Appendix: Summary of the Recursive Bayesian Inference Schemes

Abstract This appendix aims to provide a summary of the key aspects of general sequential Bayesian estimation problem and to highlight the reason for which a closed-form solution can be obtained when certain prerequisites are met. In addition, the reasons for which approximate solution are widely sought for are explained. In so doing, for the sake of the brevity the details are not presented and instead significant text and research books on the topic are cited (Bittanti 2004; Ljung 1999; Doucet et al. 2001; Haykin 2001), so that the reader can find the details on the derivation of the Bayesian algorithms used in this monograph.

While studying the dynamics of a structural system, usually we have to deal with a state space representation of it; let us consider the following state-space equation:

$$x_k = f_k(x_{k-1}) + v_k \tag{A.1}$$

$$y_k = H_k x_k + w_k \tag{A.2}$$

where f_k (■), maps the state vector x_k. over time, H_k. links the (usually unobservable or partially observable) state vector x_k to the observation y_k. The v_k and w_k denote the zero mean additive noises, which quantitatively represent the model and observation inaccuracies, respectively.

The inference problem can be viewed as recursively estimating the expected value $E[x_k|y_{1:k}]$ of the state of the system, conditioned on the observations. Provided that the initial probability density function (PDF) $p(x_0|y_0) = p(x_0)$ of the state vector is known, the problem consists in recursively estimating $p(x_k|y_{1:k})$ at the time t_k, assuming that the conditional probability density function $p(x_{k-1}|y_{1:k-1})$ is available at the time t_{k-1}. In the literature of the sequential Bayesian estimation, it is customary to estimate the state of a system in two different stages: prediction and update. In the prediction stage the well-known Chapman-Kolmogorov equation provides so called a-priory estimate of the PDF of the state at t_k (Arulampalam et al. 2002):

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1})\, p(x_{k-1}|y_{1:k-1}) dx_{k-1}. \tag{A.3}$$

In the update stage, by taking advantage of the Bayes rule, the PDF of the state is adjusted via including the information conveyed by the observation y_k (Cadini et al. 2009) in the estimation:

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$$p(x_k|y_{1:k}) = \varsigma\, p(y_k|x_k)p(x_k|y_{1:k-1}) \tag{A.4}$$

where ς is a normalizing constant which depends on the likelihood function of the observation process. The Eqs. (A.3) and (A.4) together set the basis for any recursive Bayesian inference scheme. The analytical solution of the integral in the (A.3) is not possible, except for a limited category of systems, namely systems formulated by linear state-space equations and disturbed by white Gaussian noises (Julier and Uhlmann 1997). Provided that the probability density functions of the evolution and the observation equations are Gaussian, they both are represented by the following exponential $e^{(\blacksquare)}$ form:

$$p(x_{k-1}|y_{1:k-1}) = \frac{1}{((2\pi)^n|P_{k-1}|)^{1/2}}\exp\left[-\frac{1}{2}(x_{k-1}-\hat{x}_{k-1})^T P_{k-1}^{-1}(x_{k-1}-\hat{x}_{k-1})\right] \tag{A.5}$$

$$p(x_k|x_{k-1}) = \frac{1}{((2\pi)^n|V|)^{1/2}}\exp\left[-\frac{1}{2}(x_{k-1}-f_k(x_{k-1}))^T V^{-1}(x_{k-1}-f_k(x_{k-1}))\right] \tag{A.5}$$

where, P_{k-1} and V are the covariances of the estimated state and evolution uncertainty, respectively; n is the dimension of the stare vector. Provided that the evolution equation is linear, the integral in the A.3 can be dealt with analytically, like in the Kalman filter (Kalman 1960). If the evolution equation is nonlinear and/or the probability density functions of the state and observation are not Gaussian, only an approximation of the integral in the Eq. A.3 would be available (Doucet 1997).

In case of a general nonlinear problem, one has to make recourse to approximate solutions, either by approximating the nonlinearity via successive linearization of the evolution equation (Corigliano and Mariani 2004) like in the extended Kalman filter, or via discrete approximate representation of the probability density function of the state vector (Mariani and Ghisi 2007; Doucet and Johansen 2009). The first remedy has broadly been applied to weakly nonlinear dynamic systems, and the required conditions for its stability have been extensively investigated (Ljung 1979). However, in some cases it is difficult, or even impossible to linearize the evolution equation. Moreover of severe nonlinearities may prevent the EKF from obtaining proper performance; hence, a category of filters have been developed to numerically handle the integrals in the Eq. A.3 (Kitagawa 1996). The aforementioned methods can be divided into two main categories: filters that are based on a Gaussian approximation of the probability density function of the uncertainties in the state, such as the sigma-point Kalman filter (Julier et al. 2000) and the Gaussian sum filter (Ito and Xiong 2000); filters that assume general form for the probability density function of the uncertainties in the state, such as the particle filter (Ristic et al. 2004) and the Rao-Blackwellized particle filter (Grisetti et al. 2007). Both the aforementioned categories of filters have known problems: the first class fail to provide accurate estimations in case of severely nonlinear and non-Gaussian problems; the second category can enhance the results but also drastically increase computational costs. This has motivated the researchers to develop a synergy of both approaches improving the numerical treatment of the integration in the Eq. (A.3) through an enhancement of the random quadrature procedures used by the particle filters. This notion has led to development of unscented particle filter (Van Der Merwe 2000), Gaussian mixture particle filter (Kotecha and Djuric 2003) and extended Kalman particle filter (de Freitas et al. 2000). From what preceded, one can conclude that the performances of recursive Bayesian filters in terms of computational burden and accuracy of the estimates may vary dealing with different problems. Aforementioned fact substantiates an assessment and comparison of performances once dealing with a specific problem. The Chap. 2 of this book provides and extensive study on the applications of recursive Bayesian filters to a shear-type building.

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