Appendix

1 Generalized Response Relation and Detailed Balance Condition

In this section we will derive the generalized response formula introduced in Sect. 2.2.1.2.

Let us consider a path ω for $s \in [0, t]$, and let us define the probability density

$$\mathcal{P}(\omega) \equiv \frac{\text{Prob}^h[\omega]}{\text{Prob}[\omega]} = e^{-\mathcal{A}}. \tag{1}$$

The action $\mathcal{A}$ can be decomposed in two parts, introducing the time-reversal $\mathcal{I}$ which changes ω into $(\mathcal{I}\omega)_s$ (for instance it changes the sign to the velocities)

$$\mathcal{A} = \frac{1}{2}(\mathcal{T}(\omega) - \mathcal{S}(\omega)), \tag{2}$$

where

$$\begin{aligned} \mathcal{T}(\omega) &= \mathcal{A}(\mathcal{I}\omega) + \mathcal{A}(\omega), \\ \mathcal{S}(\omega) &= \mathcal{A}(\mathcal{I}\omega) - \mathcal{A}(\omega). \end{aligned} \tag{3}$$

Then

$$\mathcal{P}(\omega) = e^{-\mathcal{T}(\omega)/2} e^{\mathcal{S}(\omega)/2}. \tag{4}$$

The anti-symmetric part $\mathcal{S}$ represents the excess in physical entropy due to perturbation, whereas the time symmetric term $\mathcal{T}$ is the excess in the time-integrated instantaneous of a quantity called dynamical activity [1, 2]. Notice that Eq. (3) can be also rewritten as

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$$\mathcal{S} = \log \left(\frac{\text{Prob}[\mathcal{I}\omega]}{\text{Prob}[\omega]} \Big/ \frac{\text{Prob}^h[\mathcal{I}\omega]}{\text{Prob}^h[\omega]} \right),$$
$$\mathcal{T} = \log \frac{\text{Prob}[\mathcal{I}\omega]\text{Prob}[\omega]}{\text{Prob}^h[\mathcal{I}\omega]\text{Prob}^h[\omega]}. \tag{5}$$

For a given observable Q, the response function can be then written

$$R_{QV}(t't') \equiv \frac{\delta\langle Q(t)\rangle_h}{\delta h(t')} = \frac{\delta\langle Q(t)e^{-\mathcal{T}/2+\mathcal{S}/2}\rangle}{\delta h(t')}$$
$$= \frac{1}{2}\left\langle Q(t)\ \frac{\delta\mathcal{S}}{\delta h(t')}\bigg|_{h=0}\right\rangle - \frac{1}{2}\left\langle Q(t)\ \frac{\delta\mathcal{T}}{\delta h(t')}\bigg|_{h=0}\right\rangle. \tag{6}$$

Let us consider a continuous-time homogeneous Markov process with transition rates $W(y|x)$ between states x and y and persistence probability of remaining in the state x for a time t, $P(x;t)$. As perturbed process, we consider the one evolving in presence of the external field $h(t)$ coupled to the observable $V(x)$. The persistence probabilities get changed into $P_h(x;t)$.

For the perturbed transition rates[1] $W_h(y|x)$ we assume the local detailed balance condition

$$W_h(y|x) = W(y|x)e^{\beta/2h(t)[V(y)-V(x)]}. \tag{7}$$

Then let us write explicitly the probability of a trajectory and of its time-reversal

$$\begin{aligned}
\text{Prob}[\omega] &= \rho(x_0)P(x_0;t_0)W(x_1|x_0)P(x_1;t_1)W(x_2|x_1)\dots \\
&\quad \times W(x_n|x_{n-1})P(x_n;t_n) \\
\text{Prob}[\mathcal{I}\omega] &= \rho(x_n)P(x_n;t_n)W(x_{n-1}|x_n)P(x_{n-1};t_{n-1})W(x_{n-2}|x_{n-1})\dots \\
&\quad \times W(x_0|x_1)P(x_0;t_0) \\
\text{Prob}^h[\omega] &= \rho(x_0)P(x_0;t_0;h(t_0))W(x_1|x_0;h(t_0))P(x_1;t_1;h(t_1)) \\
&\quad \times W_h(x_2|x_1)\dots W(x_n|x_{n-1};h(t_{n-1}))P(x_n;t_n;h(t_n)) \\
\text{Prob}^h[\mathcal{I}\omega] &= \rho(x_n)P(x_n;t_n;h(t_n))W(x_{n-1}|x_n;h(t_n)) \\
&\quad \times P(x_{n-1};t_{n-1};h(t_{n-1}))\dots W(x_0|x_1;h(t_1))P(x_0;t_0;h(t_0)),
\end{aligned} \tag{8}$$

where ρ is the distribution of the initial conditions. Now we can give an explicit meaning to the entropy and frenesy excesses of Eq. (5). Let us start by computing the entropic contribution. Enforcing the Local detailed balance condition (7) one obtains

[1] We have assumed the short hand notation $W_h(y|x) \equiv W(y|x;h(t))$ and $P_h(x;t) \equiv P(y|x;h(t))$, where it is not ambiguous.

$$\frac{\text{Prob}^h[\omega]}{\text{Prob}^h[\mathcal{I}\omega]} = \frac{\rho(x_0)}{\rho(x_n)} \exp\left\{\beta \int_0^t ds\ h(s)\dot{V}(s)\right\} \times \frac{W(x_1|x_0)W(x_2|x_1)\ldots W(x_n|x_{n-1})}{W(x_{n-1}|x_n)W(x_{n-2}|x_{n-1})\ldots W(x_0|x_1)} \tag{9}$$

and

$$\frac{\text{Prob}[\omega]}{\text{Prob}[\mathcal{I}\omega]} = \frac{\rho(x_0)}{\rho(x_n)} \frac{W(x_1|x_0)W(x_2|x_1)\ldots W(x_n|x_{n-1})}{W(x_{n-1}|x_n)W(x_{n-2}|x_{n-1})\ldots W(x_0|x_1)}. \tag{10}$$

Hence,

$$S = \log\left(\frac{\text{Prob}^h[\omega]}{\text{Prob}^h[\theta\omega]}\right) - \log\left(\frac{\text{Prob}[\omega]}{\text{Prob}[\theta\omega]}\right) = \beta \int_0^t ds\ h(s)\dot{V}(s). \tag{11}$$

The "frenetic" term can be obtained in the same way:

$$\begin{aligned} \mathcal{T} &= -\log \frac{\text{Prob}^h[\mathcal{I}\omega]\text{Prob}^h[\omega]}{\text{Prob}[\mathcal{I}\omega]\text{Prob}[\omega]} \\ &= -\log \frac{[P(x_0; t_0; h(t_0))\ldots P(x_n; t_n; h(t_n))]^2}{[\ldots P(x_n; t_n)]^2} \\ &= -2\log P(x_0; t_0; h(t_0))P(x_1; t_1; h(t_1))\ldots P(x_n; t_n; h(t_n)) \\ &\quad + 2\log P(x_0; t_0)P(x_1; t_1)\ldots P(x_n; t_n). \end{aligned} \tag{12}$$

Using the definition of persistence probability

$$P(x; t; h(t)) = e^{-\sum_{y\neq x}\int_0^t ds\ W(y|x;h(s))}, \tag{13}$$

together with the local detailed balance condition one obtains

$$\mathcal{T} = 2\int_0^t ds \left[\sum_{y\neq x} W(y|x)\left(e^{\beta/2h(s)[V(y)-V(x)]} - 1\right)\right], \tag{14}$$

and then

$$\left\langle Q(t)\ \frac{\delta\mathcal{T}}{\delta h(t')}\bigg|_{h=0}\right\rangle = \beta\langle Q(t)B(t')\rangle, \tag{15}$$

where

$$B(t) \equiv \sum_{y\neq x} W(y|x)[V(y) - V(x(t))]. \tag{16}$$

In the end, for the response function one finds

$$R_{QV}(t,t') = \frac{\beta}{2}[\langle Q(t)\dot{V}(t')\rangle - \langle Q(t)B(t')\rangle]. \tag{17}$$

2 Entropy Production for a System with Memory

Consider the following simple one-dimensional Langevin equation

$$m\ddot{x} = -\gamma\dot{x} + -h[x] + \eta, \tag{18}$$

where $\eta(t)$ is Gaussian noise of zero mean and correlation

$$\langle\eta(t)\,\eta(t')\rangle = \nu(t-t'), \tag{19}$$

with $\nu(t) = \nu(-t)$. The force term $h[x]$ contains a local in time part, denoted h_x, and a linear memory term,

$$h[x(t)] = h_x[x(t)] - \int_{-\infty}^{t} dt'\, g(t-t')\, x(t'). \tag{20}$$

Both $\nu(t)$ and $g(t)$ are left unspecified.

Since we are interested into the stationary regime, we let the initial time to $-\infty$, and the final one to $+\infty$. Under this assumption the probability of the a trajectory generated by the Langevin equation (18) is

$$\mathcal{P}\{x\} \propto \exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} dt\, dt'\, X[x(t)]\nu^{-1}(t-t')X[x(t')]\right\}, \tag{21}$$

where $\nu^{-1}(t)$ is the inverse of $\nu(t)$ defined as

$$\int_{-\infty}^{+\infty} ds\, \nu(t-s)\, \nu^{-1}(s-t') = \int_{-\infty}^{+\infty} ds\, \nu^{-1}(t-s)\, \nu(s-t') = \delta(t-t'), \tag{22}$$

and

$$X[x(t)] = m\ddot{x}(t) + \gamma\dot{x}(t) + h[x(t)]. \tag{23}$$

By going in the Fourier space,

$$x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, x(\omega) \quad \longleftrightarrow \quad x(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, x(t), \tag{24}$$

the probability (21) becomes

$$\mathcal{P}\{x\} \propto \exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\widetilde{X}[x(\omega)]\nu(\omega)^{-1}\widetilde{X}[x(-\omega)]\right\}, \tag{25}$$

where $\nu^{-1}(\omega) = 1/\nu(\omega)$, with $\nu(-\omega) = \nu(\omega)$, and

$$\widetilde{X}[x(\omega)] = \omega^2 x(\omega) - i\omega x(\omega) + h(\omega) \tag{26}$$

with

$$h(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, h[x(t)]. \tag{27}$$

Consider now the reversed trajectory $x^{\mathrm{R}}(t) = x(-t)$. Its probability follows from (25) by noticing that $x^{\mathrm{R}}(\omega) = x(-\omega)$. To compute the ratio between the probability of a trajectory x and its reversed x^{R} we then have to separate the terms even and odd under the replacement $x(\omega) \to x(-\omega)$ into (25). To this end we have to look closer to $h(\omega)$.

From its definition we have

$$h(\omega) = h_x(\omega) - \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \int_{-\infty}^{t} dt'\, g(t-t')\, x(t'). \tag{28}$$

Now

$$\begin{aligned}\int_{-\infty}^{t} dt'\, g(t-t')\, x(t') &= \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, x(\omega)\int_{-\infty}^{t} dt'\, e^{-i\omega t'}\, g(t-t') \\ &= \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\, x(\omega)\int_{0}^{\infty} dt'\, e^{i\omega t'}\, g(t'),\end{aligned} \tag{29}$$

so that

$$h(\omega) = h_x(\omega) - g(\omega)\, x(\omega), \tag{30}$$

with

$$\begin{aligned}g(\omega) &= \int_0^{\infty} dt\, e^{i\omega t}\, g(t) \\ &= \int_0^{\infty} dt'\, \cos(\omega t)\, g(t) + i\int_0^{\infty} dt\, \sin(\omega t)\, g(t) \\ &= \phi(\omega) + i\omega\, \psi(\omega),\end{aligned} \tag{31}$$

where

$$\phi(\omega) = \int_0^{\infty} dt'\, \cos(\omega t)\, g(t) \tag{32}$$

$$\psi(\omega) = \int_0^\infty dt \, \frac{\sin(\omega t)}{\omega} \, g(t), \tag{33}$$

are real even functions of ω. Collecting all terms we have

$$h(\omega) = h_x(\omega) - \phi(\omega)\, x(\omega) - i\omega\, \psi(\omega)\, x(\omega) \tag{34}$$

and (25) takes the form

$$\begin{aligned}
\mathcal{P}\{x\} &\propto \exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[-i\omega\widetilde{x}(\omega) + \widetilde{h}_x(\omega)\right]\nu(\omega)^{-1}\left[i\omega\widetilde{x}(-\omega) + \widetilde{h}_x(-\omega)\right]\right\} \\
&\propto \exp\Big\{-\frac{1}{2}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[\omega^2\widetilde{x}(\omega)\widetilde{x}(-\omega) + \widetilde{h}_x(\omega)\widetilde{h}_x(-\omega)\right]\nu(\omega)^{-1} \\
&\qquad + \frac{1}{2}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, i\omega \left[\widetilde{x}(\omega)\widetilde{h}_x(-\omega) - \widetilde{x}(-\omega)\widetilde{h}_x(\omega)\right]\nu(\omega)^{-1}\Big\},
\end{aligned} \tag{35}$$

where we have used the short-hand notation

$$\widetilde{x}(\omega) = \gamma x(\omega) + \psi(\omega)\, x(\omega), \qquad \widetilde{h}_x(\omega) = h_x(\omega) - \phi(\omega)\, x(\omega) + m\omega^2 x(\omega). \tag{36}$$

The first integral in the exponential is now even under the replacement $x(\omega) \to x(-\omega)$, while the second is odd. As a consequence the so called *entropy production* reads:

$$\begin{aligned}
\log \frac{\mathcal{P}\{x\}}{\mathcal{P}\{x^{\mathrm{R}}\}} &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, i\omega \left[\widetilde{x}(\omega)\widetilde{h}_x(-\omega) - \widetilde{x}(-\omega)\widetilde{h}_x(\omega)\right]\nu(\omega)^{-1} \\
&= -\int_{-\infty}^{+\infty} \frac{d\omega}{\pi}\, \omega \,\mathrm{Im}\left[\widetilde{x}(\omega)\widetilde{h}_x(-\omega)\right]\nu(\omega)^{-1} \\
&= -\int_{-\infty}^{+\infty} \frac{d\omega}{\pi}\, \omega \,\mathrm{Im}\left[x(\omega)h_x(-\omega)\right][\gamma + \psi(\omega)]\, \nu(\omega)^{-1},
\end{aligned} \tag{37}$$

since the term proportional to $x(\omega)\, x(-\omega)$ is projected out when one takes the imaginary part. Note that if h_x is linear in x then the entropy production vanishes for all trajectories. Taking the average over all trajectories, weighted with (25), one gets the average entropy production

$$\left\langle \log \frac{\mathcal{P}\{x\}}{\mathcal{P}\{x^{\mathrm{R}}\}} \right\rangle = -\int_{-\infty}^{+\infty} \frac{d\omega}{\pi}\, \omega \left\langle \mathrm{Im}\left[x(\omega)h_x(-\omega)\right]\right\rangle [\gamma + \psi(\omega)]\, \nu(\omega)^{-1}. \tag{38}$$

It is easy to realize, that, in the linear case, $h_x(\omega) \propto x(\omega)$ and (37) vanishes, as discussed in Sect. 3.3.2.

In a general non-equilibrium set up, the entropy production grows linearly in time, namely for large observation time T

$$\left\langle \log \frac{\mathcal{P}\{x\}}{\mathcal{P}\{x^{\mathrm{R}}\}} \right\rangle \sim \sigma T \qquad \text{for } T \gg 1, \tag{39}$$

where σ is the entropy production rate. Formally one can define

$$\left\langle \log \frac{\mathcal{P}\{x\}}{\mathcal{P}\{x^{\mathrm{R}}\}} \right\rangle = \int_{-\infty}^{t} \sigma(s) ds. \tag{40}$$

With the definition

$$K(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\gamma + \psi(\omega)]\, \nu(\omega)^{-1} \tag{41}$$

and by exploiting the properties of the Fourier Transform and ignoring sub leading terms, from (38) one arrives to the following identification:

$$\sigma_t = \int_{-\infty}^{t} dt' K(t-t') \big[\dot{x}(t) h_x(t') + \dot{x}(t') h_x(t) \big], \tag{42}$$

which coincides with the one derived in [3].

3 How to Generate Time Translational Invariant Colored Noise

Let us suppose that our purpose is to reproduce equation with memory (as, for instance in Eq. (3.78)). By exploiting the idea described in Sect. 3.3, one can starts from the Markovian problem:

$$\begin{aligned} \dot{x} &= -\alpha x + \lambda y + \sqrt{2D_x}\, \xi_x \\ \dot{y} &= -\gamma y + \mu x + \sqrt{2D_y}\, \xi_y. \end{aligned} \tag{43}$$

However, once integrated the second equation of (43) , starting from time t_0 with initial condition y_0 and substituted it in the first one, the expression for the *effective* noise of the variable x is

$$\eta(t) = \lambda y_0 g(t-t_0) + \lambda \sqrt{2D_y} \int_{t_0}^{t} ds g(t-s) \phi_y(s) + \sqrt{2D_x} \phi_x(t). \tag{44}$$

At a first sight it seems that Eq. (44) does not satisfy a time translational condition, namely $\langle \eta(t)\eta(t') \rangle \neq f(t-t')$. In order to show this, let us write down the probability distribution of the noise:

$$P[\eta|y_0] = \int \mathcal{D}\sigma\delta\left[\eta - \lambda y_0 g(t-t_0) - \lambda\sqrt{2D_y}\int_{t_0}^{t} ds g(t-s)\phi_y(s) - \sqrt{2D_x}\phi_x(t)\right]. \tag{45}$$

By introducing the *hat* variable $\hat{\eta}(t)$ and by exploiting the integral representation of the delta function, one has

$$P[\eta|y_0] = \int \mathcal{D}\hat{\eta}\mathcal{D}\phi_x\mathcal{D}\phi_y A_\eta B_{\phi_x} C_{\phi_y}, \tag{46}$$

where we have introduced the following notations:

$$A_\eta = \exp\left\{\int_{t_0}^{t_1} dt\, i\hat{\eta}(t)[\eta(t) - \lambda y_0 g(t-t_0)]\right\} \tag{47}$$

$$B_{\phi_x} = \exp\left\{-i\sqrt{2D_x}\int_{t_0}^{t_1} dt\hat{\eta}(t)\phi_x\right\} \tag{48}$$

$$C_{\phi_y} = \exp\left\{-i\lambda\sqrt{2D_y}\int_{t_0}^{t_1} dt\int_{t_0}^{t_1} dt'\hat{\eta}(t')g(t'-t)\phi_y(t)\right\}. \tag{49}$$

Now we use the identity $\langle e^{\lambda x}\rangle = e^{\frac{1}{2}\lambda^2\langle x^2\rangle}$, which is valid for Gaussian integrals, obtaining

$$\int \mathcal{D}\phi_x P[\phi_x]B_{\phi_x} = \exp\left\{-D_x\int_{t_0}^{t_1}\hat{\eta}^2(t)\right\} \tag{50}$$

$$\int \mathcal{D}\phi_y P[\phi_y]C_{\phi_y} = \exp\left\{-\lambda^2 D_y\int_{t_0}^{t_1} dt\int_{t_0}^{t_1} dt'\hat{\eta}(t)\Delta(t,t')\hat{\eta}(t')\right\}, \tag{51}$$

where we have introduced

$$\Delta(t,t') \equiv \int_{t_0}^{t_1} dt' g(t-t'')g(t'-t'') \tag{52}$$

$$= \frac{1}{2\gamma}\left[e^{-\gamma|t-t'|} - e^{-\gamma|t+t'-2t_0|}\right]. \tag{53}$$

Finally, by integrating over the $\hat{\eta}$ the following *Onsager-Machlup* probability distribution is obtained

$$P[\eta|y_0] = \exp\left\{-\frac{1}{2}\int_{t_0}^{t_1} dt\int_{t_0}^{t_1} dt' F[\eta(t)]\nu(t,t')F[\eta(t')]\right\} \tag{54}$$

with

$$F[\eta(t)] \equiv \eta(t) - \lambda y_0 g(t-t_0) \tag{55}$$

$$\nu^{-1}(t,t') = 2D_x\delta(t-t') + \frac{\lambda^2 D_y}{\gamma}\left[e^{-\gamma|t-t'|} - e^{-\gamma(t+t'-2t_0)}\right]. \tag{56}$$

As expected, expression (56) is not of the requested form: the autocorrelation is not time translational invariant and dependence of the initial condition is explicit. However, one can choose the initial condition y_0 randomly with distribution P_0. Then, the final expression for the distribution of the noise is obtained by integrating over the initial condition:

$$P[\eta] = \int dy_0 P_0(y_0) P[\eta|y_0]. \tag{57}$$

By choosing P_0 of the Gaussian form with zero mean and variance $\sigma^2 = \frac{D_y}{\gamma}$, after some calculations, one obtains a colored gaussian process whose correlation is time translational invariant, namely

$$P[\eta] = \exp\left\{-\frac{1}{2}\int_{t_0}^{t_1} dt \int_{t_0}^{t_1} dt' \eta(t)\nu(t,t')\eta(t')\right\} \tag{58}$$

with

$$\nu^{-1}(t,t') = 2D_x\delta(t-t') + \frac{\lambda^2 D_y}{\gamma} e^{-\gamma|t-t'|}. \tag{59}$$

In conclusion, from this example we learn the correct procedure to reproduce a colored noise with correlation (59) by using an auxiliary variable. In order to obtain it, it is necessary to choose the initial condition y_0 from a specific random distribution.

4 Calculation of First Two Coefficients of the Kramers-Moyal Expansion

For larger generality (whose motivation is discussed in the Conclusions), in this Appendix we discuss the case where the gas surrounding the intruder may have a non-zero average $\mathbf{u}$[2]:

$$p(\mathbf{v}) = \frac{1}{\sqrt{(2\pi T_g/m)^d}} \exp\left[-\frac{m(\mathbf{v}-\mathbf{u})^2}{2T_g}\right], \tag{60}$$

which is a simple task involving only the definition of new shifted variables

$$\mathbf{c} = \mathbf{V} - \mathbf{u} \tag{61}$$

[2] Note that in all the cases discussed in the main text, we have always taken $\mathbf{u} = 0$.

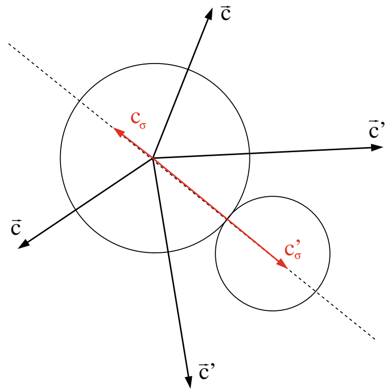

Fig. 1 An example for the change of variables $(c'_x, c'_y) \to (c_\sigma, c'_\sigma)$, introduced in Eq. (64). Such change of variable, when inverted, has two possible determinations: in this example both represented vectors $\mathbf{c}'$ yield the same (c_σ, c'_σ)

$$\mathbf{c}' = \mathbf{V}' - \mathbf{u}. \tag{62}$$

We are interested in computing

$$\begin{aligned} D_i^{(1)}(\mathbf{V}) &= \int d\mathbf{V}'(V'_i - V_i)W_{tr}(\mathbf{V}'|\mathbf{V}) \\ &= \int d\mathbf{c}'(c'_i - c_i)\chi \frac{1}{\sqrt{2\pi T_g/m}k(\epsilon)^2} \\ &\quad \times \exp\left\{-m\left[c'_\sigma + (k(\epsilon)-1)c_\sigma\right]^2/(2T_g k(\epsilon)^2)\right\}. \end{aligned} \tag{63}$$

In order to perform the integral, we make the following change of variables (see Fig. 1 for an example)

$$\begin{aligned} c_\sigma &= c_x \frac{c'_x - c_x}{\sqrt{(c'_x - c_x)^2 + (c'_y - c_y)^2}} + c_y \frac{c'_y - c_y}{\sqrt{(c'_x - c_x)^2 + (c'_y - c_y)^2}} \\ c'_\sigma &= c'_x \frac{c'_x - c_x}{\sqrt{(c'_x - c_x)^2 + (c'_y - c_y)^2}} + c'_y \frac{c'_y - c_y}{\sqrt{(c'_x - c_x)^2 + (c'_y - c_y)^2}} \end{aligned} \tag{64}$$

which implies

$$d\mathbf{c}' = dc'_x dc'_y \to dc_\sigma dc'_\sigma |J|, \tag{65}$$

where

$$|J| = \frac{|c'_\sigma - c_\sigma|}{\sqrt{c_x^2 + c_y^2 - c_\sigma^2}}\Theta(c_x^2 + c_y^2 - c_\sigma^2) \tag{66}$$

is the Jacobian of the transformation. The collision rate is then

$$r(\mathbf{V}) = \chi\sqrt{\frac{\pi}{2T_g/m}}e^{-\frac{mc^2}{4T_g}}\left[(c^2+2T_g/m)I_0\left(\frac{mc^2}{4T_g}\right)+c^2 I_1\left(\frac{mc^2}{4T_g}\right)\right], \tag{67}$$

where $I_n(x)$ are the modified Bessel functions. For $D_i^{(1)}$ we can write

$$\begin{aligned} D_i^{(1)}(\mathbf{V}) &= \chi\int_{-\infty}^{+\infty} dc_\sigma \int_{c_\sigma}^{\infty} dc'_\sigma (c'_i - c_i)|J|\frac{1}{\sqrt{2\pi T_g/m}k(\epsilon)^2} \\ &\quad\times \exp\left\{-m\left[c'_\sigma + (k(\epsilon)-1)c_\sigma\right]^2/(2T_g k(\epsilon)^2)\right\} \\ &= \chi\int_{-c}^{+c} dc_\sigma \int_{c_\sigma}^{\infty} dc'_\sigma (c'_i - c_i)\frac{c'_\sigma - c_\sigma}{\sqrt{c^2-c_\sigma^2}} \\ &\quad\times \frac{1}{\sqrt{2\pi T_g/m}k(\epsilon)^2}\exp\left\{-m\left[c'_\sigma + (k(\epsilon)-1)c_\sigma\right]^2/(2T_g k(\epsilon)^2)\right\}, \end{aligned} \tag{68}$$

where we have enforced the constraint of the theta function, namely $c_\sigma \in (-c, +c)$, with $c = \sqrt{c_x^2 + c_y^2}$. Notice that the integral in dc'_σ is lower bounded by the condition $c'_\sigma \geq c_\sigma$ which follows from the definition of c_σ. In order to compute the integral, we have to invert the transformation (64). That yields two determinations for the variables c'_x and c'_y (see Fig. 1)

$$(A)\begin{cases} c'_x - c_x = \frac{c'_\sigma - c_\sigma}{c^2}\left(c_\sigma c_x + c_y Sign(c_x)\sqrt{c^2-c_\sigma^2}\right) \\ c'_y - c_y = \frac{c'_\sigma - c_\sigma}{c^2}\left(c_\sigma c_y - c_x Sign(c_x)\sqrt{c^2-c_\sigma^2}\right) \end{cases}$$

$$(B)\begin{cases} c'_x - c_x = \frac{c'_\sigma - c_\sigma}{c^2}\left(c_\sigma c_x - c_y Sign(c_x)\sqrt{c^2-c_\sigma^2}\right) \\ c'_y - c_y = \frac{c'_\sigma - c_\sigma}{c^2}\left(c_\sigma c_y + c_x Sign(c_x)\sqrt{c^2-c_\sigma^2}\right) \end{cases}$$

Then the integral (68) can be written as

$$\begin{aligned} D_x^{(1)}(\mathbf{V}) &= \frac{1}{l_0}\int_{-c}^{c} dc_\sigma \int_{c_\sigma}^{\infty} dc'_\sigma \left[(c'_x - c_x)^{(A)} + (c'_x - c_x)^{(B)}\right]|J| \\ &\quad\times \frac{1}{\sqrt{2\pi T_g/m}k(\epsilon)^2}\exp\left\{-m\left[c'_\sigma + (k(\epsilon)-1)c_\sigma\right]^2/(2T_g k(\epsilon)^2)\right\}, \end{aligned} \tag{69}$$

yielding

$$D_x^{(1)} = -\frac{2}{3}\frac{1}{l_0}k(\epsilon)\sqrt{\frac{m\pi}{2T_g}}c_x e^{-\frac{mc^2}{4T_g}}\left[(c^2+3T_g/m)I_0\left(\frac{mc^2}{4T_g}\right)+(c^2+T_g/m)I_1\left(\frac{mc^2}{4T_g}\right)\right],$$
$$D_y^{(1)} = -\frac{2}{3}\frac{1}{l_0}k(\epsilon)\sqrt{\frac{m\pi}{2T_g}}c_y e^{-\frac{mc^2}{4T_g}}\left[(c^2+3T_g/m)I_0\left(\frac{mc^2}{4T_g}\right)+(c^2+T_g/m)I_1\left(\frac{mc^2}{4T_g}\right)\right]. \tag{70}$$

Analogously, for the coefficients $D_{ij}^{(2)}$ one obtains

$$\begin{aligned} D_{xx}^{(2)}(\mathbf{V}) &= \frac{1}{2}\frac{1}{l_0}\int_{-c}^{c}dc_\sigma\int_{c_\sigma}^{\infty}dc'_\sigma\left[\left((c'_x-c_x)^{(A)}\right)^2+\left((c'_x-c_x)^{(B)}\right)^2\right]|J| \\ &\quad\times\frac{1}{\sqrt{2\pi T_g/m}k(\epsilon)^2}\exp\left\{-m\left[c'_\sigma+(k(\epsilon)-1)c_\sigma\right]^2/(2T_gk(\epsilon)^2)\right\} \\ &= \frac{1}{2}\frac{1}{l_0}\frac{k(\epsilon)^2}{15}\sqrt{\frac{2m\pi}{T_g}}e^{-\frac{mc^2}{4T_g}} \\ &\quad\times\left\{\left[c^2(4c_x^2+c_y^2)+3T_g(7c_x^2+3c_y^2)/m+15T_g^2/m^2\right]I_0\left(\frac{mc^2}{4T_g}\right)\right. \\ &\quad\left.+\left[c^2(4c_x^2+c_y^2)+T_g(13c_x^2+7c_y^2)/m+3T_g^2/m^2\frac{-c_x^2+c_y^2}{c^2}\right]I_1\left(\frac{mc^2}{4T_g}\right)\right\}, \end{aligned} \tag{71}$$

$$\begin{aligned} D_{xy}^{(2)}(\mathbf{V}) &= \frac{1}{2}\frac{1}{l_0}\int_{-c}^{c}dc_\sigma\int_{c_\sigma}^{\infty}dc'_\sigma\left[(c'_x-c_x)^{(A)}(c'_y-c_y)^{(A)}+(c'_x-c_x)^{(B)}(c'_y-c_y)^{(B)}\right]|J| \\ &\quad\times\frac{1}{\sqrt{2\pi T_g/m}k(\epsilon)^2}\exp\left\{-m\left[c'_\sigma+(k(\epsilon)-1)c_\sigma\right]^2/(2T_gk(\epsilon)^2)\right\} \\ &= \frac{1}{2}\frac{1}{l_0}\frac{k(\epsilon)^2}{5}\sqrt{\frac{2m\pi}{T_g}}e^{-\frac{mc^2}{4T_g}}c_xc_y \\ &\quad\times\left[(c^2+4T_g/m)I_0\left(\frac{mc^2}{4T_g}\right)+\frac{c^4+2c^2T_g/m-2T_g^2/m^2}{c^2}I_1\left(\frac{mc^2}{4T_g}\right)\right]. \end{aligned} \tag{72}$$

Then we introduce the rescaled variables

$$q_x = \frac{c_x}{\sqrt{T_g/m}}\epsilon^{-1} \qquad q_y = \frac{c_y}{\sqrt{T_g/m}}\epsilon^{-1}, \tag{73}$$

obtaining

$$
\begin{aligned}
D_x^{(1)}(\mathbf{V}) = & -\frac{2}{3}\frac{1}{l_0}\sqrt{\frac{\pi}{2}}\frac{T_g}{m} q_x k(\epsilon)\epsilon e^{-\frac{\epsilon^2 q^2}{4}} \\
& \times\left[\left(\epsilon^2 q^2+3\right) I_0\left(\frac{\epsilon^2 q^2}{4}\right)+\left(\epsilon^2 q^2+1\right) I_1\left(\frac{\epsilon^2 q^2}{4}\right)\right], \\
D_{xx}^{(2)}(\mathbf{V}) = & \frac{1}{2}\frac{1}{l_0}\frac{1}{15}\sqrt{2\pi}\left(\frac{T_g}{m}\right)^{3/2} k(\epsilon)^2 e^{-\frac{\epsilon^2 q^2}{4}} \\
& \times\left\{\left[\epsilon^4 q^2(4q_x^2+q_y^2)+3\epsilon^2(7q_x^2+3q_y^2)+15\right] I_0\left(\frac{\epsilon^2 q^2}{4}\right)\right. \\
& \left.+\left[\epsilon^4 q^2(4q_x^2+q_y^2)+\epsilon^2(13q_x^2+7q_y^2)+3\frac{-q_x^2+q_y^2}{q^2}\right] I_1\left(\frac{\epsilon^2 q^2}{4}\right)\right\} \\
D_{xy}^{(2)}(\mathbf{V}) = & \frac{1}{2}\frac{1}{l_0}\frac{1}{5}\sqrt{2\pi}\left(\frac{T_g}{m}\right)^{3/2} q_x q_y k(\epsilon)^2 \epsilon^2 e^{-\frac{\epsilon^2 q^2}{4}} \\
& \times\left[\left(\epsilon^2 q^2+4\right) I_0\left(\frac{\epsilon^2 q^2}{4}\right)+\left(\frac{\epsilon^4 q^4+2\epsilon^2 q^2-2}{\epsilon^2 q^2}\right) I_1\left(\frac{\epsilon^2 q^2}{4}\right)\right].
\end{aligned}
\tag{74}
$$

Up to this last results we have not introduced any small ϵ approximation. The next step consists in assuming that $q \sim \mathcal{O}(1)$ with respect to ϵ, which is equivalent to assume that $c^2 \sim T_g/M$: this assumption must be compared to its consequences, in particular to Eq. (4.27); the assumption is good for not too small values of α and for $\gamma_g \gg \gamma_b$, i.e. when $T_{tr} \sim T_g$. When this is the case, expanding in ϵ and using that $I_0(x) \sim 1 + x^2/4$ and $I_1(x) \sim x/2$ for small x, one finds Eq. (4.22).

References

1. Baiesi, M., Boksenbojm, E., Maes, C., Wynants, B.: Nonequilibrium linear response for markov dynamics. II: Inertial dynamics. J. Stat. Phys. **139**, 492 (2010)
2. Baiesi, M., Maes, C., Wynants, B.: Nonequilibrium linear response for markov dynamics. I: Jump processes and overdamped diffusions. J. Stat. Phys. **137**, 1094 (2009)
3. Zamponi, F., Bonetto, F., Cugliandolo, L.F., Kurchan, J.: A fluctuation theorem for non-equilibrium relaxational systems driven by external forces. J. Stat. Mech. P09013 (2005)

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