Appendices

A Discrete choice models

This short appendix reviews the basic facts about discrete choice theory used in previous chapters. For a more thorough treatment on this theory and its many applications in economics, we refer to Ben-Akiva & Lerman [16] and Train [75].

Consider an agent who faces a choice among a finite number of alternatives $i = 1, \dots, n$, each one incurring a random cost $\tilde{x}_i = x_i + \epsilon_i$. Here x_i is the expected cost of of the ith alternative and the random term ϵ_i satisfies $\mathbb{E}(\epsilon_i) = 0$. Suppose that the agent observes the random variables $\tilde{x}_i$ and then chooses the alternative that yields the minimal cost: $\tilde{x}_i \leq \tilde{x}_j$ for $j = 1, \dots, n$. The expected value of the minimal cost defines a map $\varphi \colon \mathbb{R}^n \to \mathbb{R}$,

$$\varphi(x) = \mathbb{E}\big(\min \{x_1 + \epsilon_1, \dots, x_n + \epsilon_n\}\big). \tag{A.1}$$

Denote by $\mathcal{E}$ the class of maps that can be expressed in this form, where $\epsilon = (\epsilon_i)_{i=1}^n$ is a random vector with continuous distribution and $\mathbb{E}(\epsilon) = 0$. The next proposition summarizes the basic properties of the expected utility function [26, 28, 71, 78], and characterizes the choice probability of each alternative as the derivatives of φ.

Proposition A.1. *Every function $\varphi \in \mathcal{E}$ is concave and of class C^1 with $\varphi(x) \leq \min\{x_1, \dots, x_n\}$, and we have*

$$\frac{\partial \varphi}{\partial x_i}(x) = \mathbb{P}\big(\tilde{x}_i \leq \tilde{x}_j, \, \forall j = 1, \dots, n\big). \tag{A.2}$$

Proof. Let us denote $m(x) = \min\{x_1, \dots, x_n\}$. The inequality $\varphi(x) \leq m(x)$ follows at once by taking expectation in the inequality $\min_{i=1\dots n}\{x_i + \epsilon_i\} \leq x_j + \epsilon_j$. Let $F(\epsilon)$ be the joint distribution of $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ so that

$$\varphi(x) = \int_{\mathbb{R}^n} m(x + \epsilon) \, dF(\epsilon).$$

Since m is concave, the same holds for φ. To compute $\frac{\partial \varphi}{\partial x_i}$ we consider the differential quotient

$$\frac{\varphi(x + te_i) - \varphi(x)}{t} = \int_{\mathbb{R}^n} q_t(\epsilon) \, dF(\epsilon),$$

where $q_t(\epsilon) = [m(x+\epsilon+te_i) - m(x+\epsilon)]/t$. Denoting

$$A = \{\epsilon \in \mathbb{R}^n \,:\, x_i + \epsilon_i < x_j + \epsilon_j,\, \forall j \neq i\},$$
$$B = \{\epsilon \in \mathbb{R}^n \,:\, x_i + \epsilon_i \leq x_j + \epsilon_j,\, \forall j \neq i\},$$

it follows that $\lim_{t\downarrow 0^+} q_t(\epsilon) = 1_A(\epsilon)$ and $\lim_{t\uparrow 0^-} q_t(\epsilon) = 1_B(\epsilon)$. Since the convergence is monotone, we may use Lebesgue's theorem to deduce

$$\begin{aligned} D_i^+\varphi(x) &= \int_{\mathbb{R}^n} 1_A(\epsilon)\, dF(\epsilon) = \mathbb{P}(A), \\ D_i^-\varphi(x) &= \int_{\mathbb{R}^n} 1_B(\epsilon)\, dF(\epsilon) = \mathbb{P}(B), \end{aligned} \tag{A.3}$$

and since F is non-atomic we get $\mathbb{P}(A) = \mathbb{P}(B)$, so that the partial derivative $\frac{\partial\varphi}{\partial x_i}$ exists and satisfies (A.2). The C^1 character then follows since φ is concave. □

Example. The *Logit* model assumes that the ϵ_i's are independent Gumbel variables with parameter β, which gives the expected utility function

$$\varphi(x) = -\frac{1}{\beta} \ln\left(e^{-\beta x_1} + \cdots + e^{-\beta x_n}\right)$$

and the corresponding choice probabilities

$$\mathbb{P}\big(\tilde{x}_i \leq \tilde{x}_j,\, \forall j = 1,\ldots,n\big) = \frac{\exp(-\beta x_i)}{\sum_{j=1}^n \exp(-\beta x_j)}.$$

In the *Probit* model with normally distributed ϵ_i's there is no simple analytical expression for $\varphi(x)$ nor the choice probabilities.

We note that in the specification of the SUE and MTE models, all the relevant information was encapsulated in the functions φ_i^d, which are precisely of the form (A.1). Thus, we could take these functions as the primary modeling objects, without expliciting the random distributions that produced them. To this end, it is useful to have an analytic characterization of the class $\mathcal{E}$. The next result from [65] provides a complete characterization of this class.

Proposition A.2. *A function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is in $\mathcal{E}$ if and only if the following hold:*

(a) *φ is C^1 and componentwise non-decreasing;*

(b) *$\varphi(x_1+c,\ldots,x_n+c) = \varphi(x_1,\ldots,x_n) + c$;*

(c) *$\varphi(x) \to x_i$ when $x_j \to \infty$ for all $j \neq i$;*

(d) *for x_i fixed, the mapping $\frac{\partial\varphi}{\partial x_i}(x_1,\ldots,x_n)$ is a distribution with continuous density on the remaining variables.*

Proof. For $\varphi \in \mathcal{E}$, the properties (a)–(d) are direct consequences of (A.1) and (A.2) (for a proof of (a) and (b) the reader may also refer to [26, 28, 78]). To establish the converse, let us consider a random vector $\eta = (\eta_2, \ldots, \eta_n)$ with distribution function $F_\eta(x_2, \ldots, x_n) = \frac{\partial \varphi}{\partial x_1}(0, x_2, \ldots, x_n)$. We begin by noting that property (b) implies

$$\varphi(x) = x_1 - \int_a^{x_1} \left[1 - \frac{\partial \varphi}{\partial x_1}(y, x_2, \ldots, x_n)\right] dy + \varphi\big(0, x_2 - a, \ldots, x_n - a\big),$$

so that letting $a \to -\infty$ and using (c) we get

$$\varphi(x) = x_1 - \int_{-\infty}^{x_1} \left[1 - \frac{\partial \varphi}{\partial x_1}(y, x_2, \ldots, x_n)\right] dy.$$

On the other hand, setting $Y = \min\{x_2 - \eta_2, \ldots, x_n - \eta_n\}$ and using (b) we get

$$\begin{aligned}\frac{\partial \varphi}{\partial x_1}(y, x_2, \ldots, x_n) &= \frac{\partial \varphi}{\partial x_1}\big(0, x_2 - y, \ldots, x_n - y\big) \\ &= F_\eta\big(x_2 - y, \ldots, x_n - y\big) \\ &= \mathbb{P}\big(y \le Y\big),\end{aligned}$$

so that $\varphi(x) = x_1 - \int_{-\infty}^{x_1} F_Y(y)\, dy$, and then integration by parts allows to work out this expression as

$$\begin{aligned}\varphi(x) &= x_1 - \int_{-\infty}^{x_1} \big[x_1 - y\big]\, dF_Y(y) \\ &= x_1\big[1 - \mathbb{P}\big(Y \le x_1\big)\big] + \int_{-\infty}^{x_1} y\, dF_Y(y) \\ &= \int_{-\infty}^{\infty} \min\big\{x_1, y\big\}\, dF_Y(y),\end{aligned}$$

which means $\varphi(x) = \mathbb{E}(\min\{x_1, Y\}) = \mathbb{E}(\min\{x_1, x_2 - \eta_2, \ldots, x_n - \eta_n\})$. We may then conclude by taking $\epsilon_1 = 0$ and $\epsilon_i = -\eta_i$ for $i = 2, \ldots, n$. Notice that $\mathbb{E}(\epsilon) = 0$ follows from (c) and Lebesgue's theorem, while $\mathbb{P}(\epsilon = \alpha) = 0$ follows since φ is C^1. □

Remark. Condition (d) may be weakened to "*$\frac{\partial \varphi}{\partial x_1}(0, x_2, \ldots, x_n)$ is a continuous distribution on $\mathbb{R}^{n-1}$*".

B Stochastic approximation

This section provides a brief overview of stochastic approximations of differential equations. A detailed account can be found in the books by Duflo [32] and Kushner & Yin [50] (see also [12, 13, 15]). The specific material reviewed here is taken from the recent paper by Benaïm et al. [14], which extends the results from the setting of differential equations to differential inclusions.

B.1 Differential inclusions

Consider the differential inclusion

$$\frac{dx}{dt} \in F(x(t)) \tag{I}$$

where $F\colon \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is a closed set-valued map with nonempty compact convex values, satisfying the growth condition $\sup_{z\in F(x)} \|z\| \le c(1+\|x\|)$ for some $c \ge 0$. A *solution* is an absolutely continous map $x\colon \mathbb{R} \to \mathbb{R}^m$ satisfying (I) for a.e. $t \in \mathbb{R}$. Let us also consider the following notions of approximate solutions:

Perturbed Solution: An absolutely continuous map $x\colon \mathbb{R}_+ \to \mathbb{R}^m$ is called a *perturbed solution* if there is a function $t \mapsto \delta(t) \ge 0$ with $\lim_{t\to\infty} \delta(t) = 0$ and a locally integrable map $t \mapsto u(t) \in \mathbb{R}^m$ with

$$\lim_{t\to\infty} \sup_{h\in[0,T]} \left\| \int_t^{t+h} u(s)\, ds \right\| = 0, \quad \forall T \ge 0,$$

such that the following holds for a.e. $t \ge 0$:

$$\frac{dx}{dt} \in F^{\delta(t)}\big(x(t)\big) + u(t),$$

where $F^\delta(x) = \{y \in \mathbb{R}^m \,:\, \text{there exists } z \in B(x,\delta) \text{ with } d(y, F(z)) \le \delta\}$.

Discrete Approximation: A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ is called a *discrete approximation* for (I) if

$$\frac{x_{n+1} - x_n}{\gamma_{n+1}} \in F(x_n) + u_{n+1}$$

with $\gamma_n > 0$, $\gamma_n \to 0$, $\sum \gamma_n = \infty$, and $u_n \in \mathbb{R}^m$. The sequence is called a *Robbins–Monro process* with respect to a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ if γ_n is a deterministic sequence and u_n is a random vector which is $\mathcal{F}_n$-measurable with $\mathbb{E}(u_{n+1}|\mathcal{F}_n) = 0$.

Proposition B.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded discrete approximation. Let us denote $\tau_n = \sum_{i=1}^n \gamma_i$ and suppose that for all $T > 0$ we have*

$$\lim_{n\to\infty} \sup_k \left\{ \left\| \sum_{n+1}^k u_i \gamma_i \right\| \;:\; \tau_n < \tau_k \le \tau_n + T \right\} = 0. \tag{B.1}$$

Then the linearly interpolated process $t \mapsto w(t)$ defined by

$$w(t) = x_n + \frac{x_{n+1} - x_n}{\tau_{n+1} - \tau_n}\big(t - \tau_n\big), \quad \forall t \in \big[\tau_n, \tau_{n+1}\big]$$

is a perturbed solution for (I).

Proposition B.2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a Robbins–Monro discrete approximation and suppose that for some $q \in [2, \infty)$ we have*

$$\sum_n \gamma_n^{1+q/2} < \infty \text{ and } \{u_n\}_{n\in\mathbb{N}} \text{ is bounded in } L^q. \tag{$\mathrm{H_q}$}$$

Then (B.1) *holds almost surely.*

B.2 ω-limit sets and attractors

Recall that the ω-limit set of a map $t \mapsto x(t)$ is the set of all its accumulation points as $t \to \infty$. The ω-limit of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is defined similarly.

A compact set $A \subseteq \mathbb{R}^m$ is said to be *internally chain transitive* (ICT) for the dynamics (I) if for each pair $x, y \in A$, each $\epsilon > 0$, and all $T > 0$, there is a finite sequence of solutions $x_1(\cdot), \ldots, x_n(\cdot)$ and times $t_1, \ldots, t_n \in [T, \infty)$ such that

(a) $x_i(t) \in A$ for all $t \in [0, t_i]$;
(b) $\|x - x_1(0)\| < \epsilon$ and $\|x_n(t_n) - y\| < \epsilon$;
(c) $\|x_{i+1}(0) - x_i(t_i)\| < \epsilon$ for $i = 1, \ldots, n-1$.

Theorem B.3.

(a) *If $x(\cdot)$ is a bounded perturbed solution, its ω-limit set is ICT.*
(b) *If $\{x_n\}_{n\in\mathbb{N}}$ is a bounded Robbins–Monro discrete approximation satisfying* ($\mathrm{H_q}$) *for some $q \geq 2$, then almost surely its ω-limit set is ICT.*

Let $\Phi_t(x)$ be the set-valued dynamical system induced by (I), namely

$$\Phi_t(x) = \big\{x(t) : x(\cdot) \text{ solution of (I) with } x(0) = x\big\},$$

and define the ω-limit set of a point $x \in \mathbb{R}^m$ and a set $U \subseteq \mathbb{R}^m$ as

$$\omega_\Phi(x) = \bigcap_{t\geq 0} \overline{\Phi_{[t,\infty)}(x)}; \qquad \omega_\Phi(U) = \bigcap_{t\geq 0} \overline{\Phi_{[t,\infty)}(U)}.$$

A set $A \subseteq \mathbb{R}^m$ is called:

Forward precompact if $\Phi_{[t,\infty)}(A)$ is bounded for some $t \geq 0$.

Invariant if for each $x \in A$ there is a solution of (I) with $x(0) = x$ and $x(t) \in A$ for all $t \in \mathbb{R}$.

Attracting if it is compact and there exists a neighborhood $U \in \mathcal{N}_A$ such that for all $\epsilon > 0$ there is $t_\epsilon > 0$ with $\Phi_t(U) \subseteq A + B(0, \epsilon)$ for $t \geq t_\epsilon$.

Attractor if it is invariant and attracting.

Attractor free if it is invariant and contains no proper attractor for the restricted dynamics

$$\Phi_t^A(x) = \{x(t) : x(\cdot) \text{ solves (I) with } x(0) = x \text{ and } x(t) \in A \text{ for all } t \in \mathbb{R}\}.$$

Note: attractivity of Φ^A refers to neighborhoods in the trace topology of A.

Proposition B.4. *ICT's are invariant and attractor free.*

Proposition B.5. *A compact subset $A \subset \mathbb{R}^m$ is*

(a) *attracting iff there exists $U \in \mathcal{N}_A$ forward precompact with $\omega_\Phi(U) \subseteq A$;*
(b) *attractor iff there exists $U \in \mathcal{N}_A$ forward precompact with $\omega_\Phi(U) = A$;*
(c) *attractor if and only it is invariant, Lyapunov stable (for all $U \in \mathcal{N}_A$ there exists $V \in \mathcal{N}_A$ with $\Phi_t(V) \subseteq U$ for all $t \geq 0$), and its basin of attraction $B(A) = \{x : \omega_\Phi(x) \subseteq A\}$ is a neighborhood of A.*

Proposition B.6.

(a) *If A is attracting, L is invariant, and $\omega_\Phi(x) \subseteq A$ for some $x \in L$, then $L \subseteq A$.*
(b) *If A is attractor then $\Phi_t(A) \subseteq A$ for all $t \geq 0$.*

B.3 Lyapunov functions

Theorem B.7. *Let $\Lambda \subseteq \mathbb{R}^m$ be a compact set and U an open neighborhood of Λ which is forward invariant: $\Phi_t(U) \subseteq U$ for all $t \geq 0$. Let $V : U \to [0, \infty)$ be a continuous map such that $V(x) = 0$ for all $x \in \Lambda$, and $V(y) < V(x)$ for all $x \in U \setminus \Lambda$, $y \in \Phi_t(x)$, $t > 0$. Then Λ contains an attractor A with $U \subseteq B(A)$.*

Theorem B.8. *Let $\Lambda \subseteq \mathbb{R}^m$ and U an open neighborhood of Λ. Suppose that $V : U \to \mathbb{R}$ is a continuous Lyapunov function: $V(y) \leq V(x)$ for all $x \in U$, $y \in \Phi_t(x)$ and $t \geq 0$, with strict inequality if $x \notin \Lambda$. If $V(\Lambda)$ has empty interior then every ICT set A is contained in Λ and V is constant over A.*

Corollary B.9. *Under the assumptions of* Theorem B.8*, let $\{x_n\}_{n \in \mathbb{N}}$ be a bounded Robbins–Monro discrete approximation satisfying* ($\mathrm{H_q}$) *for some $q \geq 2$. Then x_n converges almost surely to Λ.*

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