Appendix A
Sample Datasets and Figures

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A. Rogers, *Applied Multiregional Demography Through Problems*,
https://doi.org/10.1007/978-3-030-38215-5

Table A.1 Demographie Data for India, 1970

Urban region								Rural region							
				Observed rates (per 1000)						Observed rates (per 1000)					
Age	Population	Births	Deaths	Outmig.	Birth	Death	Outmig.	Age	Population	Births	Deaths	Outmig.	Birth	Death	Outmig.
0	14,140,200		540,830	131,860		38.248	9.325	0	64,966,800		3,749,333	360,672		57.712	5.552
5	14,798,300		58,278	98,442		3.938	6.652	5	68,071,500		404,496	269,265		5.942	3.956
10	13,637,500		23,598	70,661		1.730	5.181	10	54,639,700		142,663	193,276		2.611	3.537
15	10,944,900	361,195	20,245	151,924	33.001	1.850	13.881	15	36,502,000	1,811,182	101,879	415,552	49.619	2.791	11.384
20	10,454,900	923,207	29,320	253,459	88.304	2.804	24.243	20	32,627,500	4,331,904	138,066	693,277	132.768	4.232	21.248
25	8,955,700	805,956	24,581	109,872	89.994	2.745	12.268	25	31,843,600	4,308,732	131,882	300,528	135.309	4.142	9.438
30	7,140,400	580,051	23,620	63,759	76.198	3.103	8.376	30	28,551,700	3,271,086	133,672	174,397	114.567	4.682	6.108
35	6,881,500	367,275	25,868	43,812	53.371	3.759	6.367	35	26,011,900	2,087,353	147,542	119,837	80.246	5.672	4.607
40	5,714,300	148,412	27,618	32,235	25.972	4.833	5.641	40	22,648,400	884,424	165,168	88,172	39.050	7.293	3.893
45	4,476,500	53,492	30,450	23,847	11.950	6.802	5.327	45	18,315,900	329,073	187,991	65,227	17.967	10.264	3.561
50	3,810,300		39,787	15,975		10.442	4.193	50	16,879,800		265,956	43,696		15.756	2.589
55	2,223,400		32,371	18,012		14.559	8.101	55	10,432,000		229,172	49,269		21.968	4.723
60	2,398,900		59,037	22,432		24.703	9.386	60	11,944,300		445,211	61,358		37.274	5.137
65	1,129,400		37,873	20,693		33.534	18.322	65	5,691,800		287,999	56,601		50.599	9.944
70+	1,907,800		139,108	33,787		72.915	17.710	70+	9,629,600		1,059,459	92,417		110.021	9.597
TOTAL	109,077,000	3,239,588	1,112,584	1,090,770	29.700	10.200	10.000	TOTAL	438,756,500	17,023,754	7,590,489	2,983,544	38.800	17.300	6.800

Source: Rogers (1985)

Table A.2 Uniregional Life Table: India, 1970

Age, x	*P(x)*	*q(x)*	*Q(x)*	*d(x)*	*L(x)*	*m(x)*	*s(x)*	*T(x)*	*e(x)*
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	0.761213	0.238787	100,000	23,879	440,303	0.054232	0.852517	4,885,568	48.86
5	0.972463	0.027537	76,121	2.096	375,366	0.005584	0.980073	4,445,265	58.40
10	0.987898	0.012102	74,025	896	367,886	0.002435	0.987558	4,069,899	54.98
15	0.987213	0.012787	73,129	935	363,309	0.002574	0.984007	3,702,013	50.62
20	0.980761	0.019239	72,194	1,389	357,498	0.003885	0.980883	3,338,704	46.25
25	0.981007	0.018993	70,805	1,345	350,664	0.003835	0.979759	2,981,206	42.10
30	0.978487	0.021513	69,460	1,494	343,566	0.004349	0.976260	2,630,542	37.87
35	0.973983	0.026017	67,966	1,768	335,410	0.005272	0.970332	2,286,976	33.65
40	0.966582	0.033418	66,198	2,212	325,459	0.006797	0.960005	1,951,566	29.48
45	0.953202	0.046798	63,986	2,994	312,442	0.009584	0.941267	1,626,107	25.41
50	0.928746	0.071254	60,991	4,346	294,092	0.014777	0.915744	1,313,665	21.54
55	0.901744	0.098256	56,645	5,566	269,312	0.020667	0.871674	1,019,573	18.00
60	0.838328	0.161672	51,080	8,258	234,753	0.035178	0.814747	750,260	14.65
65	0.786618	0.213382	42,821	9,137	191,264	0.047773	1.695266[a]	515,508	12.04
70+	0.	1.000000	33,684	33,684	324,244	0.103885	0.	324,244	9.6 :

Sources: Rogers (1985)

[a]This s(x) exceeds unity because it refers to survivorship into an open-ended age interval. Because not all members in that interval die over a period of five years, a "correction" must be incorporated into the value of s(x)

Table A.3 Fertility and Survivorship Elements of the Population Projection Matrix: India, 1970

Fertility Elements					*Survivorship Elements*			
Age,x	$b_{uu}(x)$	$b_{uv}(x)$	$b_{vu}(x)$	$b_{vv}(x)$	$s_{uu}(x)$	$s_{uv}(x)$	$s_{vu}(x)$	$s_{vv}(x)$
0					0.858696	0.036686	0.021439	0.821862
5					0.957236	0.028591	0.018073	0.960723
10	0.069883	0.006145	0.003816	0.102052	0.946748	0.044240	0.034581	0.952089
15	0.252453	0.029194	0.018773	0.368544	0.904014	0.084133	0.071981	0.910900
20	0.378781	0.032549	0.021294	0.552139	0.900593	0.085333	0.071840	0.907690
25	0.360721	0.019368	0.012798	0.525156	0.936345	0.048957	0.036911	0.941430
30	0.283405	0.011788	0.008156	0.411861	0.947717	0.035121	0.025554	0.949044
35	0.174452	0.005915	0.004314	0.253223	0.950085	0.028518	0.020215	0.948075
40	0.083302	0.002677	0.001967	0.120781	0.945377	0.025801	0.017567	0.939788
45	0.026705	0.000538	0.000481	0.038792	0.935590	0.022110	0.014353	0.923247
50					0.912065	0.027142	0.016123	0.894760
55					0.869703	0.036763	0.020871	0.844249
60					0.810955	0.052643	0.028482	0.777584
65					1.959624[a]	0.366715	0.208395	1.449770[a]

Source: Rogers (1985)

[a]These elements exceed unity because they refer to survivorship into an open-ended age interval. Because not all members in that interval leave the population over a period of five years, a "correction" must be incorporated into the value of s(x)

Table A.4 Biregional Cohort-Survival Projection of the Urban and Rural Populations of India to the Year 2000

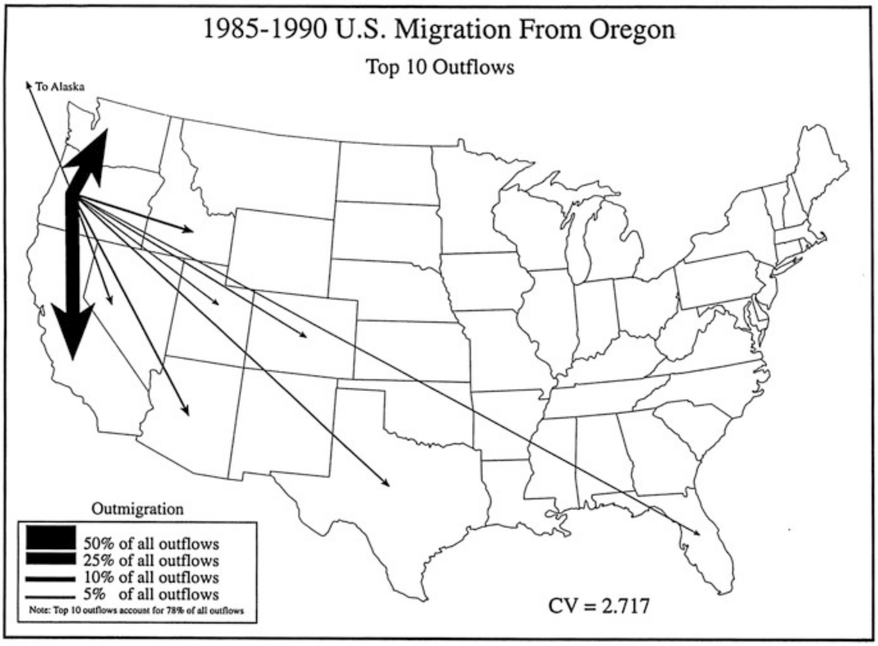
Age	*Population (in thousands)*[a]			*Age Composition*[a]		
x	*Total*	*Urban*	*Rural*	*Total*	*Urban*	*Rural*
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*
0	163,181	37,231	125,950	0.1553	0.1280	0.1657
5	127,799	31,167	96,632	0.1216	0.1072	0.1272
10	115,345	28,445	86,900	0.1098	0.0978	0.1144
15	102,424	26,237	76,187	0.0975	0.0902	0.1003
20	87,095	24,619	62,474	0.0829	0.0847	0.0822
25	74,851	22,608	52,242	0.0712	0.0777	0.0687
30	61,813	19,281	42,432	0.0588	0.0663	0.0560
35	74,163	22,884	51,279	0.0706	0.0787	0.0675
40	60,587	19,488	41,100	0.0577	0.0670	0.0541
45	41,023	13,735	27,288	0.0390	0.0472	0.0359
50	35,698	11,616	24,082	0.0340	0.0399	0.0317
55	31,563	9,182	22,381	0.0300	0.0316	0.0295
60	24,962	7,066	17,896	0.0238	0.0243	0.0236
65	19,072	5,566	13,407	0.0182	0.0191	0.0178
70+	31,118	11,676	19,443	0.0296	0.0402	0.0256
Total	1,050,692	290,799	759,892	1.0000	1.0000	1.0000
Share	1.0000	0.2768	0.7232			
Annual growth rate	0.0210	0.0274	0.0186			

[a]Slight differences are due to independent rounding
Source: Rogers (1985)

Table A.5 Age-Disaggregated Projections of Observed Populations: India

100U	r_u	m_u	$m_u/r_u x$ *100*	*T*
19.9	0.037	0.017	47.1	1970
21.6	0.035	0.015	42.8	1975
23.3	0.033	0.014	41.2	1980
27.7	0.025	0.009	33.8	2000
30.1	0.023	0.006	28.2	2020
33.8	0.019	0.004	19.8	Stability

Source: Rogers (1985)



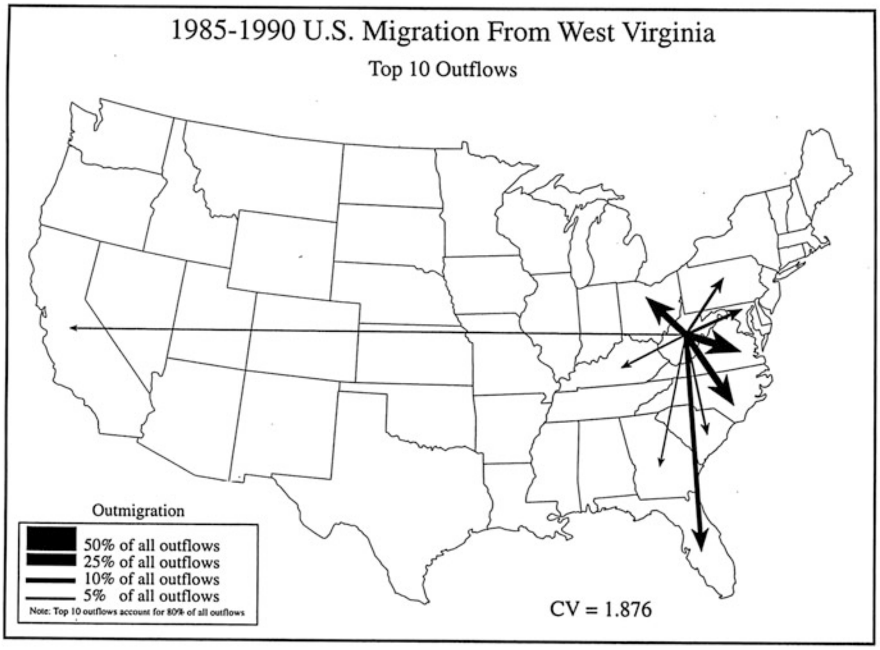


Fig. A.1 CVs for the top 10 1985 U.S outmigration flows from Oregon and West Virginia. (Source: Rogers and Raymer 1998)

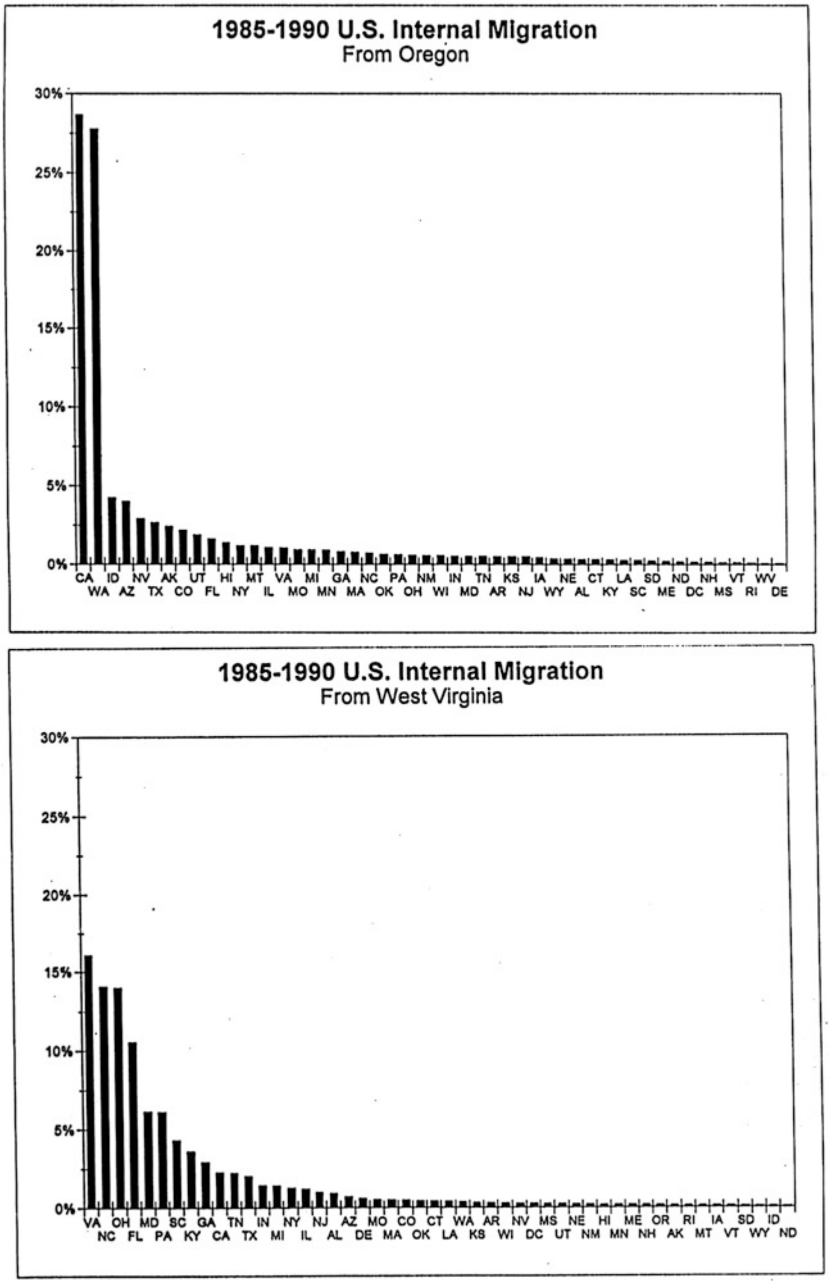


Fig. A.2 Histogram for the 1985–1990 U.S outmigration flow from Oregon and West Virginia. (Source: Rogers and Raymer 1998)

Component	Parameter	Parameter Value	Age	Pred (x)
Constant	a0	0.0030	0	0.02300
			5	0.01513
Negative exponential	a1	0.0200	10	0.01036
	alpha1	0.1000	15	0.00752
			20	0.02778
Double exponential	a2	0.0600	25	0.03643
	alpha2	0.1000	30	0.02567
	mu2	20.0000	35	0.01696
	lambda 2	0.4000	40	0.01148
			45	0.00815
			50	0.00612
			55	0.00489
			60	0.00415
			65	0.00370
			70	0.00342
			75	0.00326
			80	0.00316

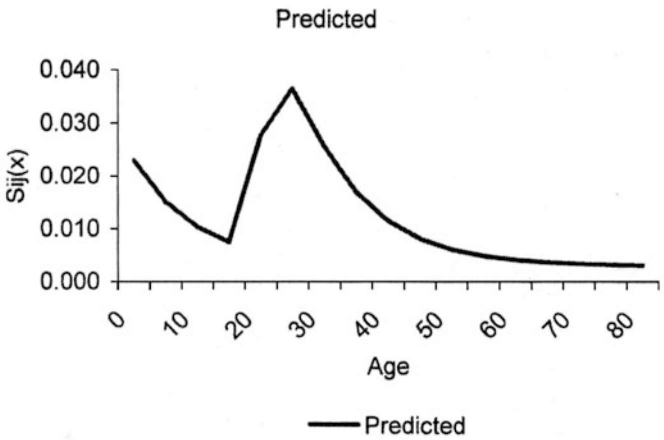


Fig. A.3 Parameters and age profile of the standard model migration schedule

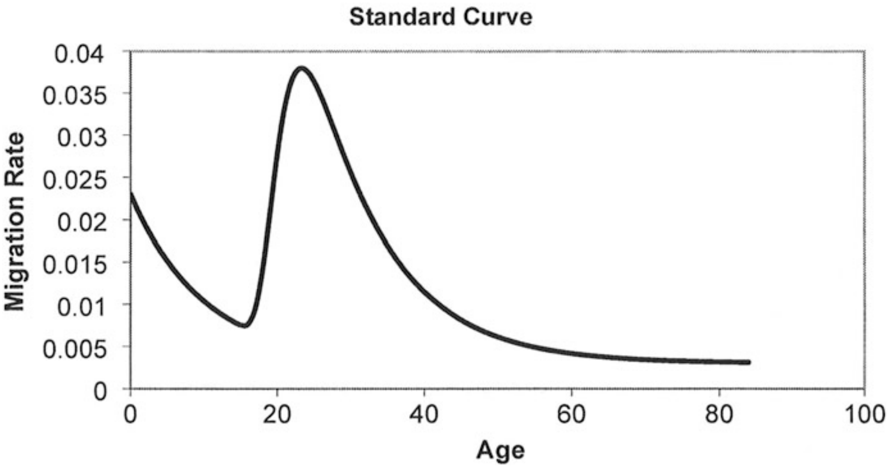


Fig. A.4 The standard migration age profile

Appendix B
An Introduction to Matrix Algebra

In this appendix we introduce the algebra of matrices. Matrices are used in such diverse fields as statistics, economics, sociology, engineering, and atomic physics. Formulating problems in matrix form often confers both a notational and an analytical advantage: (i) economy in notation often leads to insights that otherwise may have been obscured by a more complicated expression, and (ii) a matrix formulation of a problem places at our disposal a large collection of theorems that have been proved about matrices. Occasionally, these two advantages interact to suggest important conclusions that otherwise might be difficult to establish.

B.1 Definitions and Notation

A matrix is a rectangular array of numbers arranged in rows and columns, as in

$$\underset{m\times n}{\boldsymbol{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \tag{B.1}$$

A matrix does not have a numerical value; it is simply a convenient way to represent tabular arrangements of numbers These numbers are called the elements of the matrix and are represented by double-subscripted lower case letters. The two subscripts denote, respectively, the row and column position of the element in the rectangular arrangement. For example, a_{21} denotes the number that occupies the position in the second row and first column in the matrix A.

The matrix in equation (B.1) has 2 rows and 4 columns. It therefore be said to be of order 2 by 4 (generally written as $m \times n$).

Thus we may adopt the more compact form

$$\underset{m\times n}{\boldsymbol{A}} = \left[a_{ij} \right]_{m\times n} \tag{B.2}$$

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A. Rogers, *Applied Multiregional Demography Through Problems*,
https://doi.org/10.1007/978-3-030-38215-5

Two matrices are equal if, and only if, they are of the same order if their corresponding elements are equal.

$$\underset{n\times m}{\boldsymbol{A}'} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{bmatrix} \tag{B.3}$$

obtained by interchanging the rows and columns of $\boldsymbol{A}$, is defined to be the transpose of $\boldsymbol{A}$. Thus, if $\boldsymbol{A}$ is an $m \times n$ matrix, then the transpose of $\boldsymbol{A}$, $\boldsymbol{A}'$, say, is the $n \times m$ matrix $\boldsymbol{B}$ with $b_{ji} = a_{ij}$ for $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$

A matrix with only a single column is called a column vector and will be denoted with braces as

$$\underset{4\times 1}{\{\boldsymbol{a}\}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{B.4}$$

Similarly, a matrix with only a single row is called a row vector. In consonance with the notation for a transposed matrix, these will be denoted by

$$\underset{1\times 4}{\{\boldsymbol{a}\}'} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \tag{B.5}$$

A matrix with only a single row and a single column contains only one element and is simply a number or scalar.

Using the definition of column and row vectors, we may rewrite the matrix $\boldsymbol{A}$ as a set of n column vectors:

$$\underset{m\times n}{\boldsymbol{A}} = \begin{bmatrix} \{\boldsymbol{a}_1\} & \{\boldsymbol{a}_2\} & \{\boldsymbol{a}_3\} & \{\boldsymbol{a}_4\} \end{bmatrix} \tag{B.6}$$

or, alternatively, as a set of m row vectors:

$$\underset{m\times n}{\boldsymbol{A}} = \begin{bmatrix} \{\boldsymbol{a}_1\}' \\ \{\boldsymbol{a}_2\}' \\ \{\boldsymbol{a}_3\}' \\ \{\boldsymbol{a}_4\}' \end{bmatrix} \tag{B.7}$$

B.2 Simple Matrix Operations

The operations of addition, subtraction, and multiplication of ordinary algebra may be carried over to matrices with slight modification. The matrix analog of division, however, is considerably more complicated. Its description, therefore, will be postponed until later.

B.2.1 Matrix Addition

Matrix addition is only defined for matrices that are of the same order. Such matrices are said to be conformable for addition. Each element of their sum is formed by adding the elements in the corresponding positions in the matrices to be summed.

Note that matrix addition is commutative and associative, that is,

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A} \tag{B.8}$$

and

$$(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{D} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{D}) \tag{B.9}$$

Also, it is easily demonstrated that the transpose of a sum is the sum of the individual transposes:

$$(\boldsymbol{A} + \boldsymbol{B})' = \boldsymbol{A}' + \boldsymbol{B}'$$

B.2.2 Matrix Subtraction

Matrix subtraction is defined in a manner exactly analogous to that of matrix addition. We first designate by $-\boldsymbol{B}$ the matrix in which every element is the negative of the corresponding element in $\boldsymbol{B}$ and then proceed as in addition.

B.2.3 Scalar Multiplication

As in ordinary algebra, multiplication by a number or scalar may be viewed as the addition or subtraction of several identical quantities. For example, the sum of two equal matrices may be obtained by multiplying every element of one or the matrices by two.

Since scalar multiplication may be expressed in terms or addition and subtraction of matrices, it also must possess their commutative and distributive properties. Thus

$$c\boldsymbol{A} = \boldsymbol{A}c \tag{B.10}$$

and

$$c(\boldsymbol{A} + \boldsymbol{B}) = c\boldsymbol{A} + c\boldsymbol{B} \tag{B.11}$$

The two matrices

$$\boldsymbol{A} = \begin{bmatrix} 1\ 2\ 4 \\ 0\ 7\ 3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0\ 1\ 5 \\ 3\ 2\ 1 \end{bmatrix}$$

may be added to form the sum

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 1\ 3\ 9 \\ 3\ 9\ 4 \end{bmatrix}$$

Their difference is

$$\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 5 & 2 \end{bmatrix}$$

and the difference of a particular weighted combination of $\boldsymbol{A}$ and $\boldsymbol{B}$ is, for example,

$$\begin{aligned} \boldsymbol{E} &= 3\boldsymbol{A} - 2\boldsymbol{B} \\ &= \begin{bmatrix} 3 & 6 & 12 \\ 0 & 21 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 10 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ -6 & 17 & 7 \end{bmatrix} \end{aligned}$$

B.2.4 Matrix Multiplication

One might choose to define the multiplication of two matrices in a number of different ways, such as simply multiplying the corresponding elements of the two matrices. It turns out, however, that such a definition does not lead to practically useful results. Matrices have been used a great deal in problems involving simultaneous linear equations and linear transformations. As a result, matrix multiplication has been defined so as to facilitate such operations. consider, for example, the two simultaneous equations

$$\begin{aligned} 3x_1 + 5x_2 &= 15 \\ 2x_1 + 4x_2 &= 8 \end{aligned} \tag{B.12}$$

We may express these two equations more compactly in matrix form, as follows:

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \end{bmatrix} \tag{B.13}$$

The economy of notation, of course, increases with the order of the matrix. However, for equations (B.12) and (B.13) to be equivalent expressions, we need to adopt a definition of matrix multiplication that equates the left-hand sides of both equations. The following definition satisfies this requirement.

If

$$\underset{m\times n}{\boldsymbol{A}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and

$$\underset{n\times p}{\boldsymbol{B}} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

then

$$\underset{m\times p}{\boldsymbol{C}} = \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times p}{\boldsymbol{B}} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

The element in the ith row and jth column of the matrix product of $\boldsymbol{A}$. and $\boldsymbol{B}$, that is, c_{ij}, is formed by multiplying the elements of the ith row of $\boldsymbol{A}$ by the corresponding elements in the jth column of $\boldsymbol{B}$, and adding. This leads to the following two important observations:

1. Matrix multiplication is defined only for matrices that are conformable for multiplication. Two matrices are said to be conformable for multiplication if the number of columns in the first is equal to the number of rows in the second.
2. The matrix product of two matrices that are conformable for multiplication will always have as many rows as the first and as many columns as the second.

Both of the above observations may be represented schematically as

$$\left[m \times n\right]\left[n \times p\right] = \left[m \times p\right]$$

Unlike multiplication in ordinary algebra, matrix multiplication is not commutative in most instances. That is,

$$\boldsymbol{AB} \neq \boldsymbol{BA}$$

Indeed, frequently the reverse multiplication is undefined. This may be seen by reversing the matrices in equation (B.15). Thus

$$\left[n \times p\right]\left[m \times n\right]$$

and the multiplication cannot be performed since the two matrices are not conformable for multiplication. If $p = m$, the multiplication is defined: however, the matrix product has a dimension of $n \times n$ instead of the previous $m \times m$. And if $m = n$, then we satisfy a necessary condition for two matrices to be commutative, but we do not have a sufficient condition. This will be illustrated in the examples that follow.

It may be readily determined that the associative and distributive laws hold for matrix multiplication. That is,

$$(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$$

$$\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$$

$$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$$

Finally, the following not intuitively obvious result may be established for the transpose of a matrix product:

$$(\boldsymbol{AB})' = \boldsymbol{B}'\boldsymbol{A}'$$

B.3 Special Matrices

Many special types of matrices occur repeatedly in matrix analysis and accordingly have received special names. A square matrix is one in which the number of rows equals the number of columns. Such a matrix is called an "mth-order" matrix, and the elements beginning from the upper left corner and proceeding in a diagonal direction to the lower right corner (that is, $a_{11}, a_{22}, \ldots, a_{mm}$) constitute its principal diagonal. Those immediately below the principal diagonal are said to be in the principal subdiagonal. The sum of the elements along the principal diagonal of a square matrix is called the trace of that matrix.

A symmetric matrix is a square matrix that is equal to its transpose, as, for example,

$$\begin{bmatrix} 5 & 0 & 3 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

A zero or null matrix is one in which all elements are zero. It fullfills a purpose of matrix algebra analogous to that served by zero in ordinary algebra.

A diagonal matrix is a square matrix with all elements, except those on the principal diagonal, equal to zero. A scalar matrix is a diagonal matrix with all elements on the principal diagonal equal to the same scalar quantity. If this scalar is unity, the matrix is called the identity matrix and is denoted by $\boldsymbol{I}$. It serves a purpose in matrix algebra analogous to that of the number one in scalar algebra. Examples of diagonal, scalar, and identity matrices, respectively, are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the above definitions, it may be observed that premultiplication by a diagonal matrix is equivalent to multiplying the elements of each row of the premultiplied matrix by the scalar in that row of the diagonal matrix. Postmultiplication by a diagonal matrix, on the other hand, results in the multiplication of the elements of each column in the postmultiplied matrix by the scalar in that column of the diagonal matrix. It should be clear that multiplication by the identity matrix does not alter the matrix that is multiplied:

$$\underset{m\times m}{\boldsymbol{A}}\ \underset{m\times m}{\boldsymbol{I}} = \underset{m\times m}{\boldsymbol{I}}\ \underset{m\times m}{\boldsymbol{A}} = \underset{m\times m}{\boldsymbol{A}}$$

Frequently it is necessary or expedient to deal with submtrices of a large matrix. In such situations, the large matrix may be partitioned into several smaller matrices, and dashed lines may be introduced to indicate the particular partitioning scheme that is adopted. For example,

$$\underset{4\times 5}{\boldsymbol{A}} = \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right] = \left[\begin{array}{c|c} \underset{2\times 3}{\boldsymbol{A}_{11}} & \underset{2\times 2}{\boldsymbol{A}_{12}} \\ \hline \underset{2\times 3}{\boldsymbol{A}_{21}} & \underset{2\times 2}{\boldsymbol{A}_{22}} \end{array}\right]$$

Partitioning lines always run across the entire matrix, and partitioned matrices may be added, subtracted, or multiplied, provided they are appropriately partitioned, that is, conformably. Thus, for example, we may postmultiply the above partitioned matrix $\boldsymbol{A}$ by the partitioned matrix

$$\underset{5\times 3}{\boldsymbol{B}} = \left[\begin{array}{cc|c} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ \hline b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \end{array}\right] = \left[\begin{array}{c|c} \underset{3\times 2}{\boldsymbol{B}_{11}} & \underset{3\times 1}{\boldsymbol{B}_{12}} \\ \hline \underset{3\times 2}{\boldsymbol{B}_{21}} & \underset{3\times 1}{\boldsymbol{B}_{22}} \end{array}\right]$$

to form the partitioned matrix

$$\underset{4\times 3}{\boldsymbol{C}} = \underset{4\times 5}{\boldsymbol{A}}\ \underset{5\times 3}{\boldsymbol{B}} = \left[\begin{array}{c|c} \underset{2\times 3}{\boldsymbol{A}_{11}} & \underset{2\times 2}{\boldsymbol{A}_{12}} \\ \hline \underset{2\times 3}{\boldsymbol{A}_{21}} & \underset{2\times 2}{\boldsymbol{A}_{22}} \end{array}\right] \left[\begin{array}{c|c} \underset{3\times 2}{\boldsymbol{B}_{11}} & \underset{3\times 1}{\boldsymbol{B}_{12}} \\ \hline \underset{3\times 2}{\boldsymbol{B}_{21}} & \underset{3\times 1}{\boldsymbol{B}_{22}} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \underset{2\times 3}{\boldsymbol{A}_{11}}\ \underset{3\times 2}{\boldsymbol{B}_{11}} + \underset{2\times 2}{\boldsymbol{A}_{12}}\ \underset{3\times 2}{\boldsymbol{B}_{21}} & \underset{2\times 3}{\boldsymbol{A}_{11}}\ \underset{3\times 1}{\boldsymbol{B}_{12}} + \underset{2\times 2}{\boldsymbol{A}_{12}}\ \underset{3\times 1}{\boldsymbol{B}_{22}} \\ \hline \underset{2\times 3}{\boldsymbol{A}_{21}}\ \underset{3\times 2}{\boldsymbol{B}_{11}} + \underset{2\times 2}{\boldsymbol{A}_{22}}\ \underset{3\times 2}{\boldsymbol{B}_{21}} & \underset{2\times 3}{\boldsymbol{A}_{21}}\ \underset{3\times 1}{\boldsymbol{B}_{12}} + \underset{2\times 2}{\boldsymbol{A}_{22}}\ \underset{3\times 1}{\boldsymbol{B}_{22}} \end{array}\right] = \left[\begin{array}{c|c} \underset{2\times 2}{\boldsymbol{C}_{11}} & \underset{2\times 1}{\boldsymbol{C}_{12}} \\ \hline \underset{2\times 2}{\boldsymbol{C}_{21}} & \underset{2\times 1}{\boldsymbol{C}_{22}} \end{array}\right]$$

B.4 The Inverse of a Square Matrix

The reciprocal of a number, a, say, in ordinary scalar algebra, is defined to be the number, b, say, which when multiplied by a produces unity. Thus

$$ab = ba = 1$$

and

$$b = \frac{1}{a} = a^{-1}$$

Analogously, in matrix algebra, we define the reciprocal of a square matrix, $\boldsymbol{A}$, say, to be the square matrix, $\boldsymbol{B}$, say, which when multiplied by $\boldsymbol{A}$ produces the identity matrix. That is,

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$$

and

$$\boldsymbol{B} = \boldsymbol{A}^{-1}$$

The reciprocal matrix is commonly referred to as the inverse matrix, and its use in matrix algebra is analogous to division in ordinary scalar algebra.

To illustrate the use of the inverse matrix in the solution of simultaneous equations, consider a set of equations of the form

$$\underset{m\times m}{\boldsymbol{A}}\ \underset{m\times 1}{\{\boldsymbol{x}\}} = \underset{m\times 1}{\{\boldsymbol{b}\}}$$

If $\boldsymbol{A}^{-1}$ exists, we may premultiply both sides of the above equation by it to find

$$\boldsymbol{A}^{-1}\boldsymbol{A}\{\boldsymbol{x}\} = \boldsymbol{A}^{-1}\{\boldsymbol{b}\}$$
$$\boldsymbol{I}\{\boldsymbol{x}\} = \boldsymbol{A}^{-1}\{\boldsymbol{b}\}$$

or

$$\{\boldsymbol{x}\} = \boldsymbol{A}^{-1}\{\boldsymbol{b}\}$$

B.5 Determinants

Our development of a formal method for obtaining the inverse of a matrix makes use of numbers called *determinants*. Hence, we interrupt our exposition of matrix analysis at this point, in order to define what a determinant is and to list some of its properties.

B.5.1 Definitions and Notations

Associated with every square matrix, $\boldsymbol{A}$, say, is a unique number, denoted by $|\boldsymbol{A}|$, called its *determinant*. This number is formally defined as the sum of the products of all possible permutations of the elements in the matrix, such that each product contains an element form each row and from every column. Each product in the sum receives a positive sign if the number of inversions, v, say, necessary to transform the particular permutation of the second subscripts $i, j, \ldots, p$ to the natural order $1, 2, \ldots, m$ is even, and a negative sign if it is odd. That is,

$$|\underset{m\times m}{\boldsymbol{A}}| = \sum^{m!} (-1)^v a_{1i} a_{2j} \ldots a_{mp}$$

where the second subscripts take on all possible $[m! = m \cdot (m-1) \ldots (2) \cdot (1)]$ permutations of the numbers $1, 2, \ldots, m$. An inversion is carried out simply by interchanging the positions of two adjacent numbers. Thus, for example, the sequence 3, 2, 1 may be transformed into the natural order by three inversions (that is, $v = 3$).

B.5.2 Examples

(i) $|a_{11}| = a_{11}$

(ii) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{12} - a_{12}a_{21}$

(iii) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

(iv) $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 3(4) - 5(2) = 2$

(v) $\begin{vmatrix} 3 & 2 & 5 \\ 4 & 1 & 0 \\ 0 & 2 & 2 \end{vmatrix} = 3(1)(2) + 2(0)(0) + 5(4)(2) - 5(1)(0) - 3(0)(2) - 2(4)(2) = 30$

B.5.3 Solving Homogeneous Linear Equation Systems

Assume a set of homogeneous simultaneous linear equations are solvable, that is, have solutions. Let's consider the conditions that must be met for this to be true.

A solution of the linear homogeneous equation system

$$\boldsymbol{Ax} = \mathbf{0} \tag{B.14}$$

that always exists is the null vector

$$\boldsymbol{Ax} = \mathbf{0} \tag{B.15}$$

This is called the *trivial solution*, and it rarely is off interest. Rather, we are almost always seeking non-trivial solutions and these occur only when the matrix $\boldsymbol{A}$ is singular (i.e., when the determinant of $\boldsymbol{A}$ is zero. This can be seen by premultiplying both sides of equation (B.14) by $\boldsymbol{A}^{-1}$ and using Cramer's Rule to express the solution of each unknown as the ratio of two determinants for homogenous equations, the determinant in the numerical solution will be zero because of the presence of the null column vector.

Hence,

$$x_1 = \frac{0}{|\boldsymbol{A}|} \quad x_2 = \frac{0}{|\boldsymbol{A}|} \quad \ldots x_m = \frac{0}{|\boldsymbol{A}|}$$

or

$$\boldsymbol{x} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\mathbf{0} \tag{B.16}$$

A solution of the nonhomogeneous linear equation system

$$\boldsymbol{Ax} = \boldsymbol{b} \tag{B.17}$$

will exist if the ranks of the coefficient matrix $\boldsymbol{A}$ and the augmented coefficient matrix

$$\hat{\boldsymbol{A}} = [\boldsymbol{Ab}]$$

are equal. The equations are consistent if

$$\text{rank}(\boldsymbol{A}) = \text{rank}(\hat{\boldsymbol{A}})$$

If not, the equations are *inconsistent*.

References

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