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The chapters of this book on linear operators on Hilbert spaces, Fourier analysis, and probability only introduce these huge subjects, all of which have a vast literature. This bibliography gives book recommendations for readers who want to go into these topics in more depth.

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