Appendix A
Gaussian Random Variables

This appendix contains some well-known results about Gaussian random variables that have been used several times in this monograph.

1 Tail Bounds

The following pair of inequalities is collectively known as the Mills ratio bounds. For a standard Gaussian random variable g, for any $x > 0$,

$$\frac{xe^{-x^2/2}}{\sqrt{2\pi}(1+x^2)} \le \mathbb{P}(g > x) \le \frac{e^{-x^2/2}}{x\sqrt{2\pi}}. \tag{A.1}$$

The proof is not difficult, and may be found in numerous standard texts on probability and statistics. The inequalities in the above form were probably first observed by Gordon (1941). Another useful inequality is

$$\mathbb{P}(g > x) \le e^{-x^2/2}, \tag{A.2}$$

which follows simply by optimizing over θ in the inequality $\mathbb{P}(g > x) \le e^{-\theta x}\mathbb{E}(e^{\theta g})$, using the formula $\mathbb{E}(e^{\theta g}) = e^{\theta^2/2}$.

2 Size of the Maximum

It is a well-known result that the maximum of n i.i.d. Gaussian random variables is asymptotically like $\sqrt{2\log n}$, with fluctuations of order $(\log n)^{-1/2}$. In fact, much finer details are known, such as the limiting distribution of the maximum under the correct centering and scaling (see e.g. Leadbetter et al. 1983).

However, what is not so well-known (but equally easy) is that the maximum of n standard Gaussian random variables, even if they are not independent, cannot be

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much larger than $\sqrt{2\log n}$. In fact, irrespective of the covariance structure,

$$\mathbb{E}\Big(\max_{1\le i\le n} g_i\Big) \le \sqrt{2\log n}. \tag{A.3}$$

This is easily proved as follows. Suppose $g_1, \ldots, g_n$ are standard Gaussian random variables, not necessarily independent. Then for any $\beta > 0$,

$$\begin{aligned}\mathbb{E}\Big(\max_i g_i\Big) &= \frac{1}{\beta}\mathbb{E}\big(\log e^{\beta \max_i g_i}\big)\\ &\le \frac{1}{\beta}\mathbb{E}\left(\log \sum_{i=1}^n e^{\beta g_i}\right) \le \frac{1}{\beta}\log\sum_{i=1}^n \mathbb{E}\big(e^{\beta g_i}\big) = \frac{\beta}{2} + \frac{\log n}{\beta}.\end{aligned}$$

The claim is proved by taking $\beta = \sqrt{2\log n}$. Similarly,

$$\begin{aligned}\mathbb{E}\Big(\max_i |g_i|\Big) &= \frac{1}{\beta}\mathbb{E}\big(\log e^{\beta \max_i |g_i|}\big)\\ &\le \frac{1}{\beta}\mathbb{E}\left(\log \sum_{i=1}^n e^{\beta |g_i|}\right) \le \frac{1}{\beta}\log\sum_{i=1}^n \mathbb{E}\big(e^{\beta g_i} + e^{-\beta g_i}\big)\\ &= \frac{\beta}{2} + \frac{\log(2n)}{\beta}.\end{aligned}$$

Taking $\beta = \sqrt{2\log(2n)}$ gives

$$\mathbb{E}\Big(\max_i |g_i|\Big) \le \sqrt{2\log(2n)}.$$

A stronger bound is given by the following lemma.

Lemma A.1 *Let $g_1, \ldots, g_n$ be i.i.d. standard Gaussian random variables. If $n \ge 2$ then for any $p \ge 1$,*

$$\mathbb{E}\Big|\max_i g_i\Big|^p \le \mathbb{E}\max_i |g_i|^p \le C(p)(\log n)^{p/2},$$

where $C(p)$ is a constant that depends only on p.

Proof Combine the inequality $\mathbb{E}(\max_i g_i) \le \sqrt{2\log n}$ with the concentration of the maximum (see later in this Appendix), and observe that $\max|g_i|$ is the maximum of the concatenation of the vectors g and $-g$. □

Sometimes, information about the size of the maximum can be obtained from a famous comparison method due to Slepian (1962):

Lemma A.2 (Slepian's lemma) *Suppose g and h are centered Gaussian random n-vectors with $\mathbb{E}(g_i^2) = \mathbb{E}(h_i^2)$ for each i and $\mathbb{E}(g_i g_j) \geq \mathbb{E}(h_i h_j)$ for each i, j. Then for each $x \in \mathbb{R}$,*

$$\mathbb{P}\Big(\max_i g_i > x\Big) \leq \mathbb{P}\Big(\max_i h_i > x\Big).$$

In particular, $\mathbb{E} \max_i g_i \leq \mathbb{E} \max_i h_i$.

The following lemma due to Sudakov gives a simple way to compute lower bounds on the expected value of the maximum. For a proof of this result, see Talagrand (2005, Lemma 2.1.2).

Lemma A.3 Sudakov minoration *Let g be a centered Gaussian n-vector. Suppose a is a constant such that $\mathbb{E}(g_i - g_j)^2 \geq a$ for all $i \neq j$. Then*

$$\mathbb{E} \max_i g_i \geq Ca\sqrt{\log n},$$

where C is a positive universal constant.

3 Integration by Parts

Suppose g is a standard Gaussian random variable, and $f : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function. Under the assumption that $\mathbb{E}|f'(g)| < \infty$, a standard application of integration by parts gives the well-known identity

$$\mathbb{E} g f(g) = \mathbb{E} f'(g).$$

This identity can be easily generalized to n dimensions. Suppose that $g = (g_1, \ldots, g_n)$ is a centered Gaussian vector, possibly with correlations. If $f : \mathbb{R}^n \to \mathbb{R}$ is an absolutely continuous function such that $|\nabla f(g)|$ has finite expectation (where ∇f is the gradient of f and $|\cdot|$ is the Euclidean norm), then for any i,

$$\mathbb{E}\big(g_i f(g)\big) = \sum_{j=1}^{n} \mathbb{E}(g_i g_j) \mathbb{E}\big(\partial_i f(g)\big), \tag{A.4}$$

where $\partial_i f$ is the partial derivative of f in the ith coordinate. The above identity can be derived from the previous one by writing g as a linear transformation of a standard Gaussian random vector.

4 The Gaussian Concentration Inequality

The Gaussian Poincaré inequality gives a bound on the variance of a function of Gaussian random variables. In particular, it says that if the function is Lipschitz, then the variance is bounded by the square of the Lipschitz constant. For Lipschitz functions, it is possible to derive a much stronger version of the Poincaré inequality that gives a sub-Gaussian tail bound. If $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant K and g is an n-dimensional standard Gaussian random vector, then for any $x \geq 0$

$$\begin{aligned}&\mathbb{P}\big(f(g) \geq \mathbb{E}f(g) + x\big) \leq e^{-x^2/2K^2} \quad \text{and} \\ &\mathbb{P}\big(f(g) \leq \mathbb{E}f(g) - x\big) \leq e^{-x^2/2K^2}.\end{aligned} \tag{A.5}$$

This is known as the Gaussian concentration inequality. To prove this inequality, assume without loss of generality that $\mathbb{E}f(g) = 0$. Take any $\theta > 0$ and let $v = e^{\theta f}$. Let $(\cdot, \cdot)$ denote the usual inner product on $L^2(\gamma^n)$, where γ^n is the n-dimensional standard Gaussian measure. By the covariance lemma (Lemma 2.1) and the representation (2.2),

$$\begin{aligned}(f, v) &= \int_0^\infty e^{-t} \int \nabla v \cdot P_t \nabla f \, d\gamma^n \, dt \\ &= \int_0^\infty e^{-t} \int \theta e^{\theta f} \nabla f \cdot P_t \nabla f \, d\gamma^n \, dt.\end{aligned}$$

By the Cauchy-Schwarz inequality and the Lipschitzness of f, $|\nabla f \cdot P_t \nabla f| \leq K^2$ everywhere. Thus the above identity gives that for all $\theta > 0$

$$\frac{d}{d\theta}\mathbb{E}\big(e^{\theta f(g)}\big) = \big(f, e^{\theta f}\big) \leq K^2 \theta \, \mathbb{E}\big(e^{\theta f(g)}\big),$$

which, in turn, implies that

$$\mathbb{E}\big(e^{\theta f(g)}\big) \leq e^{K^2\theta^2/2}. \tag{A.6}$$

Optimizing the inequality $\mathbb{P}(f(g) \geq x) \leq e^{-\theta x}\mathbb{E}(e^{\theta f(g)})$ over θ gives the bound $\mathbb{P}(f(g) \geq x) \leq e^{-x^2/2K^2}$. The bound for the lower tail can be obtained by replacing f with $-f$.

5 Concentration of the Maximum

Let $g = (g_1, \ldots, g_n)$ be a Gaussian random vector, whose coordinates are not necessarily independent. One can express g as Ah, where h is a standard Gaussian vector and A is a matrix such that AA^T is the covariance matrix of g. Let a_i denote the ith row of A and a_{ij} denote the jth element of a_i. Then $\max_i g_i = f(h)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is the function

$$f(x) := \max_i a_i \cdot x.$$

Take any x and let i_x be the i that maximizes $a_i \cdot x$. (We ignore the set of measure zero on which the maximum is attained at multiple indices.) Then for each j, $\partial_j f(x) = a_{i_x j}$. Thus,

$$\left|\nabla f(x)\right|^2 = \sum_j a_{i_x j}^2 = \mathrm{Var}(g_{i_x}).$$

By the concentration inequality for Lipschitz functions, this implies that

$$\begin{aligned}\mathbb{P}\Big(\max_i g_i - \mathbb{E}\max_i g_i \geq t\Big) &\leq e^{-t^2/2\sigma^2},\\ \mathbb{P}\Big(\max_i g_i - \mathbb{E}\max_i g_i \leq -t\Big) &\leq e^{-t^2/2\sigma^2},\end{aligned} \tag{A.7}$$

where $\sigma^2 := \max_i \mathrm{Var}(g_i)$ and $t \geq 0$. The above inequalities were proved by Tsirelson et al. (1976), although they follow (with slightly worse constants) from the earlier works of Borell (1975) and Sudakov and Cirelson [Tsirelson] (1974).

Appendix B
Hypercontractivity

This appendix gives a proof of the hypercontractive inequality for the Ornstein-Uhlenbeck semigroup. This inequality is one of the key tools used in this monograph, so I felt that the curious reader deserves to see a proof.

The route I will follow here is by now the standard argument via the logarithmic Sobolev inequality for the Gaussian measure. The logarithmic Sobolev inequality says that if γ^n is the standard Gaussian measure on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ is an absolutely continuous function, then

$$\int f^2 \log \frac{f^2}{\int f^2 d\gamma^n} d\gamma^n \leq 2 \int |\nabla f|^2 d\gamma^n \tag{B.1}$$

whenever both sides are finite. This important property of the Gaussian measure was discovered by Gross (1975), who connected it to other properties like hypercontractivity. The logarithmic Sobolev inequality for the Gaussian measure is a consequence of the dynamical formula (2.4) from Chap. 2. The derivation goes as follows (taken from Ledoux 2001). Let $v := f^2$. Let P_t be the n-dimensional OU semigroup and let $v_t := P_t f$, so that $v_t \to \int v d\gamma$ pointwise. Let (u, w) denote the inner product of two functions u and w in $L^2(\gamma^n)$. Under mild regularity conditions on v, a simple computation gives

$$\begin{aligned}
\int f^2 \log \frac{f^2}{\int f^2 d\gamma^n} \, d\gamma^n &= \int v \log v \, d\gamma^n - \int v \, d\gamma^n \log \int v \, d\gamma^n \\
&= -\int_0^\infty \frac{\partial}{\partial t}(v_t, \log v_t) \, dt \\
&= -\int_0^\infty \left(\frac{\partial v_t}{\partial t}, 1 + \log v_t \right) dt.
\end{aligned}$$

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By the heat equation, $\partial v_t/\partial t = Lv_t$ where L is the OU generator. By the formula (2.2) from Chap. 2,

$$(Lv_t, 1 + \log v_t) = -\int \frac{|\nabla v_t|^2}{v_t}\, d\gamma^n.$$

An application of the Cauchy-Schwarz inequality and the identity $\nabla v_t = e^{-t} P_t \nabla v$ (as observed in formula (2.3) of Chap. 2) shows that

$$|\nabla v_t|^2 = e^{-2t}|P_t \nabla v|^2 \le e^{-2t} v_t P_t\left(\frac{|\nabla v|^2}{v}\right).$$

Combining the last three displays gives

$$\begin{aligned}\int f^2 \log \frac{f^2}{\int f^2 d\gamma}\, d\gamma &\le \int_0^\infty e^{-2t} \int P_t\left(\frac{|\nabla v|^2}{v}\right) d\gamma\, dt\\ &= \int_0^\infty e^{-2t} \int \frac{|\nabla v|^2}{v}\, d\gamma\, dt = \frac{1}{2}\int \frac{|\nabla v|^2}{v}\, d\gamma.\end{aligned}$$

Since $\nabla v = 2f\nabla f$, this completes the proof of (B.1).

We are now ready to prove the hypercontractive inequality for the OU semigroup. Recall that the hypercontractive inequality for the OU semigroup says that for any $p > 1$, any $t \ge 0$, and any $f \in L^p(\gamma^n)$,

$$\|P_t f\|_{L^{q(t)}(\gamma^n)} \le \|f\|_{L^p(\gamma^n)}, \tag{B.2}$$

where

$$q(t) := 1 + (p-1)e^{2t}.$$

Note that $q(t) > p$ if $t > 0$. The hypercontractive inequality for the OU semigroup was discovered by Nelson (1973) and explored in connection with the logarithmic Sobolev inequality by Gross.

We will now derive the hypercontractive inequality as a consequence of the logarithmic Sobolev inequality (following the exposition in Guionnet and Zegarlinski 2003). Take any f. Let $v := |f|$. Since $|P_t f| \le P_t v$ and $\|f\|_{L^p(\gamma^n)} = \|v\|_{L^p(\gamma^n)}$, it suffices to prove (B.2) only for non-negative f. So assume $f \ge 0$ everywhere. Let $f_t := P_t f$ and $r(t) := \int f_t^{q(t)} d\gamma^n$. Then by the heat equation and the formula (2.2) from Chap. 2,

$$\begin{aligned}r'(t) &= \int f_t^{q(t)} \frac{\partial}{\partial t}\big(q(t)\log f_t\big)\, d\gamma^n\\ &= q'(t)\int f_t^{q(t)} \log f_t\, d\gamma^n + q(t)\int f_t^{q(t)-1}\frac{\partial f_t}{\partial t} d\gamma^n\\ &= q'(t)\int f_t^{q(t)} \log f_t\, d\gamma^n + q(t)\int f_t^{q(t)-1} L f_t d\gamma^n\end{aligned}$$

$$= \frac{q'(t)}{q(t)} \int f_t^{q(t)} \log f_t^{q(t)} \, d\gamma^n - q(t)\big(q(t)-1\big) \int f_t^{q(t)-2} |\nabla f_t|^2 \, d\gamma^n.$$

Further note that $q'(t) = 2(q(t)-1)$. Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial t} \log \|f_t\|_{L^{q(t)}(\gamma^n)} \\
&\quad = \frac{\partial}{\partial t} \frac{\log r(t)}{q(t)} = -\frac{q'(t) \log r(t)}{q(t)^2} + \frac{r'(t)}{q(t) r(t)} \\
&\quad = \frac{q'(t)}{q(t)^2 r(t)} \int f_t^{q(t)} \log \frac{f_t^{q(t)}}{r(t)} \, d\gamma^n - \frac{q(t)-1}{r(t)} \int f_t^{q(t)-2} |\nabla f_t|^2 \, d\gamma^n \\
&\quad = \frac{q'(t)}{q(t)^2 r(t)} \left(\int f_t^{q(t)} \log \frac{f_t^{q(t)}}{r(t)} \, d\gamma^n - \frac{q(t)^2}{2} \int f_t^{q(t)-2} |\nabla f_t|^2 \, d\gamma^n \right).
\end{aligned}$$

The logarithmic Sobolev inequality (B.1) applied to the function $f_t^{q(t)/2}$ gives

$$\int f_t^{q(t)} \log \frac{f_t^{q(t)}}{r(t)} \, d\gamma^n \leq \frac{q(t)^2}{2} \int f_t^{q(t)-2} |\nabla f_t|^2 \, d\gamma^n.$$

Thus, $\frac{\partial}{\partial t} \log \|f_t\|_{L^{q(t)}(\gamma^n)} \leq 0$ for all t. This proves (B.2).

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