Appendix A
Convolutions and Laplace Transforms

Abstract Certain properties of the convolution and the Laplace transform are collected here for convenience.

A.1 Definitions and Basic Properties

Suppose $g : \mathbf{R} \to \mathbf{R}^+$, $g(t) = 0, t \in (-\infty, 0)$, and $F(t)$ is a distribution function concentrated on $[0, \infty)$. Define the *convolution* of $F(t)$ and $g(t)$ as the function

$$F * g(t) = \int_0^t g(t-u)dF(u), \; t \geq 0,$$

where the integration includes the endpoints. The following properties of convolution hold.

1. $F * g(t) \geq 0, \quad t \geq 0.$
2. If $g(t)$ is bounded on finite intervals so is $F * g(t)$.
3. If $g(t)$ is bounded and continuous, then $F * g(t)$ is continuous.
4. The convolution operation can be repeated: $F * (F * g)(t)$.

We denote by

$$I(t) = \begin{cases} 0, \text{ if } t < 0, \\ 1, \text{ if } t \geq 1 \end{cases}$$

the distribution function that assigns a mass 1 at 0.

For any distribution function $F(t)$ concentrated on $\mathbf{R}^+$,

$$F^{0*}(t) = I(t), \; F^{1*}(t) = F(t), \; F^{(n+1)*}(t) = F^{n*} * F(t), \; n \geq 1.$$

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Clearly, $F^{0*}(t)$ acts as an identity:

$$F^{0*} * g(t) = g(t),$$

and an associative property holds:

$$F * (F * g)(t) = (F * F) * g(t) = F^{2*} * g(t).$$

5. Convolutions of two distribution functions correspond to sums of independent random variables. Let X_1 and X_2 be independent with distribution functions $F_1(t)$ and $F_2(t)$, respectively. Then $X_1 + X_2$ has distribution function $F_1 * F_2(t)$.
6. From 5, it follows that the commutative property holds $F_1 * F_2(t) = F_2 * F_1(t)$.
7. By induction we may show that if $X_1, X_2, \ldots, X_n$ are independent random variables with common distribution function $F(t)$ then $X_1 + \cdots + X_n$ has distribution function $F^{n*}(t)$.
8. If $F_1(t)$ and $F_2(t)$ are absolutely continuous with densities $f_1(t)$ and $f_2(t)$ for $t > 0$ then $F_1 * F_2(t)$ has density for $t > 0$

$$f_1 * f_2(t) = \int_0^t f_1(t-u) f_2(u) du = \int_0^t f_2(t-u) f_1(u) du.$$

For a non-negative random variable X with distribution function $F(t)$, the Laplace transform is a function defined on $[0, \infty)$ by

$$\hat{F}(s) = \mathbf{E}\left[e^{-sX}\right] = \int_0^\infty e^{-st} dF(t), \quad s \geq 0.$$

The following properties are useful:

1. Distinct distributions have distinct Laplace transforms.
2. Suppose X_1, X_2 are independent and have distribution functions $F_1(t)$ and $F_2(t)$, respectively. Then

$$(\widehat{F_1 * F_2})(s) = \hat{F}_1(s) \hat{F}_2(s).$$

3. If $F(t)$ is a distribution function, then

$$\widehat{F^{n*}}(s) = (\hat{F}(s))^n.$$

4. For $s > 0$, $\hat{F}(s)$ has derivatives of all orders, and for any $n \geq 1$,

$$(-1)^n \frac{d^n}{ds^n} \hat{F}(s) = \int_0^\infty e^{-st} t^n dF(t).$$

Now, by monotone convergence,

$$\lim_{s\downarrow 0}(-1)^n \frac{d^n}{ds^n}\hat{F}(s) = \int_0^\infty t^n dF(t) \leq \infty.$$

In particular, if the random variable X has $F(t)$ as its distribution function, then $\mathbf{E}[X] = -\hat{F}'(0)$ and $\mathbf{E}[X^2] = \hat{F}''(0)$, and so on.

5. An integration by parts proves the following formulas

$$\int_0^\infty e^{-st} F(t)dt = \frac{\hat{F}(s)}{s}, \quad \int_0^\infty e^{-st}(1 - F(t))dt = \frac{1 - \hat{F}(s)}{s}. \tag{A.1}$$

We now extend these notions to arbitrary distributions and measures U on $[0, \infty)$. Suppose that U is a measure on $\mathbf{R}^+$. Then $U(t) = U([0, t])$ is a non-negative and nondecreasing function on $[0, \infty)$, but perhaps, $U(\infty) = U([0, \infty)) = \lim_{t\uparrow\infty} U(t) > 1$.

If there exists $a \geq 0$ such that

$$\int_0^\infty e^{-st} dU(t) < \infty$$

for $s > a$ then

$$\hat{U}(s) = \int_0^\infty e^{-st} dU(t) < \infty, \; s > a \tag{A.2}$$

is called the Laplace transform of $U(t)$. If such a does not exist, we say the Laplace transform is undefined. For detailed discussions on these topics we refer to the book of Feller [2].

A.2 Regularly Varying Functions and Tauberian Theorems

We need the following definitions.

Definition 1.1 A measurable function $L : [A, \infty) \to \mathbf{R}^+$, where $A \geq 0$, is said to be slowly varying at infinity (s.v.f.) if for every $x > 0$,

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1. \tag{A.3}$$

Definition 1.2 A measurable function $f : \mathbf{R}^+ \to \mathbf{R}^+$ is said to be regularly varying at infinity (r.v.f.) with exponent $\rho > 0 (f(t) \in RV(\rho))$ if

$$f(t) \sim t^\rho L(t), \quad t \to \infty, \tag{A.4}$$

for some s.v.f. $L(t)$.

Below, we formulate only the results for regularly varying functions that we need in the book. For a comprehensive treatment of the notion of regular variation and its applications, we refer to the books of Feller [2], Bingham et al. [1], or Seneta [3].

Theorem 1.1 (Feller [2], Theorem 1, §VIII.9) (a) *If* $Z(t) \in RV(\gamma)$ *and the integral* $Z_p^*(x) = \int_x^\infty t^p Z(t)dt$ *converges then*

$$\frac{t^{p+1}Z(t)}{Z_p^*(t)} \to \zeta,$$

where $\zeta = -(p + 1 + \gamma) \geq 0$. *Conversely, if the last relation holds true and* $\zeta > 0$ *then* Z *and* Z_p^* *are regularly varying with exponents* $\gamma = -\zeta - p - 1$ *and* $-\zeta$, *respectively.*

(b) *If* $Z \in RV(\gamma)$ *and the integral* $Z_p(x) = \int_0^x t^p Z(t)dt$ *converges then if* $p \geq -\gamma - 1$, *then*

$$\frac{t^{p+1}Z(t)}{Z_p(t)} \to \zeta,$$

where $\zeta = p + \gamma + 1$. *Conversely, if the last limit is* $\zeta > 0$ *then* Z *and* Z_p *are regularly varying with exponents* $\zeta - p - 1$ *and* ζ, *respectively.*

Theorem 1.2 (Feller [2], Lemma, §XIII.5) *If*

$$U(t) \sim \frac{t^\rho L(t)}{\Gamma(\rho + 1)}, \quad t \to \infty,$$

for $\rho > 0$ *and has* $u(t) = U'(t)$ *is eventually monotone then*

$$u(t) \sim \frac{\rho U(t)}{t}, t \to \infty.$$

Theorem 1.3 *Let* $U(t)$ *be a measure on* $\mathbf{R}^+$ *and* $\hat{U}(s)$ *be its Laplace transform defined by (A.2). Then*

$$U(\infty) := \lim_{t\to\infty} U(t) < \infty \text{ if and only if } \hat{U}(0) := \lim_{s\to 0} \hat{U}(s) < \infty. \tag{A.5}$$

If this is the case then $U(\infty) = \hat{U}(0)$.

Theorem 1.4 (Karamata's Tauberian theorem) *If $L(t)$ is slowly varying at infinity and $0 \le \rho < \infty$, then each of the relations*

$$\hat{U}(s) \sim s^{-\rho} L\left(\frac{1}{s}\right), \quad s \to 0, \tag{A.6}$$

and

$$U(t) \sim \frac{1}{\Gamma(\rho+1)} t^{\rho} L(t), \quad t \to \infty, \tag{A.7}$$

implies the other.

The proofs and certain comments can be found in the books cited above.

The next two theorems describe the limiting behavior of the sum of independent identically distributed random variables in case when their mathematical expectation is infinite.

Theorem 1.5 (Feller [2], Theorem 1, §XIII.6) *Let $\beta \in (0, 1)$ be fixed. The function $\gamma_\beta(s) = e^{s^\beta}$, $s \ge 0$ is the Laplace transform of a distribution function $G_\beta(t)$, which has the following properties:*

(a) *$G_\beta(t)$ is stable, that is, if $X_1, X_2, \ldots, X_n$ are random variables with distribution function G_β then*

$$\frac{X_1 + X_2 + \cdots + X_n}{n^{1/\beta}}$$

has the same distribution function $G_\beta(t)$.

(b) *$G_\beta(t)$ satisfies the relations:*

$$t^\beta (1 - G_\beta(t)) \to \frac{1}{\Gamma(1-\beta)}, \quad t \to \infty$$

$$\exp(t^{-\beta}) G_\beta(t) \to 0, \quad t \to \infty.$$

Theorem 1.6 (Feller [2], Theorem 2, §XIII.6) *Let $F(t)$ be a distribution function on $(0, \infty)$, i.e., ($F(0) = 0$, $F(+\infty) = 1$) and such that*

$$F^{*n}(a_n t) \to G(t), \tag{A.8}$$

for the points of continuity of G, where G is a proper distribution function, not concentrated in one point. Then

(a) *There exists a slowly varying at infinity function L and a constant β, $0 < \beta < 1$, such that*

$$1 - F(t) \sim \frac{t^{-\beta} L(t)}{\Gamma(1-\beta)}, \quad t \to \infty.$$

(b) *Conversely, if F satisfies the last relation then it is possible to choose a sequence a_n, $n = 1, 2, \ldots$, such that*

$$\frac{nL(a_n)}{a_n^{\beta}} \to 1,$$

and in this case (A.8) holds with $G(t) = G_\beta(t)$.

Remark 1.1 If $X_1, X_2, \ldots$ are independent identically distributed random variables with distribution function $F(t)$ then (A.8) means that

$$\lim_{n\to\infty} \Pr\left\{\frac{X_1 + X_2 + \cdots + X_n}{a_n} \le t\right\} = G(t).$$

This is an analog of the central limit theorem in the case of a sum of i.i.d. random variables with infinite mean.

References

1. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Cambridge University Press, Cambridge (1987)
2. Feller, W.: An Introduction to Probability Theory and its Applications, vol. II. Wiley, New York (1971)
3. Seneta, E.: Regularly Varying Functions. Springer, Berlin (1976)