Appendix
The Subgroups of the Binary Polyhedral Groups

In this Appendix, we study the structure of the subgroups of the binary polyhedral groups T^*, O^*, I^* that we refer to in the main body of the manuscript. More information on these groups may be found in [1–5].

Proposition 85

(a) *The proper subgroups of the binary tetrahedral group T^* are $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathcal{Q}_8$. Its maximal subgroups are isomorphic to $\mathbb{Z}_6$ or $\mathcal{Q}_8$, its maximal cyclic subgroups are isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_6$, and its non-trivial normal subgroups are isomorphic to $\mathbb{Z}_2$ or $\mathcal{Q}_8$.*

(b) *The proper subgroups of the binary octahedral group O^* are isomorphic to $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathcal{Q}_8$, Dic_{12}, $\mathcal{Q}_{16}$ or T^*. Its maximal subgroups are isomorphic to Dic_{12}, $\mathcal{Q}_{16}$ or T^*, its maximal cyclic subgroups are isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_6$ or $\mathbb{Z}_8$, and its non-trivial normal subgroups are isomorphic to $\mathbb{Z}_2$, $\mathcal{Q}_8$ or T^*.*

(c) *The proper subgroups of the binary icosahedral group I^* are isomorphic to $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathcal{Q}_8$, $\mathbb{Z}_{10}$, Dic_{12}, Dic_{20} or T^*, its maximal subgroups are isomorphic to Dic_{12}, Dic_{20} or T^*, its maximal cyclic subgroups are isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_6$ or $\mathbb{Z}_{10}$, and it has a unique non-trivial normal subgroup, isomorphic to $\mathbb{Z}_2$.*

Proof Recall first that if G is a binary polyhedral group, it is periodic [1] and has a unique element of order 2 that generates $Z(G)$. By periodicity, the group G satisfies the p^2-condition (if p is prime and divides the order of G then G has no subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$), which implies that every Sylow p-subgroup of G is cyclic or generalised quaternion, as well as the $2p$-condition (every subgroup of order $2p$ is cyclic).

(a) Consider first the binary tetrahedral group T^*. It is isomorphic to $\mathcal{Q}_8 \rtimes \mathbb{Z}_3$. Using the presentation given by Eq. (1.10), one may check that $T^* \setminus \mathcal{Q}_8$ consists of the eight elements of

$$\left\{ S^{-j} X^j \;\middle|\; j \in \{-1, 1\} \text{ and } S \in \{1, P, Q, PQ\} \right\},$$

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and of the eight elements of order 6 that are obtained from those of order 3 by multiplication by the unique (central) element P^2 of order 2. The proper non-trivial subgroups of T^* are isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and $\mathcal{Q}_8$. The fact that T^* has a unique element of order 2 rules out the existence of subgroups isomorphic to S_3. Since $\mathcal{Q}_8$ is a Sylow 2-subgroup of T^*, $\mathbb{Z}_8$ cannot be a subgroup of T^*. Further, since $T^*/Z(T^*) \cong A_4$, the quotient by $Z(T^*)$ of any order 12 subgroup of T^* would be a subgroup of A_4 of order 6, which is impossible. Also, any copy of $\mathbb{Z}_3$ (resp. $\mathbb{Z}_4$) is contained in a copy of $\mathbb{Z}_6$ (resp. $\mathcal{Q}_8$). The maximal subgroups of T^* are thus isomorphic to $\mathbb{Z}_6$ or $\mathcal{Q}_8$, and its maximal cyclic subgroups are isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_6$. Among these possible subgroups, it is straightforward to check that the normal non-trivial subgroups are those isomorphic to $\mathbb{Z}_2$ or $\mathcal{Q}_8$.

(b) Consider the binary octahedral group O^*, with presentation given by Eq. (2.14). Recall from Lemma 74(b) that $\langle P, Q, X\rangle$ is the unique subgroup of O^* isomorphic to T^*. The twenty-four elements of $O^* \setminus T^*$ are comprised of twelve elements of order 4 and twelve of order 8. Under the canonical projection onto $O^*/Z(O^*) \cong S_4$, these elements are sent to the six transpositions and the six 4-cycles of S_4 respectively. The squares of the elements of order 8 are the elements of T^* of order 4. Consequently, the elements of $O^* \setminus T^*$ of order 4 generate maximal cyclic subgroups. Thus O^* has three subgroups isomorphic to $\mathbb{Z}_8$. The Sylow 2-subgroups are copies of $\mathcal{Q}_{16}$, and since each copy of $\mathbb{Z}_8$ is contained in a copy of $\mathcal{Q}_{16}$ and each copy of $\mathcal{Q}_{16}$ contains a unique copy of $\mathbb{Z}_8$, it follows from Sylow's Theorems that O^* possesses exactly three (maximal and non-normal) copies of $\mathcal{Q}_{16}$, and that the subgroups of O^* of order 8 are isomorphic to $\mathbb{Z}_8$ or $\mathcal{Q}_8$.

It remains to determine the subgroups of order 12. Under the projection onto the quotient $O^*/Z(O^*)$, such a subgroup would be sent to a subgroup of S_4 of order 6, so is the inverse image under this projection of a copy of S_3, isomorphic to Dic_{12}. It is not normal because the subgroups of S_4 isomorphic to S_3 are not normal. Further it cannot be a subgroup of $\langle P, Q, X\rangle$ since projection onto $O^*/Z(O^*)$ would imply that the image of $\langle P, Q, X\rangle$, which is isomorphic to A_4, would have a subgroup of order 6, which is impossible. We thus obtain the isomorphism classes of the subgroups of O^* given in the statement, as well as the isomorphism classes of the maximal and maximal cyclic subgroups.

We now determine the normal subgroups of O^*. As we already mentioned, the subgroups of O^* isomorphic to Dic_{12} or $\mathcal{Q}_{16}$ are not normal, and the fact that each of the three cyclic subgroups of order 8 is contained in a single copy of $\mathcal{Q}_{16}$ implies that these subgroups are not normal in O^*. Clearly $Z(O^*) \cong \mathbb{Z}_2$ and $\langle P, Q, X\rangle \cong T^*$ are normal in O^*. Since T^* is normal in O^* and possesses a unique copy $\langle P, Q\rangle$ of $\mathcal{Q}_8$, this copy of $\mathcal{Q}_8$ is normal in O^*. The subgroups isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_6$ are not normal because they are contained in $\langle P, Q, X\rangle$ and are not normal there. The same is true for the subgroups isomorphic to $\mathbb{Z}_4$ and lying in $\langle P, Q, X\rangle$. Finally, under the canonical projection onto $O^*/Z(O^*)$, any subgroup of order 4 generated by an element of $O^* \setminus T^*$ is sent to a subgroup of S_4 generated by a transposition, so cannot be normal in O^*. This yields the list of isomorphism classes of normal subgroups of O^*.

(c) Finally, consider the binary icosahedral group I^* of order 120. It is well known that I^* admits the presentation $\langle S, T \mid (ST)^2 = S^3 = T^5 \rangle$, is isomorphic to the group $\mathrm{SL}(2, \mathbb{F}_5)$, and $I^*/Z(I^*) \cong A_5$. The group I^* has thirty elements of order 4 (which project to the fifteen elements of A_5 of order 2), twenty elements each of order 3 and 6 (which project to the twenty 3-cycles of A_5), and twenty-four elements each of order 5 and 10 (which project to the twenty-four 5-cycles of A_5). Its proper subgroups of order less than or equal to 10 are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathcal{Q}_8$ and $\mathbb{Z}_{10}$. The only difficulty here is the case of order 8 subgroups: I^* has no element of order 8 since under the projection onto $I^*/Z(I^*)$, such an element would project onto an element of A_5 of order 4, which is not possible. Since I^* possesses a unique element of order 2, the Sylow 2-subgroups of I^*, which are of order 8, are isomorphic to $\mathcal{Q}_8$. Any subgroup of order 15 or 30 (resp. 60) would project to a subgroup of A_5 of order 15 (resp. 30), which is not possible either. Note that I^* has no element of order 12 (resp. 20) since such an element would project to one of order 6 (resp. 10) in A_5. Since I^* has a unique element of order 2, any subgroup of order 12 (resp. 20) must thus be isomorphic to Dic_{12} (resp. Dic_{20}) using the classification of the groups of these orders up to isomorphism. Such a subgroup exists by taking the inverse image of the projection of any subgroup of A_5 isomorphic to Dih_6 (resp. Dih_{10}).

Any subgroup of I^* of order 24 projects to a subgroup of A_5 of order 12, which must be a copy of A_4. Hence any subgroup of A_5 of order 12, which is isomorphic to A_4, lifts to a subgroup of I^* isomorphic to T^*. So any subgroup of I^* of order 24 is isomorphic to T^*, and such a subgroup exists.

Let G be a subgroup of I^* of order 40, and let G' be its projection in $I^*/Z(I^*)$. Then G' is of order 20, and the Sylow 5-subgroup K of G' is normal. Now G' has no element of order 4 since I^* has no element of order 8, so $G'/K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We thus have a short exact sequence:

$$1 \to \mathbb{Z}_5 \to G' \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1$$

which splits since the kernel and the quotient have coprime orders [6, Theorem 10.5]. Since $\mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$, the action of any non-trivial element of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on $\mathbb{Z}_5$ must be multiplication by -1 (it could not be the identity, for otherwise A_5 would have an element of order 10, which is impossible), but this is not compatible with the structure of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence I^* has no subgroup of order 40. We thus obtain the list of subgroups of I^* given in the statement. The cyclic subgroups of order 3 and 5 of I^* are contained in the cyclic subgroups of order 6 and 10 respectively obtained by multiplying a generator by the central element of order 2. Thus the maximal cyclic subgroups of I^* are isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_6$ or $\mathbb{Z}_{10}$.

We now consider the maximal subgroups. Clearly, any subgroup of I^* isomorphic to Dic_{12} or T^* is maximal. Further, since T^* has no subgroup of order 12, any subgroup of I^* isomorphic to Dic_{12} is also maximal. The subgroups of I^* isomorphic to $\mathcal{Q}_8$ are its Sylow 2-subgroups, so are conjugate, and since one of these subgroups is contained in a copy of T^*, the same is true for any

such subgroup. Thus the subgroups of I^* isomorphic to $\mathcal{Q}_8$ are non maximal. Replacing $\mathcal{Q}_8$ by $\mathbb{Z}_3$ (resp. $\mathcal{Q}_8$ by $\mathbb{Z}_5$ and T^* by Dic_{20}) and applying a similar argument shows that the subgroups of I^* isomorphic to $\mathbb{Z}_6$ (resp. $\mathbb{Z}_{10}$) are also non maximal. This yields the list of the isomorphism classes of the maximal subgroups of I^* given in the statement. Finally, since A_5 is simple, the only non-trivial normal subgroup of I^* is its unique subgroup of order 2. $\square$

References

1. Adem, A., Milgram, R.J.: Cohomology of finite groups. Springer-Verlag, New York-Heidelberg-Berlin (1994)
2. Coxeter, H.S.M.: Regular complex polytopes, 2nd edn. Cambridge University Press, Cambridge (1991)
3. Coxeter, H.S.M., Moser, W.O.J.: Generators and relations for discrete groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 14. Fourth edition, Springer-Verlag, Berlin (1980)
4. Thomas, C.B.: Elliptic structures on 3-manifolds. University of Chicago Department of Mathematics, Chicago, IL, Lecture Notes in Mathematics (1983)
5. J. A. Wolf, Spaces of constant curvature, sixth edition, AMS Chelsea Publishing, vol. 372, 2011
6. Mac Lane, S.: Homology, Classics in Mathematics, Reprint of the 1975 edition. Springer-Verlag, Berlin (1995)