ ,  
 $(S \downarrow \mathbf{C}), (\mathbf{C} \downarrow T)$  comma categories, **275**.
- $\mathbf{S}(C)$  in § 9.6, set of all subalgebras of algebras in the set  $C$ , **416**, 514.
- $S_{\text{cat}}$ ,  
 $G_{\text{cat}}, P_{\text{cat}}$  category constructed from the monoid, group, or partially ordered set  $S, G, P$ , **222, 223, 224**, 227, 244, 249.
- $\mathbf{s}, \mathbf{s}_R, \mathbf{s}_d$  co-operations in coalgebras, **466, 496**.
- $\mathbf{Set}_{(\mathbb{U})}$ ,  
**Group**<sub>( $\mathbb{U}$ )</sub>, etc. explicit notation for categories of  $\mathbb{U}$ -small objects, **234**.
- $s_f, s_A$  symbol for the value of a term  $s$  at the tuple  $f$ , respectively for its action on the whole algebra  $A$ , 19–**20–32**, 39–41, 57, **411**.
- $S^{\text{gp}}$  universal enveloping group of the monoid  $S$  (obtained by adjoining inverses to all elements), **85–87**.
- $\text{SL}(n, K)$  special linear group, 44, 457–461, 472, 475, 510.
- $\subseteq, \subset, \supseteq, \supset$  inclusion relations, **122**.
- $\text{symb}_T$  map  $X \rightarrow T$  taking each element of  $X$  to the symbol representing it, 17–19, 30, 39.
- $T$ ,  
 $T_X, \cdot, \cdot^{-1}, e, T_X, \Omega$  in Chapters 2–4, the set of all terms in a set  $X$ , and given operation-symbols, 16–**17–32**, 39–43, 49, 138.
- $\otimes$  tensor product, **75–79**, 214, **500, 502**.
- $T_{\text{red}}$  in § 3.4, set of reduced group-theoretic terms, **39**.
- 2** diagram category with picture  $\cdot \rightarrow \cdot$ , **224**, 293, 313, 403.

- $\mathbb{U}$  a set-theoretic universe; see “universe” and “Axiom of Universes” in Word Index, **233**.
- $U_\Omega, U_{\mathbf{V}}$  underlying-set functors on categories of algebras (see mainly “functors: forgetful” in Word Index), **393, 408**.
- $\sqcup$  disjoint union of sets, **63, 104**.
- $\mathbf{V}(J)$  variety of  $\Omega$ -algebras defined by a set  $J$  of identities, **402**.
- $\mathbf{V} \circ \mathbf{W}$  variety equivalent to category of  $\mathbf{W}$ -algebra objects of  $\mathbf{V}$ , and to category of  $\mathbf{V}$ -algebra objects of  $\mathbf{W}$ , **523–529**.
- $\mathbf{Var}(C)$  variety generated by a set  $C$  of algebras, **402, 416–418, 420–421**.
- $\vee$  “join” (in a (semi)lattice); “or” (disjunction of propositions), **11, 173, 174, 211**.
- $\wedge$  “meet” (in a (semi)lattice); “and” (conjunction of propositions), **11, 29, 211**.
- $[x]$  frequently: equivalence class of  $x$  under some equivalence relation, **30, 129**.
- $X \cap Y, \bigcap X_i$  intersection (of  $X$  and  $Y$ ; of the sets  $X_i$ ), **22, 100**.
- $X \cup Y, \bigcup X_i$  union (of  $X$  and  $Y$ ; of the sets  $X_i$ ), **22, 100**.
- $X^I$  the set of all maps  $I \rightarrow X$ , **14**.
- $\langle X \mid Y \rangle,$   
 $\langle X \mid Y \rangle_{\mathbf{V}}$  object (of  $\mathbf{V}$ ) presented by generators  $X$ , relations  $Y$ , **51, 396, 408**.
- $[x, y]$  variously: commutator in a group or ring; Lie bracket; interval in a partially ordered set, **42, 58, 124, 312**.
- $x^y$  conjugate of  $x$  by  $y$ , i.e.,  $y^{-1}xy$ , **42**.
- $X_0 \amalg \dots \amalg X_{n-1}$  coproduct of objects in a category, **266**.
- $\mathbb{Z}$  the integers, **14**.
- $\widehat{\mathbb{Z}}_{(p)}$  the  $p$ -adic integers, **319–322**.
- $\mathbb{Z}\langle x_1, \dots, x_n \rangle, \mathbb{Z}[t], \mathbb{Z}S$  see  $K\langle t \rangle, K[t], KS$ , respectively.
- $\mathbb{Z}, \mathbb{Z}_n$  the cyclic groups of infinite order; order  $n$ , **54, 78, 322–323**.
- $0, 0_L$  additive neutral element; least element in a lattice, **88, 183, 506**.

# Word and Phrase Index

I have tried to include in this index not only the locations where terms are defined, but also all significant occurrences of the concepts in question; but it has not been easy to decide which occurrences are significant. I would welcome readers' observations on the types of cases they would find it useful to have in the index, and on entries that are erroneous, unnecessary, or missing.

Pages where terms are defined or where conventions are made relating to them are shown with boldface page numbers. (Sometimes a formal definition occurs after the first page of discussion of a topic, and sometimes more than one version of a concept is defined, leading to occasional entries such as 172-**173-187**-234.)

Terms used by other authors for which different words are used here are, if referenced, put in single quotes; e.g., 'free product', for what we call a coproduct.

In cross-references, I often truncate multiword entries; e.g., I may say "see rings" though the actual heading is "rings and  $k$ -algebras". But where this would be confusing, I add "...". E.g., under "associative algebra" I write "see rings ...".

In referring to secondary headings, I use a colon, writing "see main-heading: subheading". Within subheadings, the main current heading is abbreviated "-". In particular, "see -: subheading" points to another subheading under the current main heading.

In the few cases where an entry ends with a numeral (e.g., "matrices: with determinant 1"), I have put the numeral in double quotes, so that it cannot be mistaken for a page number.

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