# Appendix A
# Proof of Lemma 3.22

Keeping to the semisimple notation in use in Sect. 3.6, we recall that the **radial component** of $g \in G$ is defined as the unique $R(g) \in \overline{\mathfrak{a}^+}$ such that

$$g = ke^{R(g)}k'$$

for some $k, k' \in K$ (Cartan decomposition in $G$, see [27, p. 402]). Let $|X| = \sqrt{\langle X, X\rangle}$ denote the norm on $\mathfrak{g}$, $o$ the origin of $G/K$ and $d(.,.)$ its distance function.

**Lemma A.1.** *For all* $g, g' \in G$

$$\left|R(g) - R(g')\right| \leq d(g \cdot o, g' \cdot o).$$

*Remark.* This inequality is equivalent to $d(a \cdot o, a' \cdot o) \leq d(ka \cdot o; a' \cdot o)$ for all $a, a' \in \overline{\exp \mathfrak{a}^+}$ and $k \in K$.

*Proof.* [1]The classical decomposition $G = (\exp \mathfrak{p}) K$ reduces the problem to the case $g = e^X$, $g' = e^{X'}$ with $X, X' \in \mathfrak{p}$. We split the proof into two steps.

*(i)* $\left|R\left(e^X\right) - R(e^{X'})\right| \leq |X - X'|$.

Indeed, by Cartan decomposition in $\mathfrak{p}$ we may write $X = k \cdot H$, $X' = k' \cdot H'$ for some $k, k' \in K$, $H, H' \in \overline{\mathfrak{a}^+}$ and Proposition 5.18 in [28, p. 196] implies

$$\left|R\left(e^X\right) - R(e^{X'})\right| = \left|H - H'\right| \leq \left|k \cdot H - k' \cdot H'\right| = |X - X'|.$$

*(ii)* $|X - X'| \leq d(\operatorname{Exp} X, \operatorname{Exp} X') = d(e^X \cdot o, e^{X'} \cdot o)$.

Indeed $d(\operatorname{Exp} X, \operatorname{Exp} X')$ is the length of the geodesic segment $\Gamma$ joining both points. Let $\gamma$ be the corresponding curve segment in the chart Exp, joining

[1] I am indebted to Sigurður Helgason for shortening my original proof of the lemma.



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$X$ to $X'$ in $\mathfrak{p}$. Then ($\ell$ denoting length) $|X - X'| \le \ell(\gamma)$ in the Euclidean space $\mathfrak{p}$. Besides $\ell(\gamma) \le \ell(\Gamma)$ by a general property, due to Élie Cartan, of the exponential chart in a Riemannian manifold of negative curvature [27, p. 73]. Thus $|X - X'| \le \ell(\Gamma)$ and the proof is complete.

■

*Remark.* For $G/K$ Cartan's theorem is easily proved as follows. The point is to show that $|(D_X \operatorname{Exp}) V| \ge |V|$ for $X, V \in \mathfrak{p}$, with $D_X \operatorname{Exp} = D_o\tau\left(e^X\right) \circ (\operatorname{sh}\operatorname{ad} X/\operatorname{ad} X)$. By Cartan decomposition of $X$ it suffices to consider $X \in \mathfrak{a}$. In the orthogonal decomposition $\mathfrak{p} = \mathfrak{a} \oplus (\oplus_{\alpha>0}\mathfrak{p}_\alpha)$ the endomorphism $(\operatorname{ad} X)^2$ is then diagonal with respective eigenvalues $0, \alpha(X)^2$, therefore $\operatorname{sh}\operatorname{ad} X/\operatorname{ad} X = 1 + \sum_1^\infty (\operatorname{ad} X)^{2n} / (2n+1)!$ is diagonal with eigenvalues $\ge 1$ and the result follows.

*Proof of Lemma 3.22.* We now explain how Lemma A.1 implies the estimate

$$R\left(e^X e^{tH}\right) = tH + A(X) + O\left(e^{-t\mu(H)}\right)$$

as $t \to +\infty$ for $X \in \mathfrak{p}$, $H \in \mathfrak{a}^+$, with $\mu(H) := \inf_{\alpha>0} \underline{\alpha(H)}$.

First, replacing $g$ by $e^{H'}g$ and $g'$ by $e^{H'}$ with $H' \in \overline{\mathfrak{a}^+}$, the lemma gives

$$\left|R(e^{H'}g) - H'\right| \le d(e^{H'}g \cdot o, e^{H'} \cdot o) = d(o, g \cdot o). \tag{A.1}$$

Then let $e^X = k(X)e^{A(X)}e^{\Sigma X_\alpha}$ be the Iwasawa decomposition for $X \in \mathfrak{p}$, with $k(X) \in K$, $A(X) \in \mathfrak{a}$, $X_\alpha \in \mathfrak{g}_\alpha$, the sum $\Sigma$ running over all positive roots $\alpha$. Fixing $H \in \mathfrak{a}^+$ and remembering $[H, X_\alpha] = \alpha(H)X_\alpha$ we obtain

$$e^X e^{tH} = k(X)e^{tH+A(X)}e^{V(t)} \text{ with } V(t) := \sum_{\alpha>0} e^{-t\alpha(H)}X_\alpha \in \mathfrak{n}.$$

Since $\alpha(H) > 0$ for all positive roots $\alpha$, we have $tH + A(X) \in \mathfrak{a}^+$ for $t$ large enough (uniformly for $X$ in a compact subset of $\mathfrak{p}$) and (A.1) applies to $R\left(e^{tH+A(X)}e^{V(t)}\right) = R\left(e^X e^{tH}\right)$:

$$\left|R\left(e^X e^{tH}\right) - tH - A(X)\right| \le d(o, e^{V(t)} \cdot o).$$

The latter distance can be evaluated by means of the decomposition $G = (\exp\mathfrak{p})K$. Forgetting $t$ for the moment let us write $e^{sV} = e^{W(s)}k(s)$ with $W(s) \in \mathfrak{p}$ and $k(s) \in K$, smooth functions of $s \in \mathbb{R}$, hence $e^{2W(s)} = e^{sV}e^{-s\theta V}$ and $d(o, e^V \cdot o) = |W(1)|$. Disregarding a trivial case we assume $V \ne 0$. For the $s$-derivative $W'$ we obtain, with $w = \operatorname{ad} W(s)$,

$$\frac{1 - e^{-2w}}{2w} 2W' = \left(e^{-2w} - 1\right) V + (V - \theta V).$$

Since $\left(1 - e^{-2w}\right)/2w = e^{-w}(\operatorname{sh} w)/w$ is, for $W \in \mathfrak{p}$, an invertible endomorphism of $\mathfrak{p}$ it follows that $2W' = V - \theta V + [W, \cdots]$ and, by scalar product with $W$,

$$2\left\langle W(s), W'(s)\right\rangle = \langle W(s), V - \theta V\rangle\,,$$

hence $f'(s) \leq |V - \theta V|\sqrt{f(s)}$ with $f(s) = |W(s)|^2$. To integrate this, observe that $f(0) = 0$ and $f(s) > 0$ for $s \neq 0$ because $W(s) = 0$ implies $e^{sV} = e^{s\theta V} \in N \cap \theta N = \{e\}$ hence $s = 0$. Thus $\sqrt{f(s)} \leq \frac{s}{2}|V - \theta V|$ and

$$d(o, e^V \cdot o) = \sqrt{f(1)} \leq \frac{1}{2}|V - \theta V|\,.$$

For $V = V(t)$ defined above we have $|V - \theta V| \leq Ce^{-t\mu(H)}$ with $\mu(H) = \inf_{\alpha>0}\alpha(H) > 0$ and a constant $C$ uniform for all $X$ in a compact subset of $\mathfrak{p}$. This completes the proof of Lemma 3.22. ■

# Appendix B
# Proof of Theorem 3.23

Here $G/K$ is a rank one Riemannian symmetric space of the noncompact type and we use the notation of Sect. 3.7. As explained in the outline (Sect. 3.7.1) our main task is to make explicit the kernels $a, b$ in the integral formulas (Lemmas B.2 and B.5 below)

$$\int_K \varphi\left(\|xH + k\cdot yH\|\right) dk = \int_{|x-y|}^{x+y} \varphi(z)a(x,y,z)\,dz \tag{B.1}$$

$$\int_K \varphi\left(\|Z(xH, k\cdot yH)\|\right) dk = \int_{|x-y|}^{x+y} \varphi(z)b(x,y,z)\,dz, \tag{B.2}$$

where $\varphi$ is a continuous function on $[0,\infty[$, $H \in \mathfrak{a}$ with $\alpha(H) = 1$, $\|H\| = 1$ and $x, y > 0$. As usual the Haar measure $dk$ over $K$ is normalized by $\int_K dk = 1$. After identification of $\mathfrak{a} = \mathbb{R}H$ with $\mathbb{R}$ the orthogonal projection $\pi : \mathfrak{p} \to \mathfrak{a}$ is $\pi(X) = X \cdot H$ (the dot denotes here the scalar product on $\mathfrak{p}$ corresponding to the norm $\|.\|$ in (3.44)).

**Lemma B.1.** *Let $f$ be a continuous function on* $[-1,1]$. *Then*

$$\int_K f(\pi(k\cdot H))\,dk = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} f(t)\left(1-t^2\right)^{(n-3)/2} dt.$$

*Proof.* The map $k \mapsto k\cdot H = \mathrm{Ad}(k)H$ induces a diffeomorphism of $K/M$ onto the unit sphere $\Sigma$ of $\mathfrak{p}$. The classical spherical measure $d\sigma$ on $\Sigma$ is invariant under all isometries preserving the origin of $\mathfrak{p}$, therefore under the adjoint action of $K$, and $d\sigma$ corresponds to a $K$-invariant measure on $K/M$: there exists a positive constant $C$ such that

$$\int_K f(\pi(k\cdot H))\,dk = \int_{K/M} f(\pi(k\cdot H))d(kM) = C\int_\Sigma f(x_1)d\sigma(X).$$



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Here $X = (x_1, \ldots, x_n)$ are coordinates with respect to an orthonormal basis of $\mathfrak{p}$, with $H$ as the first basis vector. Given $t \in [-1, 1]$ the intersection of $\Sigma$ with the hyperplane $x_1 = t$ is (for $n \geq 3$) a $(n-2)$-dimensional sphere with radius $\sqrt{1-t^2}$ and it follows that (with another constant $C'$)

$$\int_K f(\pi(k \cdot H))\, dk = C' \int_{-1}^{1} f(t) \left(1 - t^2\right)^{(n-3)/2} dt,$$

which remains valid for $n = 2$ too. Taking $f = 1$ we obtain the value of $C'$ and the lemma. ∎

**Lemma B.2.** *Given $x, y > 0$ let $\varphi$ be continuous on $[|x-y|, x+y]$. Then, for all rank one spaces,*

$$\int_K \varphi\left(\|xH + k \cdot yH\|\right) dk = \int_{|x-y|}^{x+y} \varphi(z) a(x, y, z)\, dz$$

*with*

$$a(x, y, z) = \frac{2^{3-n}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \times z \frac{((x+y+z)(x+y-z)(x-y+z)(-x+y+z))^{(n-3)/2}}{(xy)^{n-2}}$$

*One has $a(x, y, z) > 0$ for $x, y > 0$ and $|x-y| < z < x+y$.*

*Proof.* Since $\|xH + k \cdot yH\|^2 = x^2 + y^2 + 2xy\pi(k \cdot H)$, this follows from the previous lemma with $f(t) = \varphi\left(\sqrt{x^2 + y^2 + 2xyt}\right)$ and the change of variable $t \mapsto z = \sqrt{x^2 + y^2 + 2xyt}$. ∎

*Remark.* A similar proof would give John's formula for the iterated spherical means (see [29, p. 356]), where the same factor $a(x, y, z)$ appears.

Lemma B.1 turns out to imply (B.2) too in the simple case of real hyperbolic spaces, as follows.

**Lemma B.3.** *Given $x, y > 0$ let $\varphi$ be continuous on $[|x-y|, x+y]$. Then, for $H^n(\mathbb{R})$,*

$$\int_K \varphi\left(\|Z(xH, k \cdot yH)\|\right) dk = \int_{|x-y|}^{x+y} \varphi(z) b(x, y, z)\, dz$$

*with*

$$b(x, y, z) = \frac{2^{n-3}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{(\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z)^{(n-3)/2}}{(\operatorname{sh} x \operatorname{sh} y)^{n-2}} \operatorname{sh} z\, B^{(n-3)/2}$$

*and*

$$B = \frac{1}{\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z} \times \operatorname{sh}\left(\frac{x+y+z}{2}\right) \operatorname{sh}\left(\frac{x+y-z}{2}\right) \operatorname{sh}\left(\frac{x-y+z}{2}\right) \operatorname{sh}\left(\frac{-x+y+z}{2}\right)$$

*Proof.* Here $H^n(\mathbb{R}) = G/K$ with $G = SO_0(n,1)$, $K = SO(n) \times \{1\}$, and $\mathfrak{p}$ is the set of matrices

$$Y = \begin{pmatrix} 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & y_n \\ y_1 & \cdots & y_n & 0 \end{pmatrix},$$

identified with $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ for short. We take the unit vector $H = (1, 0, \ldots, 0)$ as a basis of $\mathfrak{a}$. The adjoint action of $K$ on $\mathfrak{p}$ is the natural action of $SO(n)$ on $\mathbb{R}^n$. Since

$$e^Y = I + \frac{\operatorname{sh} y}{y} Y + \frac{\operatorname{ch} y - 1}{y^2} Y^2$$

with $y = \|Y\| = \left(\sum_1^n y_i^2\right)^{1/2}$, the equality $e^Z K = e^{xH} e^Y K$ implies (looking at the element in the last row and column) $\operatorname{ch} z = \operatorname{ch} x \operatorname{ch} y + \operatorname{sh} x \frac{\operatorname{sh} y}{y} y_1$ with $z = \|Z\|$, hence $z \in [|x-y|, x+y]$. Taking $Y = k \cdot yH$ with $y \in \mathbb{R}$ and $k \in K$ we see that $z = \|Z(xH, k \cdot yH)\|$ is given by

$$\operatorname{ch} z = \operatorname{ch} x \operatorname{ch} y + \pi(k \cdot H) \operatorname{sh} x \operatorname{sh} y.$$

Then, by Lemma B.1,

$$\int_K \varphi(z) dk = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} f(t)\left(1-t^2\right)^{(n-3)/2} dt$$

if $\varphi$ and $f$ are related by $\varphi(z) = f(t)$ and $\operatorname{ch} z = \operatorname{ch} x \operatorname{ch} y + t \operatorname{sh} x \operatorname{sh} y$. Expressing the latter integral with the variable $z$, the result now follows since

$$1 - t^2 = 4 \frac{\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z}{\operatorname{sh}^2 x \operatorname{sh}^2 y} B(x, y, z).$$

■

Our goal is now to prove Lemma B.5, giving (B.2) for the other hyperbolic spaces, by the classical technique of reduction to $SU(2,1)$ [27, Chap. IX, §3]. Let $V \in \mathfrak{g}_{-\alpha}$ and $W \in \mathfrak{g}_{-2\alpha}$ be fixed, with $\|V\| = \|W\| = \sqrt{2}$. The next lemma reduces integration of $M$-bi-invariant functions on $K$ to a 2-dimensional integral; for a similar lemma with different coordinates see Orloff [39, p. 588]. Let us recall that $\dim G/K = n = p + q + 1$ with $p = \dim \mathfrak{g}_{-\alpha}$, $q = \dim \mathfrak{g}_{-2\alpha}$ and let $g = k(g)e^{H(g)}n(g)$ denote the Iwasawa decomposition of $g \in G$.

**Lemma B.4.** *Let $k(r,\omega) := k(\overline{n})$ be the $K$-component of $\overline{n} = \exp(vV + wW) \in \overline{N} = \theta N$, where $v > 0$, $w > 0$ are related to $(r,\omega)$ by*

$$1 + re^{i\omega} = \frac{2}{1 + v^2 - 2iw}.$$

*Then, for any continuous function $f$ on $K$ such that $f(mkm') = f(k)$ for $m, m' \in M$ and $k \in K$,*

*(i) if $q > 1$ (quaternionic and exceptional hyperbolic spaces)*

$$\int_K f(k)\,dk$$
$$= \frac{2\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^1 (1-r^2)^{(p/2)-1} r^q dr \int_0^\pi f(k(r,\omega)) \sin^{q-1}\omega\, d\omega$$

*(ii) if $q = 1$ (complex hyperbolic spaces)*

$$\int_K f(k)dk = \frac{p}{2\pi}\int_0^1 (1-r^2)^{(p/2)-1} r\, dr \int_{-\pi}^{\pi} f(k(r,\omega))d\omega.$$

*Proof.* *(i)* The $M$-invariance of $f$ implies

$$\int_K f(k)dk = \int_{\overline{N}} f(k(\overline{n}))e^{-2<\rho, H(\overline{n})>} d\overline{n}$$

by a classical integral formula valid for arbitrary rank ([28] p. 198), if the Haar measure $d\overline{n}$ is suitably normalized. In the rank one case $\overline{N} = \exp(\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha})$; if $q > 1$ we may use polar coordinates $(v,\sigma)$ in $\mathfrak{g}_{-\alpha}$, resp. $(w,\tau)$ in $\mathfrak{g}_{-2\alpha}$, and obtain

$$\int_K f(k)dk$$
$$= C \int f(k(\exp(v\sigma + w\tau)))e^{-2<\rho, H(\exp(v\sigma+w\tau))>} v^{p-1} w^{q-1} dv\, dw\, d\sigma\, d\tau$$

where $C$ is a constant, $v, w$ run over $]0,\infty[$ and $\sigma, \tau$ over the unit spheres $S_1$, $S_2$ of $\mathfrak{g}_{-\alpha}$, $\mathfrak{g}_{-2\alpha}$ with measures $d\sigma, d\tau$. By a theorem of Kostant [60, p. 265] Ad $M$ acts transitively on $S_1 \times S_2$ if $q > 1$ so that, in view of $\|V\| = \|W\| = \sqrt{2}$, we have $v\sigma + w\tau = m \cdot (\frac{v}{\sqrt{2}}V + \frac{w}{\sqrt{2}}W)$ for some $m \in M$. But $M$ commutes with $A$ and normalizes $N$, therefore

$$k(m\overline{n}m^{-1}) = mk(\overline{n})m^{-1} \ , \ H(m\overline{n}m^{-1}) = H(\overline{n})$$

for $m \in M$, $\overline{n} \in \overline{N}$ and, by the $M$-invariance of $f$,

$$\int_K f(k)dk$$
$$= C' \int f(k(\exp(vV + wW)))e^{-2<\rho, H(\exp(vV+wW))>}v^{p-1}w^{q-1}dv\,dw, \tag{B.3}$$

an integral over $v > 0$, $w > 0$ with another constant $C'$.

The relation $\left(1 + re^{i\omega}\right)\left(1 + v^2 - 2iw\right) = 2$ defines a change of variables $(v, w) \mapsto (r, \omega)$ (which will be convenient to prove the next lemma), a diffeomorphism of $]0,\infty[\times]0,\infty[$ onto $]0,1[\times]0,\pi[$ inverted by

$$v = \left(\frac{1 - r^2}{1 + 2r\cos\omega + r^2}\right)^{1/2} \ , w = \frac{r\sin\omega}{1 + 2r\cos\omega + r^2}.$$

Besides

$$v\,dv\,dw = \left(1 + 2r\cos\omega + r^2\right)^{-2} r\,dr\,d\omega.$$

As usual let $\theta$ denote the Cartan involution of $\mathfrak{g}$. To compute the Iwasawa decomposition of $\overline{n} = \exp(vV + wW)$ it suffices to work in the Lie subalgebra of $\mathfrak{g}$ generated by $V$, $W$, $\theta V$ and $\theta W$, a method known as $SU(2,1)$-reduction since this subalgebra corresponds to a Lie subgroup of $G$ isomorphic to $SU(2,1)$. By [27, Chap. IX, Theorem 3.8] we have (remembering our choice (3.44) of the norm and $\|V\| = \|W\| = \sqrt{2}$)

$$\begin{aligned} e^{-2<\rho, H(\overline{n})>} &= \left(\left(1 + v^2\right)^2 + 4w^2\right)^{-(p/2)-q} \\ &= 2^{-p-2q}\left(1 + 2r\cos\omega + r^2\right)^{(p/2)+q} \end{aligned}$$

and the integral formula follows, with $C'$ given by the case $f = 1$.

*(ii)* For $q = 1$ the group Ad $M$ acts transitively on the unit sphere of $\mathfrak{g}_{-\alpha}$ by Kostant's theorem and trivially on $\mathfrak{g}_{-2\alpha}$. The integral (B.3) now runs over $v > 0$ and $w \in \mathbb{R}$ and the change $(v, w) \mapsto (r, \omega)$ is a diffeomorphism of $(]0,\infty[\times\mathbb{R}) \setminus ([1,\infty[\times\{0\})$ onto $]0,1[\times] - \pi, \pi[$. The result follows as above. ■

**Lemma B.5.** *Given* $x, y > 0$ *let* $\varphi$ *be continuous on* $[|x-y|, x+y]$*. Then, for all rank one spaces,*

$$\int_K \varphi\left(\|Z(xH, k \cdot yH)\|\right) dk = \int_{|x-y|}^{x+y} \varphi(z) b(x, y, z)\, dz$$

*with*

$$b(x,y,z) = \frac{2^{n-3}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{(\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z)^{(p/2)-1}}{(\operatorname{sh} x \operatorname{sh} y)^{n-2}}$$
$$\times \operatorname{sh} z\, (\operatorname{ch} z)^q\, B^{(n-3)/2}\, {}_2F_1\left(1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; B\right)$$

*and*

$$B = \frac{1}{\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z}$$
$$\times \operatorname{sh}\left(\frac{x+y+z}{2}\right) \operatorname{sh}\left(\frac{x+y-z}{2}\right) \operatorname{sh}\left(\frac{x-y+z}{2}\right) \operatorname{sh}\left(\frac{-x+y+z}{2}\right)$$

*One has* $b(x, y, z) > 0$ *for* $x, y > 0$ *and* $|x-y| < z < x+y$*.*

*Proof.* For $q = 0$ the hypergeometric factor is 1 and the result is given by Lemma B.3, where $n = p + 1$.

We shall prove the lemma for $q > 1$; the case $q = 1$ is similar with minor changes.

*(i)* Since the function $k \mapsto z = \|Z(xH, k \cdot yH)\|$ is $M$-bi-invariant we only need to compute it, by the previous lemma, for $k = k(r, \omega)$. In order to find $z \geq 0$ such that

$$e^{xH} k(r, \omega) e^{yH} = k' e^{zH} k''$$

for some $k', k'' \in K$, we use $SU(2,1)$-reduction again. By [27, Chap. IX, Theorem 3.1] the Lie subalgebra of $\mathfrak{g}$ generated by $V$, $W$, $\theta V$ and $\theta W$ contains $H$ and is isomorphic to $su(2,1)$. Under this isomorphism $H, V, W$ respectively correspond to

$$H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, V_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, W_0 = \begin{pmatrix} i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -i \end{pmatrix}.$$

All computations can now be performed in $SU(2,1)$ with the maximal compact subgroup $S(U(2) \times U(1))$; we use subscripts 0 for all notions relative to this group. Let $\overline{n}_0 = k_0 e^{tH_0} n_0$ be the Iwasawa decomposition of

$$\overline{n}_0 = \exp(vV_0 + wW_0) = \begin{pmatrix} 1 - \frac{v^2}{2} + iw & v & -\frac{v^2}{2} + iw \\ -v & 1 & -v \\ \frac{v^2}{2} - iw & -v & 1 + \frac{v^2}{2} - iw \end{pmatrix},$$

with

$$k_0 = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & u^{-1} \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2), \; ad - bc = u, \; |u| = 1.$$

Applying $\overline{n}_0$ to the (column) vector $\mathbf{v} = {}^t(1, 0, 1)$ it is easily checked, since $H_0\mathbf{v} = \mathbf{v}$ and $n_0\mathbf{v} = \mathbf{v}$, that

$$au = \frac{1 - v^2 + 2iw}{1 + v^2 - 2iw} = re^{i\omega}$$

with $\left(1 + re^{i\omega}\right)\left(1 + v^2 - 2iw\right) = 2$ as in the previous lemma.

Besides, a look at the matrix element on third column and third row of $e^{xH_0}k_0e^{yH_0} = k_0'e^{zH_0}k_0''$ gives (with $u'$, $u''$ corresponding to $k_0'$, $k_0''$)

$$uu'^{-1}u''^{-1} \operatorname{ch} z = \operatorname{ch} x \operatorname{ch} y + au \operatorname{sh} x \operatorname{sh} y.$$

Setting $e^{i\psi} := uu'^{-1}u''^{-1}$ we conclude, first in $SU(2,1)$ then in $G$, that $e^{xH}k(r,\omega)e^{yH} = k'e^{zH}k''$ implies

$$e^{i\psi} \operatorname{ch} z = \operatorname{ch} x \operatorname{ch} y + re^{i\omega} \operatorname{sh} x \operatorname{sh} y \tag{B.4}$$

for some $\psi \in \mathbb{R}$.

*(ii)* Keeping $x, y > 0$ fixed let us look at the map $(r,\omega) \mapsto (z,\psi)$ defined by (B.4), with $0 < r < 1$, $0 < \omega < \pi$ and $z \geq 0$, $-\pi \leq \psi \leq \pi$. Taking the imaginary part we have

$$\operatorname{ch} z \sin\psi = r \sin\omega \operatorname{sh} x \operatorname{sh} y, \tag{B.5}$$

thus $\sin\psi > 0$ and $0 < \psi < \pi$. Taking the modulus, $\operatorname{ch}(x - y) < \operatorname{ch} z < \operatorname{ch}(x + y)$, thus $|x - y| < z < x + y$. With $B$ defined in the lemma it is readily checked that

$$B = \frac{(\operatorname{ch}(x+y) - \operatorname{ch} z)(\operatorname{ch} z - \operatorname{ch}(x-y))}{4 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z} \tag{B.6}$$

$$1 - 2B = \frac{\operatorname{ch}^2 x + \operatorname{ch}^2 y + \operatorname{ch}^2 z - 1}{2 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z}, \tag{B.7}$$

thus $0 < B < 1/2$. Furthermore (B.4) implies $|e^{i\psi}\operatorname{ch} z - \operatorname{ch} x \operatorname{ch} y|^2 = r^2 \operatorname{sh}^2 x \operatorname{sh}^2 y$, that is

$$2 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z \; (\cos\psi - 1 + 2B) = \left(1 - r^2\right) \operatorname{sh}^2 x \operatorname{sh}^2 y. \tag{B.8}$$

Therefore $\cos\psi > 1 - 2B > 0$ and we conclude that $(r, \omega) \mapsto (z, \psi)$ is a diffeomorphism of $]0, 1[\times]0, \pi[$ onto the open set defined by

$$|x - y| < z < x + y \,, 0 < \psi < \pi/2 \,, \; \cos\psi > 1 - 2B(x, y, z).$$

By Lemma B.4 the integral $I = \int_K \varphi\,(\|Z(xH, k \cdot yH)\|)\, dk$ is

$$I = C \int_0^\pi \varphi(z)(1 - r^2)^{(p/2)-1}\, (r \sin\omega)^{q-1}\, r\, dr\, d\omega$$

with $z = z(r, \omega)$ given by (B.4) and $C$ by the previous lemma. Using (B.5), (B.8) and $\operatorname{sh} z \operatorname{ch} z \; dz\, d\psi = \operatorname{sh}^2 x \operatorname{sh}^2 y \;\; r\, dr\, d\omega$ the integral becomes

$$I = C \frac{(2 \operatorname{ch} x \operatorname{ch} y)^{(p/2)-1}}{(\operatorname{sh} x \operatorname{sh} y)^{n-2}}$$
$$\times \int \varphi(z)(\operatorname{ch} z)^{(p/2)-1+q} \operatorname{sh} z\, dz \int (\cos\psi - 1 + 2B)^{(p/2)-1}\, (\sin\psi)^{q-1}\, d\psi.$$

Considering the domain of $\psi$ for a given $z$ it is now natural to introduce a variable $t$, running over $]0, 1[$, such that

$$\cos\psi = 1 - 2tB.$$

Then $\sin^2\psi = 4tB(1 - tB)$ and the integral with respect to $\psi$ becomes

$$2^{(p/2)+q-2} B^{(n-3)/2} \int_0^1 t^{(q/2)-1}(1 - t)^{(p/2)-1}(1 - tB)^{(q/2)-1} dt$$
$$= 2^{(p/2)+q-2} \frac{\Gamma(p/2)\Gamma(q/2)}{\Gamma((n-1)/2)} B^{(n-3)/2}\, {}_2F_1\left(1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; B\right)$$

by Euler's integral representation of the hypergeometric function. Since $0 < B < 1/2$ the left-hand side is strictly positive. This implies the lemma, with $b(x, y, z) > 0$ for $|x - y| < z < x + y$. ■

*Remarks.* *(a)* Similar computations appear in the study of generalized translation operators by Flensted-Jensen and Koornwinder ([22], or [34] §7.1) in the more general framework of Jacobi functions.

*(b)* The easy Lemma B.2 may be viewed as a flat limit of Lemma B.5. Indeed on the left-hand side of the integral formula $\varepsilon^{-1}Z(\varepsilon X, \varepsilon Y)$ tends to $X+Y$ as $\varepsilon$ tends to 0 whereas, replacing $x, y, z$ by $\varepsilon x, \varepsilon y, \varepsilon z$ respectively in the right-hand side, $B$ tends to 0, the hypergeometric factor to 1 and $\varepsilon b(\varepsilon x, \varepsilon y, \varepsilon z)$ to $a(x, y, z)$.

*Proof of Theorem 3.23.* We now combine Lemmas B.2 and B.5. The function

$$\frac{b}{a}(x,y,z) = \left(\frac{\operatorname{ch} z}{\operatorname{ch} x \operatorname{ch} y}\right)^{q/2} \frac{\sigma(z)}{(\sigma(x)\sigma(y))^{n-2}}\, {}_2F_1\left(1-\frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; B\right)$$
$$\times \left(\sigma\left(\frac{x+y+z}{2}\right)\sigma\left(\frac{x+y-z}{2}\right)\sigma\left(\frac{x-y+z}{2}\right)\sigma\left(\frac{-x+y+z}{2}\right)\right)^{(n-3)/2}$$

(where $\sigma(t) = \operatorname{sh} t/t$ and $B$ is defined in the previous lemma) is continuous on the set of $(x, y, z) \in \mathbb{R}^3$ such that $|x-y| \le z \le x+y$. Indeed $0 \le B < 1/2$ in this domain by (B.6), (B.7), and the hypergeometric factor is continuous. Taking $\varphi$ continuous on $[|x-y|, x+y]$ we may therefore replace $\varphi(z)$ by $\varphi(z)(b/a)(x, y, z)$ in Lemma B.2, whence

$$\int_K \varphi\left(\|Z(xH, k\cdot yH)\|\right) dk = \int_{|x-y|}^{x+y} \varphi(z)\frac{b}{a}(x,y,z)a(x,y,z)dz$$
$$= \int_K \varphi\left(\|xH + k\cdot yH\|\right)\frac{b}{a}(x, y, \|xH + k\cdot yH\|)dk.$$

Our claim follows, as explained in the outline (Sect. 3.7.1), with

$$e(X,Y) = \frac{j(X)j(Y)}{j(X+Y)}\frac{b}{a}\left(\|X\|, \|Y\|, \|X+Y\|\right).$$

We now specialize to $j = J^{1/2}$. In the rank one case any $X \in \mathfrak{p}$ may be written as $X = k\cdot xH$ with $x \ge 0$, $k \in K$. Since $\alpha(xH) = x = \|X\|$ the Jacobian of Exp is

$$J(X) = J(xH) = \left(\frac{\operatorname{sh} x}{x}\right)^p \left(\frac{\operatorname{sh} 2x}{2x}\right)^q = \sigma(x)^{n-1}(\operatorname{ch} x)^q.$$

For $j = J^{1/2}$ we thus obtain

$$e(X,Y) = A(x,y,z)^{(n-3)/2}\, {}_2F_1\left(1-\frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; B(x,y,z)\right)$$

as claimed, with $x = \|X\|$, $y = \|Y\|$, $z = \|X+Y\|$. Clearly $e(k\cdot X, k\cdot Y) = e(X, Y)$ for $k \in K$.

Besides, $A$ and $B$ are analytic functions of $(x, y, z) \in \mathbb{R}^3$, even with respect to each variable, therefore define analytic functions of $(x^2, y^2, z^2)$. Thus $A\left(\|X\|, \|Y\|, \|X+Y\|\right)$ is an analytic function of $(X, Y) \in \mathfrak{p}\times\mathfrak{p}$ and the same holds for $B$. Since $|x - y| \leq z \leq x + y$ for the chosen values we have $0 \leq B(x, y, z) < 1/2$, which implies analyticity of the hypergeometric factor too. The theorem now follows from Proposition 3.16. ■

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