Conclusions

We have presented an overview of some recent results in the computation of solutions of Generalized Nash Equilibrium Problems. Game Theory is a venerable field with a long history and a wealth of results that is not possible to summarize in a few pages. What we have tried to do is to give a glimpse of some recent algorithmic developments from the point of view of mathematical programming. There are very many aspects and approaches that we did not even mention in these notes. Restricting attention to algorithmic issues, we want to at least mention [6, 7, 8, 31, 35, 44, 45, 51, 55, 56, 57, 58, 68, 70, 111] as an interesting sample of the available literature.

The interest of the mathematical programming community in game theory is possibly one of the big novelties in the field, and it promises to change many points of view, with its emphasis on numerical tractability and the emergence of new priorities. The attention of optimizers to Game Theory has also been fuelled by numerous engineering applications that required the effective computation of equilibria. The development of new models, different from traditional ones, has brought new challenges that need to be addressed. To attack these new problems a strong background in optimization techniques is needed. These notes were addressed precisely to optimizers that want to expand their interests in the hope to provide good, albeit somewhat informal and incomplete, entry point into this fascinating world.

Appendix

Elements of variational inequality theory

In this appendix we give a very concise introduction to variational inequalities, which is an essential tool used in our approach to the analysis and solution of games. We do not go much beyond some basic definitions and properties and refer the reader to [39] for a broader and deeper analysis of variational inequalities and also for proofs of all statements in this appendix.

Let a convex closed subset K of $\mathbb{R}^n$ be given, together with a function $F\colon K \to \mathbb{R}^n$. The *variational inequality* (VI) (K, F) is the problem of finding a point x belonging to K such that

$$F(x)^T(y - x) \geq 0, \quad \forall y \in K. \tag{A.1}$$

The set of all solutions of a VI (K, F), i.e., the set of all points x satisfying (A.1) is denoted by $\mathrm{SOL}(K, F)$. Note that, if we assume, as we shall always assume from now on, that F is continuous on K, it is readily seen that $\mathrm{SOL}(K, F)$ is a (possibly empty) closed set. The simplest geometric interpretation of a VI or, more precisely, of (A.1), is that a point $x \in K$ is a solution if an only if $F(x)$ forms a non-obtuse angle with all vectors of the form $y - x$, for all y in K.

VIs provide a unified mathematical model for a host of applied equilibrium problems and include many special cases that are important in their own right. Although some of these special cases can and should be dealt with directly and using more specific tools, it is important to list them here, in order to understand the nature of VIs.

- If $K = \mathbb{R}^n$, it is easy to see that solving the VI (K, F) is equivalent to finding the solution of the system of equations $F(x) = 0$. In fact, loosely speaking, we are looking for a point x such that (A.1) is satisfied for all $y \in \mathbb{R}^n$. But since in this case $y - x$ can be any vector, it is readily seen that the only possibility is that $F(x)$ be zero.
- If $F = \nabla f$ for some continuously differentiable convex function $f\colon K \to \mathbb{R}$, it can be shown that a point x is a solution of the VI (K, F) if and only if it

is a minimum point of the constrained, convex optimization problem

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } x \in K. \end{aligned}$$

In fact, it is known that the *minimum principle* states that a point $x \in K$ is a minimum point of this optimization problem if and only if $\nabla f(x)^T(y-x) \geq 0$ for all $y \in K$, which shows our equivalence.

- If K is the nonnegative orthant, i.e., if $K = \mathbb{R}^n_+$, then a point x is a solution of the VI $(\mathbb{R}^n_+, F)$ if and only if

$$0 \leq x \perp F(x) \geq 0,$$

where by $x \perp F(x)$ we mean that the two vectors are perpendicular, that is, $x^T F(x) = 0$. This problem is known as the *nonlinear complementarity problem* and is often denoted by NCP(F). The *linear* complementarity problem (i.e., the case in which $F(x) = Ax - b$) was introduced by Cottle as a tool to study in a unified way linear programs, quadratic problems, and bimatrix games. We refer the reader to [22] and [39] for a detailed history and analysis of this problem.

- If K is a cartesian product of the form $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+$, then we have a problem that can be viewed as a mixture of a system of equations and a nonlinear complementarity problem. Assume that $x = (u, v)$ with $u \in \mathbb{R}^{n_1}$, $v \in \mathbb{R}^{n_2}$, and F is partitioned accordingly as $F = (H, G)$, with $H\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $G\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$. It can be shown that the VI (K, F) is in this case equivalent to the problem of finding a (u, v) such that

$$\begin{aligned} H(u, v) &= 0, \\ 0 \leq v \perp G(u, v) &\geq 0. \end{aligned}$$

This problem is know as the *mixed complementarity problem* and, as we will see shortly, it is strictly related to KKT systems.

A solution of a VI can be characterized, under some standard assumptions, by its KKT conditions, that are formally very similar to the KKT conditions of an optimization problem. In order to describe these KKT conditions, we assume that the set K is defined through a system of equalities and inequalities

$$K = \{x \in \mathbb{R}^n \,:\, g(x) \leq 0,\ Ax = b\},$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^m$ is a vector of continuously differentiable convex functions and $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$. Note that, fine points apart, if K must be convex, equality constraints involved in its definitions must be linear. In this setting, the following theorem holds.

Theorem A.1. *Let the set K be defined as above. The following statements hold:*

(a) *Let $x \in \mathrm{SOL}(K, F)$. If any standard constraint qualification holds at x (for example, linear independence of active constraints, Mangasarain–Fromovitz constraint qualifications, or Abadie condition) then there exist multipliers $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$ such that*

$$\begin{aligned} &F(x) + \nabla g(x)\lambda + A^T\mu = 0, \\ &Ax = b, \\ &0 \le \lambda \perp g(x) \le 0. \end{aligned} \tag{A.2}$$

(b) *Conversely, if a triple (x, λ, ν) satisfies system* (A.2)*, then x solves the VI (K, F).*

Note that the KKT system (A.2) is very similar to the KKT system of an optimization problem, but it is important to observe that, assuming that some constraint qualification holds and assuming that the set K is defined as above, the KKT system is totally equivalent to the VI, in the sense that a point x is a solution of the KKT system if and only if $x \in \mathrm{SOL}(K, F)$. This is in contrast to optimization problems where, in order to have a similar equivalence, one must assume that the problem is convex. As a further remark, it may be interesting to note that the KKT system (A.2) is an instance of a mixed complementarity problem; just take

$$u = \begin{pmatrix} x \\ \mu \end{pmatrix}, \quad v = \lambda,$$

$$H(u, v) = \begin{pmatrix} F(x) + \nabla g(x)\lambda + A^T\mu \\ Ax - b \end{pmatrix}, \quad G(u, v) = -g(x).$$

The first important issue one has to analyze when studying VIs is, obviously, existence of solutions. The basic result one can establish parallels the famous Weierstrass theorem for optimization problems, although it should be noted that while the Weierstrass theorem can be proved using elementary tools, the proof of the following theorem, providing the basic existence result for VIs, is rather sophisticated and by no means trivial.

Theorem A.2. *Let a VI (K, F) be given and assume that K is convex and compact and that F is continuous on K. Then* $\mathrm{SOL}(K, F)$ *is nonempty and compact.*

This theorem plays a central role in establishing existence of a solution to a VI, but it is not (directly) applicable to all those cases in which the set K is unbounded; for example, it is of no use when dealing with (mixed) nonlinear complementarity problems. There are obviously many ways to deal with the difficulty of an unbounded set K – here we only mention the possibility to introduce special classes of functions that, among other things, have some bearing to existence issues.

Let a function $F\colon K \to \mathbb{R}^n$ be given, with K a closed convex subset of $\mathbb{R}^n$. We say that F is

- *monotone* on K if

$$(F(x) - F(y))^T (y - x) \geq 0, \quad \forall x, y \in K;$$

- *strictly monotone* on K if

$$(F(x) - F(y))^T (y - x) > 0, \quad \forall x, y \in K,\ x \neq y;$$

- *strongly monotone* on K if a positive constant α exists such that

$$(F(x) - F(y))^T (y - x) \geq \alpha \|x - y\|^2, \quad \forall x, y \in K.$$

Two simple considerations can illustrate better the nature of these definitions. In the unidimensional case, $n = 1$, a function is monotone if and only if it is nondecreasing, and strictly monotone if and only if it is increasing. In this setting, strongly monotone functions can be seen, roughly speaking, as increasing functions whose slope is bounded away from zero. In the general case of $n \geq 1$, there is an important relation to convex functions that also illustrates the various definitions of monotonicity. Let $f\colon K \to \mathbb{R}$ be a continuously differentiable function on K. Then the gradient ∇f is (strictly, strongly) monotone on K if and only if f is (strictly, strongly) convex on K. In a sense, monotonicity plays a role in the VI theory that is very similar to that of convexity in optimization theory. Having this in mind, the following theorem is rather "natural".

Theorem A.3. *Let a VI (K, F) be given, with K closed and convex and F continuous on K. Then:*

(a) *If F is monotone on K then* $\mathrm{SOL}(K, F)$ *is a (possibly empty) closed, convex set.*

(b) *If F is strictly monotone on K then the VI (K, F) has at most one solution.*

(c) *If F is strongly monotone on K then the VI (K, F) has one and only one solution.*

Note that point (c) in the above theorem guarantees existence (and uniqueness) of a solution even in the case of unbounded sets K. In view of the importance of monotonicity, it is important to have some easy way to check whether a function is monotone. The following proposition gives an easy test for C^1 functions.

Proposition A.4. *Let K be a closed convex subset of $\mathbb{R}^n$ and let $F\colon K \to \mathbb{R}^n$ be a C^1 function. The:*

(a) *F is monotone on K if and only if $JF(x)$ is positive semidefinite for all $x \in K$.*

(b) *If $JF(x)$ is positive definite for all $x \in K$ then F is strictly monotone on K.*

(c) *F is strongly monotone on K if and only if a positive constant α exists such that $JF(x) - \alpha I$ is positive semidefinite for every $x \in K$.*

Note that in case (b) we only have a sufficient condition for strict monotonicity. To see that the reverse implication does not hold in general, consider the function x^3, which is strictly monotone on $\mathbb{R}$. Its Jacobian (the derivative in this case) is given by x^2 and it is clear that at the origin $x^2 = 0$, so that the Jacobian is not everywhere positive definite.

Computation of the solutions of a VI is a complex issue. In the main text we describe some projection-type algorithms that, although quite simple, are, at least theoretically, very interesting. Although there exist other types of algorithms – see [39] – that might practically outperform projection-type algorithms, these latter algorithms are most interesting, since they are, on the one hand, simple, and on the other hand, they match well with the decomposition schemes that are of paramount interests in the solution of games. The use of more complex VI solution methods in the solution of GNEPs is certainly a topic that should be addressed in the near future.

Bibliography

[1] Adida E. and Perakis G.: *Dynamic pricing and inventory control: Uncertainty and competition.* Part A: *Existence of Nash equilibrium.* Operations Research Center, Sloan School of Management, MIT, Technical Report (2006).

[2] Adida E. and Perakis G.: *Dynamic pricing and inventory control: Uncertainty and competition.* Part B: *An algorithm for the normalized Nash equilibrium.* Operations Research Center, Sloan School of Management, MIT, Technical Report (2006).

[3] Altman, E. and Altman, Z.: *S-modular games and power control in wireless networks.* IEEE Transactions on Automatic Control **48(5)** (2003).

[4] Altman E., Boulogne T., El-Azouzi R., Jiménez T., and Wynter L.: *A survey on networking games in telecommunications.* Computers & Operations Research **33** (2006), 286–311.

[5] Altman E. and Wynter L.: *Equilibrium, games, and pricing in transportation and telecommunication networks.* Networks and Spatial Economics **4** (2004), 7–21.

[6] Antipin A.S.: *Solution methods for variational inequalities with coupled constraints.* Computational Mathematics and Mathematical Physics **40** (2000), 1239–1254.

[7] Antipin A.S.: *Solving variational inequalities with coupling constraints with the use of differential equations.* Differential Equations **36** (2000), 1587–1596.

[8] Antipin A.S.: *Differential equations for equilibrium problems with coupled constraints.* Nonlinear Analysis **47** (2001), 1833–1844.

[9] Arrow K.J. and Debreu G.: *Existence of an equilibrium for a competitive economy.* Econometrica **22** (1954), 265–290.

[10] Aubin J.-P. and Frankowska H.: *Set-Valued Analysis.* Birkhäuser, Boston (1990).

[11] Başar T. and Olsder G.J.: *Dynamic noncooperative game theory.* Academic Press London/New York, Second edition (1989) (Reprinted in SIAM Series "Classics in Applied Mathematics", 1999).

[12] Bassanini A., La Bella A., and Nastasi A.: *Allocation of railroad capacity under competition: A game theoretic approach to track time pricing.* In Trans-

portation and Networks Analysis: Current Trends (edited by M. Gendreau and P. Marcotte), 1–17, Kluwer Academic Publisher, Dordrecht (2002).

[13] Baye M.R., Tian G., and Zhou J.: *Characterization of existence of equilibria in games with discontinuous and non-quasiconcave payoffs.* The Review of Economic Studies **60** (1993), 935–948.

[14] Bensoussan A.: *Points de Nash dans le cas de fonctionnelles quadratiques et jeux différentiels linéaires à N personnes.* SIAM Journal of Control **12** (1974), 460–499.

[15] Berman A. and Plemmons R.J.: *Nonnegative matrices.* In the Mathematical Sciences SIAM Classics in Applied Mathematics No. 9, Philadelphia (1994).

[16] Berridge S. and Krawczyk J.B.: *Relaxation algorithms in finding Nash equilibria.* Economic working papers archives (1997), http://econwpa.wustl.edu/eprints/comp/papers/9707/9707002.abs.

[17] Bertrand J.: Review of "*Théorie mathématique de la richesse sociale*" by Léon Walras and "*Recherches sur les principes mathématiques de la théorie des richesses*" by Augustin Cournot. Journal des Savants, 499–508 (1883).

[18] Bertsekas D.P. and Tsitsiklis J.N.: *Parallel and distributed computation: Numerical methods.* Athena Scientific, Cambridge (1997).

[19] Breton M., Zaccour G., and Zahaf M.: *A game-theoretic formulation of joint implementation of environmental projects.* European Journal of Operational Research **168** (2005), 221–239.

[20] Chung M., Hande, P., Lan T., and Tan C.W.: *Power control.* In Wireless Cellular Networks, Now Publisher Inc. (2008).

[21] Contreras J., Klusch M.K., and Krawczyk J.B.: *Numerical solution to Nash-Cournot equilibria in coupled constraints electricity markets.* IEEE Transactions on Power Systems **19** (2004), 195–206.

[22] Cottle R.W., Pang J.-S., and Stone R.E.: *The linear complementarity problem.* Academic Press (1992).

[23] Cournot, A.A.: *Recherches sur les principes mathématiques de la théorie des richesses.* Hachette, Paris (1838).

[24] Dasgupta P. and Maskin E.: *The existence of equilibrium in discontinuous economic games, I: Theory.* Rev. Econ. Stud. **53** (1986), 1–26.

[25] Dasgupta P. and Maskin E.: *The existence of equilibrium in discontinuous economic games, II: Applications.* Rev. Econ. Stud. **53** (1986), 27–41.

[26] Debreu G.: *A social equilibrium existence theorem.* Proceedings of the National Academy of Sciences **38** (1952), 886–893.

[27] Debreu G.: *Theory of values.* Yale University Press, New Haven, USA (1959).

[28] Demyanov V.F., Di Pillo G, and Facchinei F.: *Exact penalization via Dini and Hadamard conditional derivatives.* Optimization Methods and Software **9** (1998), 19–36.

[29] Di Pillo G. and Facchinei F.: *Exact barrier function methods for Lipschitz programs.* Applied Mathematics and Optimization **32** (1995), 1–31.

[30] Dreves A., Facchinei F., Kanzow C., and Sagratella S.: *On the solution of the KKT conditions of generalized Nash equilibrium problems.* In preparation (2010).

[31] Dreves A. and Kanzow C.: *Nonsmooth optimization reformulations characterizing all solutions of jointly convex generalized Nash equilibrium problems.* Technical report, Institute of Mathematics, University of Würzburg, Würzburg, Germany (2009).

[32] Drouet L., Haurie A., Moresino F., Vial J.-P., Vielle M., and Viguier L.: *An oracle based method to compute a coupled equilibrium in a model of international climate policy.* Computational Management Science **5** (2008), 119–140.

[33] Ehrenmann, A.: *Equilibrium problems with equilibrium constraints and their application to electricity markets.* Ph.D. Dissertation, Judge Institute of Management, The University of Cambridge, Cambridge, UK, Cambridge, UK (2004).

[34] Facchinei F., Fischer A., and Piccialli V.: *On generalized Nash games and variational inequalities.* Operations Research Letters **35** (2007), 159–164.

[35] Facchinei F., Fischer A., and Piccialli V.: *Generalized Nash equilibrium problems and Newton methods.* Mathematical Programming **117** (2009), 163–194.

[36] Facchinei F. and Kanzow C.: *Generalized Nash equilibrium problems.* 4OR **5** (2007), 173–210.

[37] Facchinei F. and Kanzow C.: *Penalty methods for the solution of generalized Nash equilibrium problems.* SIAM Journal on Optimization **20(5)** (2010), 2228–2253.

[38] Facchinei F. and Lampariello L.: *Partial penalization for the solution of generalized Nash equilibrium problems.* Journal of Global Optimization **50** (2011), 39–57.

[39] Facchinei F. and Pang J.-S.: *Finite-dimensional variational inequalities and complementarity problems.* Springer, New York (2003).

[40] Facchinei F. and Pang J.-S.: *Exact penalty functions for generalized Nash problems.* In G. Di Pillo and M. Roma, editors, Large-Scale Nonlinear Optimization, 115–126, Springer (2006).

[41] Facchinei F. and Pang J.-S.: *Nash equilibria: The variational approach.* In "Convex optimization in signal processing and communications" (Palomar D.P. and Eldar Y. eds.), 443–493. Cambridge University Press (2010).

[42] Facchinei F., Piccialli V., and Sciandrone M.: *Decomposition algorithms for generalized potential games.* Computational Optimization and Applications **50** (2011), 237–262.

[43] Facchinei F. and Sagratella S.: *On the computation of all solutions of jointly convex generalized Nash equilibrium problems.* Optimization Letters **5** (2011), 531–547.

[44] Flåm S.D.: *Paths to constrained Nash equilibria.* Applied Mathematics and Optimization **27** (1993), 275–289.

[45] Flåm S.D. and Ruszczyński A.: *Noncooperative convex games: Computing equilibrium by partial regularization.* IIASA Working Paper 94–42 (1994).

[46] Fukushima, M.: *Restricted generalized Nash equilibria and controlled penalty algorithm.* Computational Management Science **8** (2011), 201–218.

[47] Fukushima M. and Pang J.-S.: *Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games.* Computational Management Science **2** (2005), 21–56.

[48] Gabriel S.A., Kiet S., and Zhuang J.: *A mixed complementarity-based equilibrium model of natural gas markets.* Operations Research **53** (2005), 799–818.

[49] Gabriel S. and Smeers Y.: *Complementarity problems in restructured natural gas markets.* In Recent Advances in Optimization, Lectures Notes in Economics and Mathematical Systems **563** (2006), 343–373.

[50] Garcia C.B. and Zangwill W.I.: *Pathways to solutions, fixed points, and equilibria.* Prentice-Hill, New Jersey (1981).

[51] Gürkan G. and Pang J.-S.: *Approximations of Nash equilibria.* Mathematical Programming **117** (2009), 223–253.

[52] Harker P.T.: *Generalized Nash games and quasi-variational inequalities.* European Journal of Operational Research **54** (1991), 81–94.

[53] Harker P.T. and Hong S.: *Pricing of track time in railroad operations: An internal market approach.* Transportation Research B-Meth. **28** (1994), 197–212.

[54] Haurie A. and Krawczyk J.-B.: *Optimal charges on river effluent from lumped and distributed sources.* Environmental Modelling and Assessment **2** (1997), 93–106.

[55] von Heusinger A. and Kanzow C.: SC^1 *optimization reformulations of the generalized Nash equilibrium problem.* Optimization Methods and Software **23** (2008), 953–973.

[56] von Heusinger A. and Kanzow C.: *Optimization reformulations of the generalized Nash equilibrium problem using Nikaido-Isoda-type functions.* Computational Optimization and Applications **43** (2009), 353–377.

[57] von Heusinger A. and Kanzow C.: *Relaxation methods for generalized Nash equilibrium problems with inexact line search.* Journal of Optimization Theory and Applications **143** (2009), 159–183.

[58] von Heusinger A., Kanzow C., and Fukushima M.: *Newton's method for computing a normalized equilibrium in the generalized Nash game through fixed point formulation.* Technical report, Institute of Mathematics, University of Würzburg, Würzburg, Germany (2009).

[59] Hobbs B., Helman U., and Pang J.-S.: *Equilibrium market power modeling for large scale power systems.* IEEE Power Engineering Society Summer Meeting 2001. 558–563. (2001).

[60] Hobbs B. and Pang J.-S.: *Nash-Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints.* Operations Research **55** (2007), 113–127.

[61] Hotelling H.: *Game theory for economic analysis.* The Econometrics Journal **39** (1929), 41–47.

[62] Huang J., Berry R., and Honig M.L.: *Distributed interference compensation for wireless networks.* IEEE Journal on Selected Areas in Communications **24(5)** (2006), 1074–1084.

[63] Ichiishi T.: *Game theory for economic analysis.* Academic Press, New York (1983).

[64] Jiang H.: Network capacity management competition. Technical report, Judge Business School at University of Cambridge, UK (2007).

[65] Jorswieck E.A., Larsson E.G., Luise M., and Poor H.V. (guest editors): *Game theory in signal processing and communications.* Special issue of the IEEE Signal Processing Magazine, 26 (2009).

[66] Kesselman A., Leonardi S., and Bonifaci V.: *Game-theoretic analysis of internet switching with selfish users.* Proceedings of the First International Workshop on Internet and Network Economics, WINE 2005, Lectures Notes in Computer Science **3828** (2005), 236–245.

[67] Krawczyk J.-B.: *An open-loop Nash equilibrium in an environmental game with coupled constraints.* In "Proceedings of the 2000 Symposium of the International Society of Dynamic Games", 325–339, Adelaide, South Australia (2000).

[68] Krawczyk J.-B.: *Coupled constraint Nash equilibria in environmental games.* Resource and Energy Economics **27** (2005), 157–181.

[69] Krawczyk J.-B.: *Numerical solutions to coupled-constraint* (*or generalised Nash*) *equilibrium problems.* Computational Management Science **4** (2007), 183–204.

[70] Krawczyk J.-B. and Uryasev S.: *Relaxation algorithms to find Nash equilibria with economic applications.* Environmental Modelling and Assessment **5** (2000), 63–73.

[71] Kubota, K. and Fukushima, M.: *Gap function approach to the generalized Nash equilibrium problem.* Journal of Optimization Theory and Applications **144** (2010), 511–531

[72] Kulkarni A.A. and Shanbhag U.V.: *On the variational equilibrium as a refinement of the generalized Nash equilibrium.* Manuscript, University of Illinois at Urbana-Champaign, Department of Industrial and Enterprise Systems Engineering.

[73] Leshem A. and Zehavi E.: *Game theory and the frequency selective interference channel – A tutorial.* IEEE Signal Processing Magazine **26** (2009), 28–40.

[74] Luo Z.-Q. and Pang J.-S.: *Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines.* EURASIP Journal on Applied Signal Processing (2006), 1–10.

[75] Luo Z.-Q. and Zhang S.: *Spectrum management: Complexity and duality.* Journal of Selected Topics in Signal Processing **2(1)** (2008), 57–72.

[76] MacKenzie A.B. and Wicker S.B.: *Game theory and the design of self-configuring, adaptive wireless networks.* IEEE Communication Magazine **39(11)** (2001), 126–131.

[77] Monderer D. and Shapley L.S.: *Potential games.* Games and Economic Behavior **14** (1996), 124–143.

[78] Monteiro R.D. C. and Pang J.-S.: *A potential reduction Newton method for constrained equations.* SIAM Journal on Optimization **9** (1999), 729–754.

[79] Morgan J. and Scalzo V.: *Existence of equilibria in discontinuous abstract economies.* Preprint 53-2004, Dipartimento di Matematica e Applicazioni R. Caccioppoli, Napoli (2004).

[80] Nabetani K., Tseng P., and Fukushima M.: *Parametrized variational inequality approaches to generalized Nash equilibrium problems with shared constraints.* Computational Optimization and Applications **48** (2011), 423–452.

[81] Nash J.F.: *Equilibrium points in n-person games.* Proceedings of the National Academy of Sciences **36** (1950), 48–49.

[82] Nash J.F.: *Non-cooperative games.* Annals of Mathematics **54** (1951), 286–295.

[83] von Neumann J.: *Zur Theorie der Gesellschaftsspiele.* Mathematische Annalen **100** (1928), 295–320.

[84] von Neumann J. and Morgenstern O.: *Theory of games and economic behavior.* Princeton University Press, Princeton (1944).

[85] Nisan N., Roughgarden T., Tardos E., and Vazirani V.V. (Edited by): *Algorithmic game theory.* Cambridge University Press (2007).

[86] O'Neill D., Julian D., and Boyd S.: *Seeking Foschini's genie: optimal rates and powers in wireless networks.* IEEE Transactions on Vehicular Technology, 2004.

[87] Outrata J.V., Kočvara M., and Zowe J.: *Nonsmooth approach to optimization problems with equilibrium constraints.* Kluwer Academic Publishers, Dordrecht (1998).

[88] Pang J.-S.: *Computing generalized Nash equilibria.* Manuscript, Department of Mathematical Sciences, The Johns Hopkins University (2002).

[89] Pang J.-S. and Fukushima M.: *Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games.* Computational Management Science **2** (2005), 21–56 (erratum: ibid. **6** (2009), 373–375.)

[90] Pang J.-S., Scutari G., Facchinei F., and Wang C.: *Distributed power allocation with rate constraints in gaussian parallel interference channels.* IEEE Transactions on Information Theory **54** (2008), 3471–3489.

[91] Pang J.-S., Scutari G., Palomar D.P., and Facchinei F.: *Design of cognitive radio systems under temperature-interference constraints: A variational inequality approach.* IEEE Transactions on Signal Processing **58** (2010), 3251–3271.

[92] Puerto J., Schöbel A., and Schwarze S.: *The path player game: Introduction and equilibria.* Preprint 2005-18, Göttingen Georg-August University, Göttingen, Germany (2005).

[93] Rao S.S., Venkayya V.B., and Khot N.S.: *Game theory approach for the integrated design of structures and controls.* AIAA Journal **26** (1988), 463–469.

[94] Reny P.J.: *On the existence of pure and mixed strategy Nash equilibria in discontinuous games.* Econometrica **67** (1999), 1026–1056.

[95] Robinson S.M.: *Shadow prices for measures of effectiveness. I. Linear Model.* Operations Research **41** (1993), 518–535.

[96] Robinson S.M.: *Shadow prices for measures of effectiveness. II. General Model.* Operations Research **41** (1993), 536–548.

[97] Rockafellar R.T. and Wets R.J.-B.: *Variational analysis.* Springer Verlag, Berlin (1998).

[98] Rosen J.B.: *Existence and uniqueness of equilibrium points for concave n-person games.* Econometrica **33** (1965), 520–534.

[99] Schmit L.A.: *Structural synthesis – Its genesis and development.* AIAA Journal **19** (1981), 1249–1263.

[100] Scotti S.J.: *Structural design using equilibrium programming formulations.* NASA Technical Memorandum 110175 (1995).

[101] Scutari G., Facchinei F., Pang J.-S., and Palomar D.P.: *Monotone communication games.* In preparation.

[102] Scutari A., Palomar D.P., and Barbarossa S.: *Asynchronous iterative waterfilling for Gaussian frequency-selective interference channels.* IEEE Transactions on Information Theory **54** (2008), 2868–2878.

[103] Scutari A., Palomar D.P., and Barbarossa S.: *Optimal linear precoding strategies for wideband noncooperative systems based on game theory* – Part I: *Nash equilibria.* IEEE Transactions on Signal Processing **56** (2008), 1230–1249.

[104] Scutari A., Palomar D.P., and Barbarossa S.: *Optimal linear precoding strategies for wideband noncooperative systems based on game theory* – Part II: *Algorithms.* IEEE Transactions on Signal Processing **56** (2008), 1250–1267.

[105] Scutari A, Palomar D.P., and Barbarossa S.: *Competitive optimization of cognitive radio MIMO systems via game theory.* In “Convex optimization in signal processing and communications” (Palomar D.P. and Eldar Y. eds.), 387–442. Cambridge University Press (2010).

[106] Scutari G., Palomar D.P., Pang J.-S., and Facchinei F.: *Flexible design of cognitive radio wireless systems: From game theory to variational inequality theory.* IEEE Signal Processing Magazine **26(5)** (2009), 107–123.

[107] Sun L.-J. and Gao Z.-Y.: *An equilibrium model for urban transit assignment based on game theory.* European Journal of Operational Research **181** (2007), 305–314.

[108] Tian G. and Zhou J.: *Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization.* Journal of Mathematical Economics **24** (1995), 281–303.

[109] Tidball M. and Zaccour G.: *An environmental game with coupling constraints.* Environmental Modeling Assessment **10**, 153–158 (2005).

[110] Topkis D.: *Supermodularity and complementarity.* Princeton University Press (1998).

[111] Uryasev S. and Rubinstein R.Y.: *On relaxation algorithms in computation of noncooperative equilibria.* IEEE Transactions on Automatic Control **39** (1994), 1263–1267.

[112] Vincent T.L.: *Game theory as a design tool.* ASME Journal of Mechanisms, Transmissions, and Automation in Design **105** (1983), 165–170.

[113] Vives X.: *Nash equilibrium with strategic complementarities.* Journal of Mathematical Economics **19** (1994), 305–321.

[114] Walras L.: *Éléments d’économie politique pure.* Lausanne (1900).

[115] Wei J.-Y. and Smeers Y.: *Spatial oligopolistic electricity models with cournot generators and regulated transmission prices.* Operations Research **47** (1999), 102–112.

[116] Yu W., Ginis G., and Cioffi J.M.: *Distributed multiuser power control for digital subscriber lines.* IEEE Journal on Selected Areas in Communications **20** (2002), 1105–1115.

[117] Zhou J., Lam W.H.K., and Heydecker B.G.: *The generalized Nash equilibrium model for oligopolistic transit market with elastic demand.* Transportation Research B-Meth. **39** (2005), 519–544.