Appendix A
Mathematical Background

A.1 Approximations

Whenever there is an approximation sign in an equation of the main text, one of the approximations listed here will be used. The approximations will, therefore, be used without referring to any of the particular equations below.

A.1.1 Power Series Approximation

One version of Newton's generalized binomial theorem is (Rottmann 2003):

$$(1+x)^{n/m} = 1 + \frac{n}{m}x - \frac{n(m-n)}{2!m^2}x^2 + \frac{n(m-n)(2m-n)}{3!m^3}x^3 + \cdots \tag{A.1}$$

Often used approximations are based on keeping only the first two or three terms and are valid when $x \ll 1$:

$$\frac{1}{1+x} = (1+x)^{-1} \approx 1 - x + x^2 - x^3 + \cdots \tag{A.2}$$

$$\sqrt{1+x} = (1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots \tag{A.3}$$

$$1/\sqrt{1+x} = (1+x)^{-1/2} \approx 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots \tag{A.4}$$

S. Holm, *Waves with Power-Law Attenuation*,
https://doi.org/10.1007/978-3-030-14927-7

A.1.2 McLaurin Series for Trigonometric Functions

The argument is always expressed in radians in these formulas:

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots \tag{A.5}$$

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots \tag{A.6}$$

$$\tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 - \cdots \tag{A.7}$$

The small angle approximations use only a single term:

$$\sin\theta \approx \tan\theta \approx \theta. \tag{A.8}$$

$$\cos\theta \approx 1 \tag{A.9}$$

When the argument $\theta < 0.2$ radians $\approx 11.50\,^\circ$ the error in $\sin\theta$ is less than 0.7%, the error in $\tan\theta$ is less than 1.4%, and the error in $\cos\theta$ is less than 2%. In practice the approximate formulas are useful up to approximately 0.25 radians or 15 °.

A.2 Vector Operators

The gradient is a vector which points in the direction of greatest increase of a scalar field:

$$\text{grad}\,\mathbf{u} = \nabla\mathbf{u} = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{i} + \frac{\partial u}{\partial z}\mathbf{k}. \tag{A.10}$$

The divergence of a vector field is a scalar:

$$\text{div}\,\mathbf{u} = \nabla\cdot\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \tag{A.11}$$

The curl is a vector that describes the rotation of a 3-D vector field:

$$\text{curl}\,\mathbf{u} = \nabla\times\mathbf{u}. \tag{A.12}$$

The Laplacian of a function that varies in several dimensions is a scalar:

$$\nabla^2 u = \Delta u = \text{div grad}\,u = (\nabla\cdot\nabla)u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{A.13}$$

while the Laplacian of a vector is a vector:

$$\nabla^2 \mathbf{u} = \left(\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z\right). \tag{A.14}$$

A.3 Fourier Transform

The definition of the temporal Fourier transform as used in this book is:

$$F(\omega) = \mathscr{F}(f(t)) = \int_{\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t, \tag{A.15}$$

and the inverse is:

$$f(t) = \mathscr{F}^{-1}(F(\omega)) = \frac{1}{2\pi} \int_{\infty}^{\infty} F(\omega)\, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega. \tag{A.16}$$

The spatiotemporal Fourier transform is:

$$F(\mathbf{k}, \omega) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} f(\mathbf{x}, t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} d\mathbf{x}\mathrm{d}t, \tag{A.17}$$

with inverse

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{\infty}^{\infty} \int_{\infty}^{\infty} F(\mathbf{k}, \omega)\, \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} d\mathbf{k}\mathrm{d}\omega, \tag{A.18}$$

A.3.1 Differentiation Property

The Fourier transform of a differentiation is used whenever the dispersion relation for a wave equation is needed. It can be found in this way:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2\pi} \int_{\infty}^{\infty} F(\omega) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega\right] = \frac{1}{2\pi} \int_{\infty}^{\infty} F(\omega) \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega. \tag{A.19}$$

This means that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} [\mathrm{i}\omega F(\omega)] \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega = \mathscr{F}^{-1}[\mathrm{i}\omega F(\omega)] \tag{A.20}$$

and for repeated differentiation:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) \quad \leftrightarrow \quad (\mathrm{i}\omega)^n F(\omega) \tag{A.21}$$

A.3.2 Convolution and Differentiation

The derivative of a convolution, $f(t) * g(t)$, can be found by using the property that a convolution is equivalent to a multiplication in the Fourier domain, and combine that with the differentiation property from (A.21):

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(t) * g(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}^{-1}\left(F(\omega)G(\omega)\right) = \mathscr{F}^{-1}\left(\mathrm{i}\omega F(\omega)G(\omega)\right). \tag{A.22}$$

Thus the differentiation can be performed on either of the functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f * g) = \frac{\mathrm{d}f}{\mathrm{d}t} * g = f * \frac{\mathrm{d}g}{\mathrm{d}t}. \tag{A.23}$$

A.3.3 Exponential Function

The Fourier transform of a one-sided exponential, $\mathrm{e}^{-t/\tau}$, for $t, \tau > 0$ is:

$$F(\omega) = \int_0^\infty \mathrm{e}^{-t/\tau}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \frac{-1}{1/\tau + \mathrm{i}\omega}\mathrm{e}^{-(1/\tau+\mathrm{i}\omega)}\Big|_0^\infty = \frac{1}{1/\tau + \mathrm{i}\omega} \tag{A.24}$$

so

$$\mathrm{e}^{-t/\tau}, \quad t, \tau > 0 \quad \leftrightarrow \quad \frac{1}{\omega_0 + \mathrm{i}\omega}, \quad \omega_0 = 1/\tau. \tag{A.25}$$

Its Laplace transform is found by substituting $s = \mathrm{i}\omega$, and is $1/(\omega_0 + s)$. It is everywhere nonnegative for $s \geq 0$. This is an alternative to finding the sign of all derivatives, (3.12), for checking that a function is completely monotone according to (3.14) or Schilling et al. (2012, Def. 1.3).

A.3.4 Power Law

Finding the Fourier transform of a one-sided power law, $t^{-\alpha}$, for $t > 0$ starts with:

$$F(\omega) = \int_0^\infty t^{-\alpha}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t. \tag{A.26}$$

Substitution of $x = \mathrm{i}\omega t$ gives:

$$F(\omega) = (\mathrm{i}\omega)^{\alpha-1}\int_0^\infty x^{-\alpha}\mathrm{e}^{-x}\mathrm{d}x. \tag{A.27}$$

This expression is similar to the definition of the gamma function, a generalization of the factorial where $\Gamma(n+1) = n!$ for integer n:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \tag{A.28}$$

which is defined for all complex numbers, z, except the nonpositive integers. By letting $z = 1 - \alpha$ the result is

$$\frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \; t > 0 \quad \leftrightarrow \quad (i\omega)^{\alpha-1}, \tag{A.29}$$

which is valid for $\alpha \neq 1, 2, 3, \ldots$. As was the case for the exponential function of the previous section, the Laplace transform, $s^{\alpha-1}$ (Gradshteyn and Ryzhik 2014, 12.13-3), valid for $\alpha < 1$, is nonnegative for $s \geq 0$ showing that the power-law function is completely monotone.

A.3.5 Mittag-Leffler Function

The Mittag-Leffler function, E_α, is a generalization of the exponential function. It is defined as:

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \; 0 < \alpha \leq 1. \tag{A.30}$$

For $\alpha = 1$ it is the exponential function, $\exp(t)$.

The relaxation modulus for the fractional Zener model in Sect. 5.2.2 is expressed with a Mittag-Leffler function with a transformed argument:

$$e_\alpha(t) = E_\alpha(-t^\alpha) \sim \begin{cases} \exp\left[\frac{-t^\alpha}{\Gamma(1+\alpha)}\right] \approx 1 - \frac{t^\alpha}{\Gamma(1+\alpha)}, & t \to 0 \\ \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, & t \to \infty \end{cases} \tag{A.31}$$

The approximation for small time is a stretched exponential first proposed by Kohlrausch in 1854 and rediscovered by Williams and Watts in 1970. It is, therefore, also called the Kohlrausch–Williams–Watt function (Garrappa et al. 2016). For large values of t the Mittag-Leffler function approaches a power law (Mainardi 2014).

The following integral property was shown by Mittag-Leffler:

$$\int_0^\infty e^{-u} E_\alpha(u^\alpha z) du = \frac{1}{1-z}, \; \alpha > 0. \tag{A.32}$$

By substituting $u = \mathrm{i}\omega t$ and $u^\alpha z = -\lambda t^\alpha$, the Fourier transform can be found to be (Mainardi and Gorenflo 2000):

$$\mathrm{e}_\alpha(t;\lambda) = \mathrm{E}_\alpha(-\lambda t^\alpha), \;\; t > 0 \quad \leftrightarrow \quad \frac{(\mathrm{i}\omega)^{\alpha-1}}{(\mathrm{i}\omega)^\alpha + \lambda}. \tag{A.33}$$

This result could also have been found from the series expansion in (A.30) combined with the Fourier transform of the power law (A.29) applied to each term (Mainardi 2010, Appendix E).

As expected, the Fourier transform of a decaying exponential, (A.25), is recovered for the case of $\alpha = 1$ and $\lambda = \omega_0 = 1/\tau$.

The Laplace transform of the Mittag-Leffler function is $s^{\alpha-1}/\left(s^\alpha + \lambda\right)$. It is non-negative for $s \geq 0$ as are those for the exponential and power-law function of the previous section. This shows that the Mittag-Leffler function is a completely monotone function also.

The check that the sign of the derivatives of higher and higher orders alternate, (3.12), is simple to do for the exponential and power-law functions, but much harder for the Mittag-Leffler function. Unlike the two other functions, the check of the sign of the Laplace transform is the simplest way to check for complete monotonicity for Mittag-Leffler function.

A.4 Fractional Calculus: Time-Domain Interpretation

This is not a rigorous mathematical presentation of fractional calculus as in Podlubny (1999), but rather one which covers just enough in order to apply fractional calculus to physical problems. There are two interpretations of the fractional derivative, the first is in the frequency-domain building on the Fourier transform, and the second is an interpretation in the form of a convolution of an ordinary derivative and a causal memory function.

Although the Fourier definition of (1.15) is adequate for most of this book, its interpretation in the time domain is also given here.

A.4.1 Elementary Functions

The fractional derivative of some elementary functions are first listed. They are what one would expect that a formal definition should yield, and they are found by extending the standard formulas for integer derivatives to cover the non-integer cases as well (Sokolov et al. 2002), (Herrmann 2014, Chap. 3), and (Meerschaert and Sikorskii 2012, Chap. 2).

A.4.1.1 Exponential

Direct extension of the formula for the integer-order derivative gives:

$$\frac{d^\alpha}{dx^\alpha} e^{kx} = k^\alpha e^{kx}, \quad k \geq 0 \tag{A.34}$$

A.4.1.2 Sinusoid

This result follows from the result for the exponential function when k is allowed to be imaginary:

$$\frac{d^\alpha}{dx^\alpha} \sin kx = k^\alpha \sin\left(kx + \frac{\pi}{2}\alpha\right), \quad k \geq 0 \tag{A.35}$$

A.4.1.3 Power Law

$$\frac{d^\alpha}{dx^\alpha} x^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}, \quad x \geq 0, \quad k \neq -1, -2, -3, \ldots \tag{A.36}$$

which agrees with the rule for integer-order derivatives when the gamma function is taken as the non-integer extension of the factorial:

$$\frac{d^n}{dx^n} x^k = \frac{k!}{(k-n)!} x^{k-n}. \tag{A.37}$$

A.4.2 Convolution Interpretation: Two Flavors

The Fourier property will now be used to develop a time-domain interpretation of the fractional derivative. It starts with an integer order which is the nearest integer above the fractional order, α. The amount that m is above α is the fractional part where $m - \alpha > 0$. The Fourier transform definition of (1.15) can then be decomposed into an integer part and a fractional part:

$$\mathscr{F}\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega) = (i\omega)^m F(\omega)(i\omega)^{\alpha-m}. \tag{A.38}$$

The first part has as its inverse transform an ordinary derivative. But how can the second fractional part be interpreted in terms of an inverse Fourier transform? In this case, the property of (A.29) can be rewritten so that the kernel has the same form as above:

$$h(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, t > 0 \Leftrightarrow H(\omega) = (i\omega)^{\alpha-m}. \tag{A.39}$$

This result is inserted back into (A.38) using the property that a product in the frequency domain is the same as a convolution in the time domain. The result is that the fractional derivative can be interpreted as a convolution of a derivative of order m which is the nearest integer above α, and a memory function, properly scaled by the gamma function:

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{\mathrm{d}^{m} f(t)}{\mathrm{d}t^{m}} * \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}. \tag{A.40}$$

The convolution above can be written in two different ways, depending on the order of the operations. First, for terminology, the general expression for the fractional derivative of order α is written as:

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = {}_a\mathrm{D}_t^{\alpha} f(t), \tag{A.41}$$

where a and t are limits in the defining integral. If the order is negative, $\alpha < 0$, then this is a fractional integration.

A.4.2.1 Riemann–Liouville Fractional Derivative

The Riemann–Liouville fractional derivative of order $\alpha \in R$, $m-1 \leq \alpha < m$ is:

$${}_a\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau \tag{A.42}$$

Here the convolution operation of (A.40) is the first operation to be performed followed by the integer-order derivation. This definition yields the results given for the elementary function in Sect. A.4.1.

This definition results in a peculiarity. The derivative of a constant, unity, can be found by setting $k = 0$ in the power law of (A.36). The result is:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} 1 = \frac{1}{1-\alpha} x^{-\alpha}, \tag{A.43}$$

which differs from the expected value of 0. This discrepancy is resolved with the Caputo derivative.

A.4.2.2 Caputo Fractional Derivative

This derivative is credited to Caputo (1967), but Rossikhin (2010) points out that already Gerasimov in 1948 introduced it and that it also exists in Bland (1960). Even (Liouville 1832, Sect. III (6), page 10) gives its definition.[1]

The order of operations is reversed in the Caputo fractional derivative. This is allowed because of the differentiation property of a convolution of Appendix A.3.2:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau. \tag{A.44} $$

First an integer-order derivative of order m is performed, and then convolution with the power-law memory kernel takes place. When the function is a constant, the first integer-order derivative will ensure that the result is zero, thus giving the expected result of 0, opposite to what the Riemann–Liouville derivative does.

The fractional derivative is not a local operator as the integer derivative is. It is a function of the entire history and therefore it requires much more computations than an ordinary derivative. This property is independent of which definition is used. Here the Caputo derivative is interpreted in this light, assuming that the lower limit $a = -\infty$:

$$ {}_{-\infty}^{\ C}\mathrm{D}_t^\alpha f(t) = \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial^m f(t)}{\partial t^m} * g_{m-\alpha}(t). \tag{A.45} $$

The memory function is then:

$$ g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, \ t > 0. \tag{A.46} $$

Example A.1 Fractional derivative of order 0.1

In this example the Caputo derivative of order $0 \le \alpha < 1$, i.e., $m = 1$, will be interpreted as the order approaches either limit. The memory function of (A.46) then reduces to:

$$ g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \ t > 0. \tag{A.47} $$

This function is called the kernel of the Abel operator in Rabotnov (1980, Sect. I.5), Koeller (1984). The Caputo fractional derivative, (A.44), in this case is:

$$ \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial f(t)}{\partial t} * g_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{f^{(1)}(t)}{(t-\tau)^\alpha} \mathrm{d}\tau. \tag{A.48} $$

[1] Thanks to Dumitru Baleanu for drawing my attention to this reference.

As $\alpha \to 0$ this should approach the function itself and that can be seen as follows:

$$\left.\frac{\partial^\alpha f(t)}{\partial t^\alpha}\right|_{\alpha=0} = \int_{-\infty}^{t} \frac{f^{(1)}(t)}{(t-\tau)^0}\mathrm{d}\tau = f(t). \quad \text{(A.49)}$$

As $\alpha \to 1$, the Caputo derivative should approach the first-order derivative as seen in the following way:

$$\left.\frac{\partial^\alpha f(t)}{\partial t^\alpha}\right|_{\alpha=1} = \int_{-\infty}^{t} f^{(1)}(t)\delta(\tau)\mathrm{d}\tau = f^{(1)}(t). \quad \text{(A.50)}$$

The memory kernel of (A.48) or the Abel operator's kernel with reversed time is illustrated in Fig. A.1. Its interpretation is:

- As α approaches 1 the fractional derivative is supposed to turn into an integer-order derivative. This can be seen by letting $\alpha = 1 - \varepsilon^+ < 1$ and as $\Gamma(\varepsilon^+) \to \infty$ for $\varepsilon^+ \to 0$, the memory kernel approaches an impulse and the memory is gone as expected.
- As α approaches 0, the kernel turns into an ordinary integral, i.e., one with infinite memory.

The last case as well as intermediate ones are illustrated in Fig. A.1.

A.4.2.3 Comparison of the Riemann–Liouville and the Caputo Definitions

In addition to the different handling of constants, the difference between the two becomes evident whenever numerical time-domain implementations are required. The Riemann–Liouville derivative turns out to require initialization of derivatives of non-integer orders. This is different for the Caputo fractional derivative which will require initialization of integer-order derivatives: $f^{(k)}(0), k = 0, 1, \ldots, m-1$. They are the ones that usually have physical meaning and therefore the Caputo definition is often simpler to use in numerical implementations.

The fact that different definitions of the fractional derivative may give different results could explain why it has taken a long time for the fractional derivative to be accepted.

The fractional derivative may be extended to the case of a complex order of differentiation (Nigmatullin et al. 2007) or to a variable order (Lorenzo and Hartley 2002; Ostalczyk and Rybicki 2008). These are interesting extensions but are beyond the scope of this book.

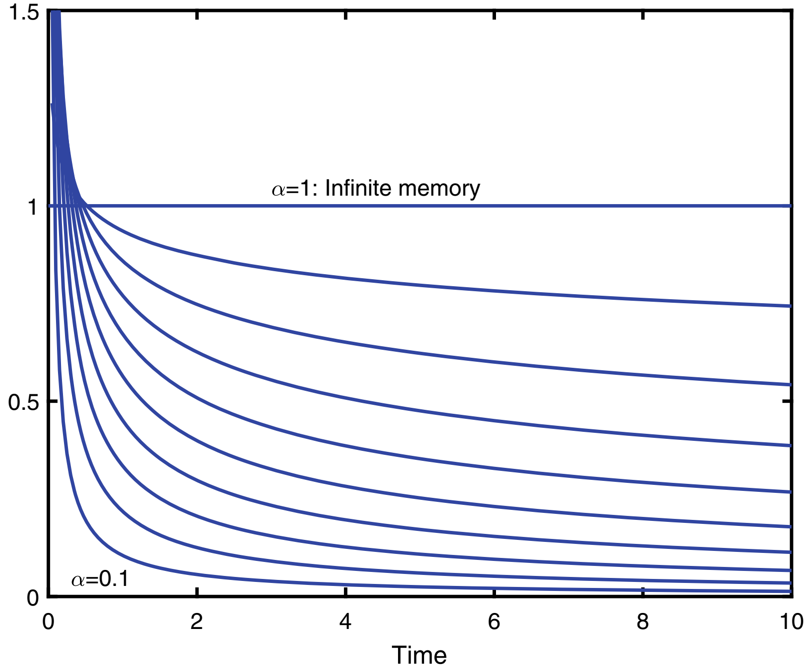

Fig. A.1 Power-law memory kernel in the convolution function of (A.48). The curves illustrate values of α from 1 in the upper curve to 0.1 in the lower curve in increments of 0.1 (inspired by Treeby and Cox 2010)

A.4.3 Fractional Integral

The fractional integral of order α may be written either with its own symbol or as a negative-order derivative, showing that it is the inverse of a fractional derivative. It is also defined via the Fourier relationship:

$$\mathscr{F}\{\mathrm{I}^{\alpha}[f(t)]\} = \mathscr{F}\{\frac{\partial^{-\alpha} f(t)}{\partial t^{-\alpha}}\} = (\mathrm{i}\omega)^{-\alpha}\mathscr{F}\{f\}. \tag{A.51}$$

The integral representation is:

$$\mathrm{I}^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau, \quad \alpha > 0. \tag{A.52}$$

which is similar to both the Riemann–Liouville and Caputo derivatives with $m = 0$. Unlike for the fractional derivative, there is no need to divide the order into an integer part, m, and a fractional part, $m - \alpha$ in the fractional integral. This formula is a generalization to non-integer α of the Cauchy formula for repeated integration:

$$I^n[f(t)] = \frac{1}{(n-1)!}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}}\mathrm{d}\tau, \quad n = 1, 2, \dots. \tag{A.53}$$

The Cauchy formula may also be used as a basis for deriving the Riemann–Liouville definition of the fractional derivative (Oldham and Spainer 1974, Sect. 3.2).

A.4.4 The Fractional Laplacian

This is a spatial fractional derivative of order α written as $(-\nabla^2)^{\alpha/2}$. The Fourier transform of $(-\nabla^2)^{\alpha/2} f(x)$ is $-|k|^\alpha F(k)$. Unlike the temporal one, there is no preferred direction and it can, therefore, be expressed as a convolution with the Riesz potential $|x-\xi|^{-\alpha}$. For further details consult (Chen and Holm 2004).

A.5 Sign Convention in Fourier Transform

There are two sign conventions that are possible for the Fourier transform. The choice of either one affects the results and makes comparisons difficult as signs may change in the formulas. Therefore the consequences of the two sign conventions for fractional derivatives are listed in Table A.1.

Convention 1 is used in this book and among others in these papers: Geertsma and Smit (1961), Stoll (1977), Fellah and Depollier (2000), Wismer (2006), Holm and Sinkus (2010), Caputo et al. (2011), Holm and Näsholm (2011), Näsholm and Holm (2011), Zhang et al. (2012), Fellah et al. (2013), Holm and Näsholm (2014) as well as in the book Mainardi (2010).

The opposite sign, that of convention 2, is used among others in Johnson et al. (1987), Nachman et al. (1990), Szabo (1994), Chen and Holm (2003, 2004), Kelly et al. (2008), Treeby and Cox (2010), Kowar et al. (2011) as well as in the books Podlubny (1999), Hovem (2012).

To complicate matters further some papers may even use a hybrid of the two conventions shown here, e.g. Biot (1956) which uses $u_3(x,t) = \exp(\mathrm{i}(kx+\omega t))$ for a plane wave.

A.6 Bernstein Functions

This section was written in collaboration with Martin B. Holm and is taken from Holm and Holm (2017).

Table A.1 Sign conventions for Fourier transform. Convention 1 is used in this book. A shorter version of this table was first published in Holm and Näsholm (2014)

Property	Convention 1	Convention 2
Fourier transform:	$\int u(x,t)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathrm{d}x\mathrm{d}t$	$\int u(x,t)\mathrm{e}^{\mathrm{i}(\omega t-kx)}\mathrm{d}x\mathrm{d}t$
$\partial^\eta u_1/\partial t^\eta$	$(\mathrm{i}\omega)^\eta u_1$	$(-\mathrm{i}\omega)^\eta u_2$
$\partial^\eta u_1/\partial x^\eta$	$(-\mathrm{i}k)^\eta u_1$	$(\mathrm{i}k)^\eta u_2$
$(-\nabla^2)^{\eta/2}$	$-\lvert k\rvert^\eta$	$-\lvert k\rvert^\eta$
Plane wave:	$u_1(x,t)=\mathrm{e}^{\mathrm{i}(\omega t-kx)}$	$u_2(x,t)=\mathrm{e}^{\mathrm{i}(kx-\omega t)}$
Wave number:	$k=\beta_k-\mathrm{i}\alpha_k$	$k=\beta_k+\mathrm{i}\alpha_k$
Attenuated wave:	$\mathrm{e}^{-\alpha_k x}\mathrm{e}^{\mathrm{i}(\omega t-\beta_k x)}$	$\mathrm{e}^{-\alpha_k x}\mathrm{e}^{\mathrm{i}(\beta_k x-\omega t)}$

A.6.1 Definition and Representation of a Complete Bernstein Function

A function $f:(0,\infty)\to R$ is a Bernstein function if $f(t)\geq 0$, all derivatives exist, and the sign of the derivatives alternate as in (4.29). A Bernstein function has the following representation

$$f(t)=a+bt+\int_0^\infty(1-\mathrm{e}^{tr})\hat{\mu}(\mathrm{d}r). \tag{A.54}$$

where $a,b\geq 0$ and $\hat{\mu}$ is a measure on $(0,\infty)$ satisfying $\int_0^\infty \min\{1,r\}\hat{\mu}(\mathrm{d}r)<\infty$.

A Bernstein function is said to be complete if the Lévy measure $\hat{\mu}$ has a completely monotone density. A complete Bernstein function $f:(0,\infty)\to R$ has the following representation (Schilling et al. 2012, Remark 6.4):

$$f(t)=a+bt+t\int_0^\infty\frac{1}{t+r}\sigma(\mathrm{d}r), \tag{A.55}$$

where $a,b\geq 0$, $a=f(0)$, $b=\lim_{t\to\infty}f(t)/t$ and σ is a measure on $(0,\infty)$ such that $\int_0^\infty 1/(1+r)\sigma(\mathrm{d}r)<\infty$.

A.6.2 Proof of Complete Bernstein Property of Wavenumber

The proof of the numbered points in Sect. 4.3.1 proceeds in these steps (Seredyńska and Hanyga 2010).

1. $G(t)\in LICM$ (Locally integrable CM) $\Rightarrow \tilde{G}(s)\in CM$. If $G(t):[0,\infty)\to[0,\infty)$ is completely monotone and locally integrable, then by Schilling et al. (2012, Theorem 1.4), the Laplace transform $\tilde{G}(s)$ is completely monotone.

2. $\tilde{G}(s) \in CM \Rightarrow s\tilde{G}(s) \in CBF$. If $\tilde{G}(s) \in CM$, then it is has a Stieltjes representation: $\tilde{G}(s) = \frac{a}{s} + b + \int_0^\infty \frac{1}{s+r}\mu(\mathrm{d}r)$ where $a, b \geq 0$ are nonnegative constants and μ is a measure on $(0, \infty)$ such that $\int_0^\infty 1/(1+r)\mu(\mathrm{d}r) < \infty$, see Schilling et al. (2012, Def. 2.1). It follows that $s\tilde{G}(s) = a + bs + \int_0^\infty \frac{s}{s+r}\mu(\mathrm{d}r)$ is a complete Bernstein function since it is equivalent to (A.55).
3. $s\tilde{G}(s) \in CBF \Rightarrow \sqrt{s\tilde{G}(s)} \in CBF$. Schilling et al. (2012, Corr. 7.6) states that if f_1, f_2 are CBF, then $f_1(f_2)$ is CBF. Since s^α with $0 \leq \alpha \leq 1$ is CBF, $\sqrt{s\tilde{G}(s)}$ is also CBF.
4. $\sqrt{s\tilde{G}(s)} \in CBF \Rightarrow s/\sqrt{s\tilde{G}(s)} \in CBF$. Schilling et al. (2012, Theorem 6.2) states that $\sqrt{s\tilde{G}(s)}/s$ is a Stieltjes function if $\sqrt{s\tilde{G}(s)}$ is CBF. Furthermore, the inverse of a Stieltjes function is a CBF by Schilling et al. (2012, Theorem 7.3). As a result, $s/\sqrt{s\tilde{G}(s)}$ is CBF.

References

M.A. Biot, Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range. J. Acoust. Soc. Am. **28**(2), 168–178 (1956)

D. Bland, *The Theory of Linear Viscoelasticity* (Pergamon Press, Oxford, 1960)

M. Caputo, J.M. Carcione, F. Cavallini, Wave simulation in biologic media based on the Kelvin-Voigt fractional-derivative stress-strain relation. Ultrasound Med. Biol. **37**(76), 996–1004 (2011)

M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II. Geophys. J. Int. **13**(5), 529–539 (1967)

W. Chen, S. Holm, Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency. J. Acoust. Soc. Am. **115**(4), 1424–1430 (2004)

W. Chen, S. Holm, Modified Szabo's wave equation models for lossy media obeying frequency power law. J. Acoust. Soc. Am. **114**(5), 2570–2574 (2003)

M. Fellah, Z.E.A. Fellah, F. Mitri, E. Ogam, C. Depollier, Transient ultrasound propagation in porous media using Biot theory and fractional calculus: Application to human cancellous bone. J. Acoust. Soc. Am. **133**(4), 1867–1881 (2013)

Z.E.A. Fellah, C. Depollier, Transient acoustic wave propagation in rigid porous media: A time-domain approach. J. Acoust. Soc. Am. **107**(2), 683–688 (2000)

R. Garrappa, F. Mainardi, G. Maione, Models of dielectric relaxation based on completely monotone functions. Fract. Calc. Appl. Anal. **19**(5), 1105–1160 (2016)

J. Geertsma, D.C. Smit, Some aspects of elastic wave propagation in fluid-saturated porous solids. Geophysics **26**(2), 169–181 (1961)

I.S. Gradshteyn, I.M. Ryzhik (2014), *Table of Integrals, Series, and Products* (Academic, New Jersey, 2014). Fourth ed. by Y.V. Geronimus, M.Y. Tseytlin, edited by A. Jeffrey

R. Herrmann, *Fractional Calculus: An Introduction for Physicists* (World Scientific, Singapore, 2014)

S. Holm, M.B. Holm, Restrictions on wave equations for passive media. J. Acoust. Soc. Am. **142**(4), (2017)

S. Holm, R. Sinkus, A unifying fractional wave equation for compressional and shear waves. J. Acoust. Soc. Am. **127**, 542–548 (2010)

S. Holm, S.P. Näsholm, A causal and fractional all-frequency wave equation for lossy media. J. Acoust. Soc. Am. **130**(4), 2195–2202 (2011)

S. Holm, S.P. Näsholm, Comparison of fractional wave equations for power law attenuation in ultrasound and elastography. Ultrasound. Med. Biol. **40**(4), 695–703 (2014)
J.M. Hovem, *Marine Acoustics: The Physics of Sound in Underwater Environments* (Peninsula publishing, Los Altos, 2012)
D.L. Johnson, J. Koplik, R. Dashen, Theory of dynamic permeability and tortuosity in fluid-saturated porous media. J. Fluid Mech. **176**, 379–402 (1987)
J.F. Kelly, R.J. McGough, M.M. Meerschaert, Analytical time-domain Green's functions for power-law media. J. Acoust. Soc. Am. **124**(5), 2861–2872 (2008)
R. Koeller, Applications of fractional calculus to the theory of viscoelasticity. J. Appl. Mech. **51**(2), 299–307 (1984)
R. Kowar, O. Scherzer, X. Bonnefond, Causality analysis of frequency-dependent wave attenuation. Math. Meth. Appl. Sci. **34**(1), 108–124 (2011)
J. Liouville, Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions (Memo on some questions of geometry and mechanics and on a new kind of calculus to resolve these questions). J. Ec. Polytech. **13**, 1–400 (1832)
C.F. Lorenzo, T.T. Hartley, Variable order and distributed order fractional operators. Nonlinear Dyn. **29**(1–4), 57–98 (2002)
F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelesticity: An Introduction to Mathematical Models* (Imperial College Press, London, 2010)
F. Mainardi, On some properties of the Mittag-Leffler function $\mathrm{E}_\alpha(-t^\alpha)$, completely monotone for $t>0$ with $0<\alpha<1$, in *Discrete and Continuous Dynamical Systems, Series B* (2014), pp. 2267–2278
F. Mainardi, R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes. J. Comput. Appl. Math. **118**(1), 283–299 (2000)
M.M. Meerschaert, A. Sikorskii, *Stochastic Models for Fractional Calculus*, vol. 43 (Walter de Gruyter, Berlin, 2012)
A.I. Nachman, J.F. Smith III, R.C. Waag, An equation for acoustic propagation in inhomogeneous media with relaxation losses. J. Acoust. Soc. Am. **88**, 1584–1595 (1990)
R. Nigmatullin, A. Arbuzov, F. Salehli, A. Giz, I. Bayrak, H. Catalgil-Giz, The first experimental confirmation of the fractional kinetics containing the complex-power-law exponents: Dielectric measurements of polymerization reactions. Phys. B **388**(1–2), 418–434 (2007)
S.P. Näsholm, S. Holm, Linking multiple relaxation, power-law attenuation, and fractional wave equations. J. Acoust. Soc. Am. **130**(5), 3038–3045 (2011)
K.B. Oldham, J. Spanier, *The Fractional Calculus*. Mathematics in Science and Engineering, vol. 111 (Academic, New York, 1974)
P. Ostalczyk, T. Rybicki, Variable-fractional-order dead-beat control of an electromagnetic servo. J. Vib. Control **14**(9–10), 1457–1471 (2008)
I. Podlubny, *Fractional Differential Equations* (Academic, New York, 1999)
Y.N. Rabotnov, *Elements of Hereditary Solid Mechanics* (Mir Publishers, 1980). (Russian edition, Moscow, 1977)
Y.A. Rossikhin, Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids. Appl. Mech. Rev. **63**(1), 010701–1–010701–12 (2010)
K. Rottmann, *Matematisk formelsamling (Mathematical formulas)* (Spektrum forlag, 2003)
R.L. Schilling, R. Song, Z. Vondracek, *Bernstein Functions: Theory and Applications* (Walter de Gruyter, Berlin, 2012)
M. Seredyńska, A. Hanyga, Relaxation, dispersion, attenuation, and finite propagation speed in viscoelastic media. J. Math. Phys. **51**(9), 092901–1–16 (2010)
I.M. Sokolov, J. Klafter, A. Blumen, Fractional kinetics. Phys. Today **55**(11), 48–54 (2002)
R.D. Stoll, Acoustic waves in ocean sediments. Geophysics **42**(4), 715–725 (1977)
T.L. Szabo, Time domain wave equations for lossy media obeying a frequency power law. J. Acoust. Soc. Am. **96**, 491–500 (1994)
B.E. Treeby, B.T. Cox, Modeling power law absorption and dispersion for acoustic propagation using the fractional Laplacian. J. Acoust. Soc. Am. **127**, 2741–2748 (2010)

M.G. Wismer, Finite element analysis of broadband acoustic pulses through inhomogenous media with power law attenuation. J. Acoust. Soc. Am. **120**, 3493–3502 (2006)

X. Zhang, W. Chen, C. Zhang, Modified Szabo's wave equation for arbitrarily frequency-dependent viscous dissipation in soft matter with applications to 3D ultrasonic imaging. Acta Mech. Solida Sin. **25**(5), 510–519 (2012)

Appendix B
Wave and Heat Equations

The purpose of this appendix is to show that in the viscous case, the acoustic and elastic theories both lead to constitutive equations of the Kelvin–Voigt type. They can be found in (B.21) for the acoustic case as well as (B.40) and (B.42) for the elastic case. When the elastic case can be separated into a compressional and a shear mode, both of them, therefore, lead to a viscous wave equation or Kelvin–Voigt wave equation, see (B.44) and (B.45).

The heat equations for conduction and convection will also be discussed, leading to the Fourier heat law and the Newton law of cooling.

Three different elastic moduli will be used in these derivations, the bulk modulus, K, the shear modulus, μ, and Young's modulus, E_Y. They are illustrated in Fig. B.1. All three of them are defined as the ratio of a change of pressure and a fractional change of dimension.

B.1 The Acoustic Wave Equation

Let the pressure and the density be composed of a static value plus a perturbation:

$$p' = p_0 + p, \tag{B.1}$$

$$\rho' = \rho_0 + \rho. \tag{B.2}$$

The derivation of the lossless acoustic wave equation starts from Euler's equation. It expresses ***conservation of momentum***:

$$\rho' \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho' \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p', \tag{B.3}$$

where $\mathbf{v}$ is the fluid velocity which relates to displacement, $\mathbf{u}$ via $\mathbf{v} = \partial \mathbf{u}/\partial t$. The operator ∇ is the grad operator. Here the total, material, or substantial time derivative

S. Holm, *Waves with Power-Law Attenuation*,
https://doi.org/10.1007/978-3-030-14927-7

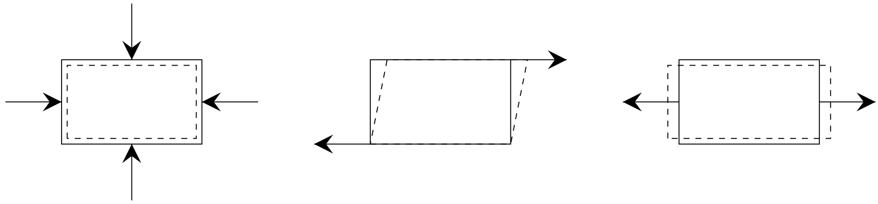

Fig. B.1 Forces and deformations that define the elastic moduli. From left to right: pressure producing a change of volume defining the bulk modulus, K; shear forces producing an angle of shear defining the shear modulus, μ; linear tension giving rise to extension, defining Young's modulus, E_Y

which is connected with the moving substance, is:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{B.4}$$

The second conservation equation is the ***conservation of mass*** principle expressed in the equation of continuity. It states that the net influx of matter into a volume element is reflected in a change of density:

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' \mathbf{v}) = \frac{\partial \rho'}{\partial t} + \rho' \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho' = \frac{\mathrm{D}\rho'}{\mathrm{D}t} + \rho' \nabla \cdot \mathbf{v} = 0, \tag{B.5}$$

where $\nabla \cdot$ is the div operator. Euler's equation and the equation of continuity are illustrated and explained in more detail in Kinsler (1999, Chap. 5).

The derivation of the acoustic wave equation builds on pressure, p, and density, ρ, variations around equilibrium or static values p_0 and ρ_0. The variations are assumed to be small and with a fluid velocity which is much smaller than the speed of sound:

$$p \ll p_0, \quad \rho \ll \rho_0, \quad |\mathbf{v}| \ll c. \tag{B.6}$$

Therefore Euler's equation may be linearized by equating the material and partial derivatives. Likewise the continuity equation may be linearized by neglecting the gradient of the density. The approximate equations are:

$$\textbf{\textit{Linearized conservation of momentum:}} \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p, \tag{B.7a}$$

$$\textbf{\textit{Linearized equation of continuity:}} \quad \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0. \tag{B.7b}$$

The third equation is the ***constitutive equation*** or equation of state. It comes from a special case of the ideal gas equation, $p' V^{\gamma} = C$, where γ is the adiabatic gas constant or heat capacity ratio, and C is a constant. Since the density is inverse proportional to volume, V, the gas law can be rewritten as $(p_0 + p)/p_0 = ((\rho_0 + \rho)/\rho_0)^{\gamma}$. Hooke's law is a linearization around the static pressure p_0:

$$p = K\frac{\rho}{\rho_0}, \quad K = \gamma p_0. \tag{B.8}$$

K is the bulk modulus, the inverse of the compressibility as illustrated in the left-hand image of Fig. B.1. This derivation builds on Landau and Lifshitz (1987, Chaps. I §1-2 and VIII §64) and Hamilton and Blackstock (2008, Chap. 3).

Substituting the divergence of (B.7a) in the time derivative of (B.7b) gives a second order equation involving p and ρ. Replacing ρ by p using (B.8) results in the lossless wave equation of (2.1) in p:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0, \tag{B.9}$$

where the speed of sound is:

$$c_0 = \sqrt{K/\rho_0}. \tag{B.10}$$

This derivation assumes that the medium is homogeneous so that the density, ρ, or the bulk modulus, K, do not vary in space. The generalization to an inhomogeneous medium can be found in Sect. 9.2.

B.1.1 The Navier–Stokes Equation and Viscosity

In Landau and Lifshitz (1987, Chap. II §15) it is argued that the most general ***constitutive equation*** under the assumptions of a Newtonian fluid and when the viscous contribution is independent of uniform rotation, is:

$$\sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right) + \zeta_B\delta_{ij}\frac{\partial v_k}{\partial x_k}. \tag{B.11}$$

The stress σ_{ij} and the strain ε_{ij} are second order tensors with nine components ($i, j = 1, 2, 3$) (Kolecki 2002). The Einstein notation is also used where the Kronecker delta, δ_{ij}, is zero if the indices are different and 1 if they are the same. The k is a dummy variable which implies a summation over all three spatial dimensions.

The two new constants η and ζ_B are the coefficients of viscosity. The shear viscosity is η, and in Landau and Lifshitz (1987), ζ_B is called the second viscosity. It is more precise to call it the bulk viscosity (Hamilton and Blackstock 2008, Chap. 3), and therefore it has been given a subscript B here. There is also an argument in Sect. B.2.1 for why that is more correct. From thermodynamic considerations, both $\eta > 0$ and $\zeta_B > 0$.

The link to the acoustics case can be seen by noting that in the lossless case, stress is the negative of pressure:

$$\sigma_{ij} = -p\delta_{ij}. \tag{B.12}$$

Landau and Lifshitz (1987) also show how (B.11) generalizes Euler's equation to the Navier–Stokes equation:

$$\rho_0 \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\zeta_B + \frac{1}{3}\eta \right) \nabla(\nabla \cdot \mathbf{v}), \quad \text{(B.13)}$$

with two more terms on the right-hand side compared to (B.3).

The Navier–Stokes equation is simplified in the case of an incompressible fluid, where $\nabla \cdot \mathbf{v} = 0$:

$$\rho_0 \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \eta \nabla^2 \mathbf{v}. \quad \text{(B.14)}$$

In that case it is only the shear viscosity which matters.

In order to simplify the ***constitutive equation***, an expression derived from the conservation of mass equation is needed. Starting with (B.7b), using the definition of velocity $\mathbf{v} = \partial \mathbf{u}/\partial t$, and integrating with respect to time gives:

$$\frac{\rho}{\rho_0} + \nabla \cdot \mathbf{u} = 0. \quad \text{(B.15)}$$

Dilatation is the relative change in volume (Royer and Dieulesaint 2000, Sect. 1.2.1.3). It can, when the volume is expressed as a static volume plus a small change (as in (B.2)), be expressed with density as:

$$\varepsilon = \frac{V}{V_0} = \frac{V' - V_0}{V_0} = \frac{\frac{m}{\rho_0 + \rho} - \frac{m}{\rho_0}}{m/\rho_0} \approx -\rho/\rho_0, \quad \text{(B.16)}$$

where m is the mass. Thus

$$\varepsilon = \nabla \cdot \mathbf{u}. \quad \text{(B.17)}$$

As in Royer and Dieulesaint (2000), the symbol for the 3-D dilatation is similar to that of the strain tensor, but the presence of indices makes it clear when the tensor is meant. In the general case where there is shear also, the dilatation is the sum of the diagonal elements, $\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, i.e., the trace of the strain tensor. The strain tensor is (Royer and Dieulesaint 2000, Sect. 3.1.1.1):

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad \text{(B.18)}$$

In an isotropic and homogeneous medium, the two terms on the right-hand side are the same. This equation also leads to:

$$\frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \frac{\partial \varepsilon_{ij}}{\partial t} \quad \text{(B.19)}$$

In this case, (B.11) simplifies to:

$$\sigma_{ij} = -p\delta_{ij} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = -p\delta_{ij} + 2\eta \frac{\partial \varepsilon_{ij}}{\partial t}. \tag{B.20}$$

In combination with (B.8) this gives the constitutive equation in the near incompressible case where the bulk viscosity can be neglected. In Landau and Lifshitz (1987, Chap. II §15) it is called the practically incompressible case:

$$\sigma_{ij} = K\varepsilon_{kk}\delta_{ij} + 2\eta \frac{\partial \varepsilon_{ij}}{\partial t}. \tag{B.21}$$

This is an expression of the Kelvin–Voigt equation. The resulting wave equation is given as a special case of the compressional part of an elastic wave equation in Sect. B.2.2.

B.1.2 Typical Media

B.1.2.1 Air

According to the ideal gas law, pressure varies with absolute temperature, T', and absolute volume, V', as $p' = nRT'/V$ where $R = 8314.51$ J/(mol K) is the universal gas constant and n is the number of moles. The density is also $\rho' = nM/V'$ where M is the molar mass of the gas, which for dry air is 0.0289645 kg/mol. Around the equilibrium point where $p' = p_0$ and $\rho' = \rho_0$, combination with (B.10) gives:

$$c_0 = \sqrt{\frac{K}{\rho_0}} = \sqrt{\gamma \frac{p_0}{\rho_0}} = \sqrt{\frac{\gamma k T'}{m}}, \tag{B.22}$$

where $k = 1.38064852 \cdot 10^{-23}$ J/K is Boltzmann's constant and the ratio of specific heats is $\gamma = c_p/c_v$ where c_p and c_v are the specific heat capacities under constant pressure and constant volume conditions respectively. The ratio is also called the adiabatic index and for an ideal diatomic gas it is $\gamma = 1.4$.

Newton's Derivation of the Speed of Sound

It is impressive that already in 1687, Newton was close to deriving (B.22). In fact Laplace said of him: *"The way at which he arrives at this [formula] is one of the most remarkable traits of his genius."* Chandrasekhar (1995, Sect. 154). Newton assumed Boyle's isothermal gas law rather than an adiabatic one which amounts to letting $\gamma = 1$ in (B.22). This derivation can be found in Newton (1687, Book II, Propositions 47–50).

He computed the value in air to be 968 ft/s (295 m/s) in the first edition of Principia in 1687 and 979 ft/s (298 m/s) in 1713 (Whiteside 1964). In 1713 the best measurement was 1142 ft/s or 348 m/s. Assuming that 343 m/s is the correct value, his latest value was 15% too low, while it should have been 18% too low. It took until 1816 for the correct derivation to be given by Laplace.

Example B.1 **Speed of sound in air.** Letting temperature, T, be in °C rather than K by $T' = 273.15 + T$ results in

$$c_0 = \sqrt{\frac{273.15 \cdot \gamma k}{m}(1 + \frac{T}{273.15})} \approx 331.3 + 0.606 \cdot T. \tag{B.23}$$

This gives a value of 343.2 m/s at 20 °C and 340.4 m/s at 15 °C.

Normal pressure is $p_0 = 1.01 \cdot 10^5$ Pa and according to (B.10) this gives a bulk modulus of $K = 1.414 \cdot 10^5$ Pa. At 15 °, the density can be found from (B.8) and the speed of sound to be $\rho_0 = K/c_0^2 = 1.22$ kg/m^3.

B.1.2.2 Water

In water the bulk modulus is approximately K = 2.2 GPa and the density is $\rho_0 = 1000$ kg/m^3 giving $c_0 = 1483$ m/s. Seawater is denser because of the salt and the bulk modulus and density also vary with temperature and other parameters.

Example B.2 **Speed of sound in seawater.** An empirical formula for the speed of sound is the nine-term equation of Mackenzie (1981):

$$\begin{aligned} c_0 = {} & 1448.96 + 4.591T - 5.304 \cdot 10^{-2}T^2 + 2.374 \cdot 10{-}4T^3 \\ & + 1.340(S - 35) + 1.630 \cdot 10^{-2}D + 1.675 \cdot 10^{-7}D^2 \\ & - 1.025 \cdot 10^{-2}T(S - 35) - 7.139x10^{-13}TD^3, \end{aligned} \tag{B.24}$$

where T is temperature in °C, S is salinity in parts per thousand and D is depth in meters. It will give values in the range from 1435.2 to 1535.7 m/s with the parameter values for the various oceans of Fig. 2.11.

The formula will even predict the surprisingly high value of 1855 m/s for the Dead Sea ($S = 330$ ppt, $T = 23$ °C, $D = 0$). This is not far from the measured value of 1840 m/s in Dec. 2006 (Beaudoin et al. 2011), although the formula may not have been intended to be accurate for such a high salinity.

Lev Davidovich Landau (1908–1968) was a Soviet physicist, born in Baku in the Russian empire, now Azerbaijan. He is said to have learned to differentiate at the age of 12 and to integrate at the age of 13. Few physicists have mastered all branches of physics like him and he wrote his ten-volume Course of Theoretical Physics mainly with Evgeny Mikhailovich Lifshitz as coauthor, starting in the late 30s. They received the Lenin prize in 1962 for this series. Three of the volumes are cited in this book (Landau and Lifshitz 1976, 1987; Landau et al. 1986).

Landau spent one year in prison in 1938–39 presumably accused of being a German spy, an unlikely charge given that his background was Jewish. He was an atheist and socialist, but had little sympathy for Stalin's regime. In 1962 he suffered a car crash from which he never recovered mentally and this also prevented him from being present at the awarding of his Nobel Prize later that year for a mathematical theory of superfluidity. Image: Wikipedia/Post of Azerbaijan.

B.2 The Elastic Wave Equation

Linearization of the Cauchy momentum equation expresses ***conservation of linear momentum*** (Parker et al. 2005; Kolsky 1963):

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \boldsymbol{\sigma}. \tag{B.25}$$

The second equation is the linearized ***conservation of mass*** principle expressed in the equation of continuity given in (B.18).

The third equation is the ***constitutive equation*** which relates strain and stress in a linear-elastic, isotropic medium:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{B.26}$$

where λ and μ are the first and second Lamé parameters, $\mu = G$ is also the shear modulus as illustrated in the middle of Fig. B.1. Its inverse is the compliance.

The constitutive equation can also be expressed with K, the bulk modulus, which is a measure of the material's resistance to uniform compression:

$$K = \lambda + 2\mu/3. \tag{B.27}$$

The ***constitutive equation*** then becomes (Landau et al. 1986, Chap. I §4):

$$\sigma_{ij} = K\varepsilon_{kk}\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right). \tag{B.28}$$

Landau et al. also give an argument for why the moduli of compression and rigidity, K and μ, are positive.

Derivation of the elastic wave equation To derive the wave equation, substitute ε from (B.18) into the constitutive equation of (B.26). Then differentiate with respect to the spatial variable. Finally replace the relation for the spatial derivative of $\sigma(t)$ with that of (B.25) and the result is the wave equation in the variable $\mathbf{u}$ which is the displacement vector:

$$(\lambda + \mu)\frac{\partial^2 u_k}{\partial x_k \partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} = \rho_0\frac{\partial^2 u_i}{\partial t^2}, \tag{B.29}$$

or

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) = \rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{B.30}$$

It is useful to separate this wave equation into its components.

Separation into compressional and shear components Following Landau et al. (1986, Chap. III, §22) one can separate out wave equations for the two components by representing the displacement as a sum of compressional and shear components:

$$\mathbf{u} = \mathbf{u}_c + \mathbf{u}_s. \tag{B.31}$$

Inserted into (B.30) this gives

$$(\lambda + \mu)\nabla(\nabla\cdot(\mathbf{u}_c + \mathbf{u}_s)) + \mu\nabla^2(\mathbf{u}_c + \mathbf{u}_s) = \rho_0\left(\frac{\partial^2\mathbf{u}_c}{\partial t^2} + \frac{\partial^2\mathbf{u}_s}{\partial t^2}\right). \tag{B.32}$$

Due to the nature of the particle displacement the shear component is divergence-free and the compressional component is curl-free:

$$\nabla\cdot\mathbf{u}_s = 0, \quad \nabla\times\mathbf{u}_c = 0. \tag{B.33}$$

In order to find the compressional wave equations, one takes the divergence of both sides of (B.32), using the first property of (B.33), giving:

$$\nabla\cdot\left(\mu\nabla^2\mathbf{u}_c + (\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}_c)\right) = \nabla\cdot(\lambda + 2\mu)\nabla^2\mathbf{u}_c = \nabla\cdot\rho_0\left(\frac{\partial^2\mathbf{u}_c}{\partial t^2}\right). \tag{B.34}$$

or

$$\nabla\cdot\left(\rho_0\frac{\partial^2\mathbf{u}_c}{\partial t^2} - (\lambda + 2\mu)\nabla^2\mathbf{u}_c\right) = 0. \tag{B.35}$$

Due to the second property of (B.33), the curl of the above expression is also zero. If both the curl and the divergence of a vector are zero, then that vector is identical to zero, thus giving the compressional wave equation:

$$\nabla^2 \mathbf{u}_c - \frac{1}{c_{0,c}^2}\frac{\partial^2 \mathbf{u}_c}{\partial t^2} = 0, \quad c_{0,c}^2 = \frac{\lambda + 2\mu}{\rho_0} = \frac{K + 4\mu/3}{\rho_0}, \tag{B.36}$$

where (B.27) has been used. In a similar way, the shear wave equation can be found. Taking the curl of (B.32), using the last property in (B.33) and the property that the curl of any gradients is zero, gives:

$$\nabla \times \left(\rho_0 \frac{\partial^2 \mathbf{u}_s}{\partial t^2} - \mu \nabla^2 \mathbf{u}_s\right) = 0. \tag{B.37}$$

Since the divergence of the expression in the parenthesis is zero also, the wave equation for the shear component is found as:

$$\nabla^2 \mathbf{u}_s - \frac{1}{c_{0,s}^2}\frac{\partial^2 \mathbf{u}_s}{\partial t^2} = 0, \quad c_{0,s}^2 = \frac{\mu}{\rho_0}. \tag{B.38}$$

Comparing the expressions for propagation speed, (B.36) and (B.38), and keeping in mind that the minimum value for K is zero, it is evident that in any medium the compressional wave velocity is always larger than the shear wave velocity, since

$$\frac{c_{0,c}}{c_{0,s}} = \sqrt{\frac{K + 4\mu/3}{\mu}} \geq \sqrt{\frac{4}{3}}. \tag{B.39}$$

In Norris (2017) it is also shown that there is a relationship between the attenuation coefficients and the phase velocities of the two modes that has to be satisfied in order for the material to be passive.

Why the Compressional Velocity Depends on Both Moduli
"The velocity of waves of distortion (shear) depends only on the density and shear modulus of the medium, and it might appear at first sight that the velocity of waves of dilatation (compressional) should depend only on the density and the bulk modulus K. However, $K = \lambda + \frac{2}{3}\mu$, so that the velocity of dilatation waves $[(K + \frac{4}{3}\mu)/\rho_0]^{1/2}$ and thus the shear modulus as well as the bulk modulus is involved. The physical reason for this is that in the propagation of waves of dilatation the medium is not subjected to a simple compression, but to a combination of compression and shear. For consider a small cube of material in the path of a plane wave of dilatation traveling in the direction of the x-axis; its cross-sectional area normal to the x-direction will not alter during the passage of the wave, whilst its x-dimension will be changed. There is thus a change in the **shape** of the element as well as in its volume, and the resistance of the medium to shear as well as its compressibility come into play." Quoted from Kolsky (1963, Chap. II).

B.2.1 Viscoelasticity

In Landau et al. (1986, Chap. V, §34), the ***constitutive equation*** in a linear viscoelastic, isotropic medium is given. The viscous contributions are given in terms of the gradient of the velocity, i.e. $\partial v_i/\partial x_k$ and it is said that this means that that is the same as the time derivatives of the components of the strain tensor. This can be seen from (B.19). Thus the ***constitutive equation*** is the viscous extension of (B.28):

$$\sigma_{ij} = K\varepsilon_{kk}\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right) + \zeta_B\frac{\partial\varepsilon_{kk}}{\partial t}\delta_{ij} + 2\eta\left(\frac{\partial\varepsilon_{ij}}{\partial t} - \frac{1}{3}\frac{\partial\varepsilon_{kk}}{\partial t}\delta_{ij}\right), \tag{B.40}$$

where η and ζ_B are called the two coefficients of viscosity by Landau and Lifshitz. This is an expression of the Kelvin–Voigt equation. Note the similarity with (B.11) after the application of (B.19) where one of the main differences is that $\mu = 0$ in a fluid in the acoustics case.

According to Karim and Rosenhead (1952), ζ_B is the bulk viscosity (or volume viscosity). It is related to the second viscosity in the same way as the bulk modulus and the first Lamé parameter are related, (B.27):

$$\zeta_B = \zeta + 2\eta/3. \tag{B.41}$$

Using that expression, one gets an expression with the Lamé parameters, as an extension of (B.26):

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} + \zeta\frac{\partial\varepsilon_{kk}}{\partial t}\delta_{ij} + 2\eta\frac{\partial\varepsilon_{ij}}{\partial t}. \tag{B.42}$$

This is also an expression of the Kelvin–Voigt equation. Here the first viscosity, η, is the shear viscosity and the second one is ζ.

The description given here shows that the bulk viscosity appears in a constitutive equation in a similar way as the bulk modulus, and the second viscosity in a similar way as the first Lamé parameter. Another parallel is that just as the two moduli of compression and rigidity, K and μ, are positive, so are the two coefficients of viscosity, η and ζ_B, positive also.

Starting with the lossless elastic wave equation of (B.30), the two viscous moduli of (B.42) lead to a viscoelastic wave equation (Bercoff et al. 2004):

$$\left(\lambda + \mu + (\zeta + \eta)\frac{\partial}{\partial t}\cdot\right)\nabla(\nabla\cdot\mathbf{u}) + \left(\mu + \eta\frac{\partial}{\partial t}\cdot\right)\nabla^2\mathbf{u} = \rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{B.43}$$

This wave equation can also be separated into a compressional and a shear wave equation, similar to the development of (B.36) and (B.38):

$$\nabla^2\mathbf{u}_p - \frac{1}{c_{0,c}^2}\frac{\partial^2\mathbf{u}_p}{\partial t^2} + \tau_c\frac{\partial}{\partial t}\nabla^2\mathbf{u}_p = 0, \quad \tau_c = \frac{\zeta + 2\eta}{\lambda + 2\mu} = \frac{\zeta_B + 4\eta/3}{K + 4\mu/3}, \tag{B.44}$$

and

$$\nabla^2 \mathbf{u}_s - \frac{1}{c_{0,s}^2}\frac{\partial^2 \mathbf{u}_s}{\partial t^2} + \tau_s \frac{\partial}{\partial t}\nabla^2 \mathbf{u}_s = 0, \quad \tau_s = \frac{\eta}{\mu}. \tag{B.45}$$

The compressional and shear wave speeds are given in (B.36) and (B.38), respectively. When separated in this way, both the compressional and the shear wave equations have the form of the Kelvin–Voigt wave equation of (2.36).

With the description based on the bulk modulus and the bulk viscosity from (B.27) and (B.41), an alternative wave equation compared to (B.43) is as follows:

$$\left(K + \mu/3 + (\zeta_B + \eta/3)\frac{\partial}{\partial t}\cdot\right)\nabla(\nabla\cdot\mathbf{u}) + \left(\mu + \eta\frac{\partial}{\partial t}\cdot\right)\nabla^2\mathbf{u} = \rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{B.46}$$

Compare the viscous terms with (B.13).

B.2.2 Pressure Waves in Fluids and Tissue

In the general case, the propagation velocity for a pressure wave is:

$$c_{0,c}^2 = (\lambda + 2\mu)/\rho_0 = (K + 4\mu/3)/\rho_0. \tag{B.47}$$

Fluids do not support shear forces, so $c_0^2 = c_{0,c}^2 = K/\rho_0 = \lambda/\rho_0$ which is as expected when comparing to the acoustics case of (B.10). This is a good approximation for tissue also as $K = \lambda + 2\mu/3 \approx \lambda$ when $\lambda \gg \mu$.

Young's modulus, E_Y, is the relationship between stress (force per unit area) and strain (proportional deformation) in a material as illustrated in the right-hand figure of Fig. B.1. It relates to the Lamé parameters as

$$E_Y = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}. \tag{B.48}$$

In tissue the Young's modulus, using (B.38), is therefore:

$$E_Y \approx 3\mu = 3\rho_0 c_{0,S}^2. \tag{B.49}$$

This is a relationship which is fundamental in ultrasound elastography in order to convert from the measured shear wave speed to shear modulus or Young's modulus.

The wave equation in the acoustics case (Sect. B.1.1) is a special case of the pressure wave equation, (B.44) when the shear modulus is zero. Therefore the time constant due to viscosity will be

$$\tau_c = (\zeta_B + 4\eta/3)/K. \tag{B.50}$$

Letting also $\zeta_B = 0$ implies, via (B.41), that $\zeta = -2\eta/3$ (Liebermann 1949). This is the case at least for ideal monatomic gases (Karim and Rosenhead 1952). This is an alternative, and less strict way to get rid of the bulk viscosity term than the assumption of incompressibility of Sect. B.1.1. In that case the time constant reduces to $\tau_c = 4\eta/3K$.

B.2.3 Viscous Waves

As pointed out in Bhatia (1967, Sect. 4.4), a nonzero shear viscosity implies that there is a shear wave even in a fluid. Starting with (B.45) and inserting for the velocity of sound and the time constant gives

$$\mu\nabla^2\mathbf{u}_s - \rho_0\frac{\partial^2\mathbf{u}_s}{\partial t^2} + \eta\frac{\partial}{\partial t}\nabla^2\mathbf{u}_s = 0. \tag{B.51}$$

In a fluid the shear modulus is zero so the first term vanishes:

$$\rho_0\frac{\partial^2\mathbf{u}_s}{\partial t^2} - \eta\frac{\partial}{\partial t}\nabla^2\mathbf{u}_s = 0. \tag{B.52}$$

This is a diffusion equation giving as a result a highly attenuated wave which is called a viscous wave. Its properties are discussed in Sect. 5.6.1 and its solution is plotted in the lower right-hand part of Fig. 2.4.

B.2.4 Typical Media

B.2.4.1 Human Tissue

In human tissue at ultrasound frequencies (typ. 2–10 MHz), the speed of sound is about 1540 m/s with some variation between tissue types. The density is slightly above that of water, e.g., 1060 kg/m^3 for muscle. According to (B.10), the effective bulk modulus at these frequencies is around $K = 2.5$ GPa.

The shear modulus, μ, is much smaller and may vary between 1 and 100 kPa, giving a shear velocity which varies from about 1 to 30 m/s according to (B.49).

B.2.4.2 Sub-bottom Sediments

In water-saturated sandy sediments the speed of sound for the compressional wave has been measured to be between 1.1 and 1.16 times the value of the water. Likewise values of density range between 1.8 and 2.1 times that of water (Chotiros 2017).

B.3 The Electromagnetic Wave Equation

Maxwell's equations on differential form are:

$$\text{Gauss' Law:} \quad \nabla \cdot \mathbf{D} = \rho_f \tag{B.53a}$$

$$\text{Gauss' Law for magnetism:} \quad \nabla \cdot \mathbf{B} = 0 \tag{B.53b}$$

$$\text{Faraday's Law:} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{B.53c}$$

$$\text{Ampere's Law:} \quad \nabla \times \mathbf{H} = \mathbf{J_f} + \frac{\partial \mathbf{D}}{\partial t} \tag{B.53d}$$

The equivalent of the ***constitutive equations*** are:

$$\text{Electrical:} \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E} \tag{B.54a}$$

$$\text{Magnetic:} \quad \mathbf{H} = \mathbf{B}/\mu \tag{B.54b}$$

Here **E** is the electric vector field, measured in volts per meter or newtons per coulomb and **P** is the polarization field, which in a vacuum is zero. **D** is the electric displacement field, a vector field that accounts for the effects of free and bound charge in materials. It is measured in coulomb per squared meter. Further **B** is the magnetic field in Tesla, T, or Newton per meter per Ampere and **H** is the magnetic field measured in Amperes per meter. The volume charge density is ρ_f and $\mathbf{J_f}$ is the displacement current.

The permittivity or dielectric constant is ε and μ is the permeability. The free space (vacuum) values are ε_0 and μ_0 and general materials are usually characterized by the relative values, i.e., $\varepsilon = \varepsilon_r \varepsilon_0$ and $\mu = \mu_r \mu_0$.

The insensitivity of these equations to absolute motion plus the fact that the Michelson–Morley experiment could not detect any change in the speed of light with movement of the earth were instrumental in developing the theory of special relativity. Both factors are discussed on the first page of the paper where the special theory of relativity was first given (Einstein 1905).

Deriving the electromagnetic wave equation The wave equation is found by assuming that $\rho_f = 0$ and $\mathbf{J_f} = 0$. Then insert the constitutive equations into (B.53a), (B.53d), in order to eliminate **D** and **H**. Then the curl operation is taken on the curl equations, (B.53c), (B.53d). The two vector identities

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}, \tag{B.55a}$$

$$\nabla^2 \mathbf{V} = \nabla \cdot (\nabla \mathbf{V}), \tag{B.55b}$$

where **V** is any vector field are applied to the result and one obtains two wave equations of the form of (2.1), one for **E**, the electric field, and one for **B**, the magnetic field.

The electromagnetic equations go back to James Clerk Maxwell. In 1865 he formulated them rather differently with some twenty equations in twenty unknowns. It was the self-taught electrical engineer Oliver Heaviside who after several years of work in 1884 (after the death of Maxwell) recast them to the much simpler form of (B.53) and (B.54) in which we know them today.

Maxwell's equations signify the first unification result in physics, as they unified the electrical and magnetic fields. In addition, by showing that the propagation speed of an electromagnetic wave was the same as that measured for light, Maxwell hypothesized correctly that light also was an electromagnetic wave.

The speed of propagation in these wave equations is:

$$c = \sqrt{\frac{1}{\varepsilon\mu}}. \tag{B.56}$$

Example B.3 **Speed of light in vacuum.** For vacuum with $\varepsilon = \varepsilon_0 = 8.854 \cdot 10^{-12}$ F/m and $\mu = \mu_0 = 4\pi \cdot 10^{-7}$ H/m, this gives $c_{vac} \approx 3 \cdot 10^8$ m/s.

Example B.4 **Speed of light and index of refraction in water.** Water has $\varepsilon_r = 1.77$ for visible light (much higher for microwave and lower frequencies, see Example 2.2), giving a speed of propagation of $c_w = c_{vac}/\sqrt{1.77} \approx 2.3 \cdot 10^8$ m/s. The index of refraction, often used in optics, is

$$n = c_{vac}/c = \sqrt{\varepsilon_r \mu_r}, \tag{B.57}$$

which in the case of water is $n = \sqrt{1.77} \approx 1.33$.

B.4 Heat Equations

Heat transfer can take place by several different mechanisms. Conduction is heat transfer through collisions of particles and movement of electrons in a body. Advection is heat transfer by bulk flow of a fluid. Convection is heat transfer by the movement of fluids, and which involves the combined processes of conduction and advection. It is the dominant heat transfer mechanism in liquids and gases. Finally, radiation is heat transfer by the emission of photons of electromagnetic waves caused by thermal motion of charged particles. Only conduction and convection will be treated here.

B.4.1 Fourier's Law of Heat Conduction

In heat conduction, objects are in contact with each other and the heat flux is described by the Fourier heat law:

$$\mathbf{q} = -\kappa_h \nabla T, \tag{B.58}$$

where

- $\mathbf{q}$ is vector heat flux density in [W/m^2] or heat flow rate per unit area. One component is given by the heat energy Q [J] by $q_x = (1/A)\mathrm{d}Q/\mathrm{d}T$.
- κ_h is the thermal conductivity in [W/m/K]. It is assumed to be independent of temperature.
- T is temperature in [K].

In order to find the partial differential equation which the temperature follows, the Fourier heat law is combined with the law of conservation of energy. It follows from (4.3) and (4.4) when there is no acoustic wave ($u = 0$):

$$\rho_0 c_v \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q}, \tag{B.59}$$

Combined with (B.58) this gives the parabolic heat equation. This is a diffusion equation similar to the one discussed in Sect. 5.6.1:

$$\nabla^2 T - \frac{\rho_0 c_v}{\kappa_h} \frac{\partial T}{\partial t} = 0, \tag{B.60}$$

where the phase velocity is proportional to the square root of frequency, $c_{ph} = \sqrt{2\kappa_h \omega/(\rho_0 c_v)}$, and thus the solution has components with infinite velocity when there are transient phenomena. This limitation of the heat law is discussed in Sect. 4.4.2.

B.4.1.1 Cattaneo Heat Law

Cattaneo modified the heat law to give it a finite propagation speed (Cattaneo 1948). Its background is explained in Müller and Ruggeri (1998, Sect. 2.1) and it consists of adding a new term on the left-hand side:

$$\mathbf{q} + \tau_C \frac{\partial \mathbf{q}}{\partial t} = -\kappa_h \nabla T, \tag{B.61}$$

where τ_C is the Cattaneo relaxation time. Combined with energy conservation, it leads to the telegrapher's equation of Sect. 5.5, the hyperbolic heat equation:

$$\nabla^2 uT - \tau_C \frac{\rho_0 c_v}{\kappa_h} \frac{\partial^2 T}{\partial t^2} - \frac{\rho_0 c_v}{\kappa_h} \frac{\partial T}{\partial t} = 0. \tag{B.62}$$

Its phase velocity increases with frequency until it reaches a finite asymptotic value, $c_{ph}(\infty) = \sqrt{\kappa_h/(\rho_0 c_v \tau_C)}$.

In Rubin (1992) and Bai (1995) it is pointed out that this heat equation gives unphysical results when the size of the domain is smaller than a critical length which is on the order of a thermal energy carrier's mean free path. Seemingly, the modification to get a finite heat wave propagation velocity may solve one physical inconsistency while introducing others.

B.4.1.2 The Fractional Heat Equation

A more general modification of the Fourier heat law is due to Gurtin and Pipkin (1968) where a causal memory kernel, the heat-flux relaxation function, is convolved with the temperature gradient:

$$\mathbf{q}(t) = -\int_0^t K(t-\tau)\nabla T(\tau) d\tau. \tag{B.63}$$

Further in Povstenko (2009) it was assumed that the memory kernel could be a power law:

$$K(t-\tau) = \frac{\tau_{th}^{1-\gamma} \kappa_h}{\Gamma(\gamma-1)} (t-\tau)^{\gamma-2}, \quad 1 < \gamma \le 2 \tag{B.64}$$

where τ_{th} is a thermal relaxation time which is characteristic for the medium. This modifies the heat law with a fractional integral:

$$\mathbf{q}(t) = -\tau_{th}^{1-\gamma} \kappa_h \mathrm{I}^{\gamma-1} \nabla T(t), \tag{B.65}$$

where $\mathrm{I}^{\gamma-1}$ is the fractional integral of order $\gamma - 1$ and γ is in the range from 1 to 2. As γ approaches unity, this heat law becomes identical to the Fourier heat law.

The temperature will follow the fractional diffusion-wave equation of Sect. 5.6.

$$\nabla^2 T - \frac{\rho_0 c_v}{\kappa_h} \frac{\partial^\gamma T}{\partial t \gamma} = 0, \tag{B.66}$$

and the phase velocity will be proportional to $\omega^{1/2-\gamma/2}$, (Sect. 5.6.2), which grows slower than $\sqrt{\omega}$. Therefore this particular heat equation only reduces the magnitude of the infinite velocity problem, but it does not eliminate it. Some of the more complex fractional extensions of the heat equation will do that and in Compte and Metzler (1997), Zhang et al. (2014) four other such equations are analyzed.

The simple fractional heat equation which may be related to anomalous heat diffusion in an inhomogeneous medium, will be used in Sect. 7.1 to demonstrate that power law attenuation can be due to this particular kind of heat diffusion.

B.4.2 Newton's Law of Cooling

When there is heat convection from a warm surface to a surrounding cooler fluid, there is only a thin boundary layer where the temperature varies. It is the property of this thermal boundary layer near the surface which determines the flow of heat and the diffusion process taking place in this layer resembles thermal conduction. With this in mind, the Fourier heat law can be reformulated for convection by expressing the temperature gradient by the difference between a surface and its surroundings (Burmeister 1993, Chap. 1).

The Fourier heat law of (B.58) in 1-D when heat transfer takes place over a surface area A, and when the temperature gradient is the difference between the surface temperature and that of the environment, T_0, is:

$$\frac{\partial Q}{\partial t} = -\kappa_h A \frac{\partial T}{\partial x} = -hA\Delta T, \quad \Delta T = T - T_0 \tag{B.67}$$

where the convective heat transfer coefficient, h, has unit [W/m^2/K].

Thermal energy is proportional to temperature, $Q = CT$, where C is total heat capacity in [J/K] or when expressed in terms of the specific heat [J/K/kg] and mass, $Q = mc_pT$. When dealing with temporal derivatives of temperature, only the surface temperature varies, so the temporal derivative of T is the same as that of ΔT. Thus the Fourier heat law can be rewritten as

$$\frac{\partial T}{\partial t} = \frac{\partial \Delta T}{\partial t} = -\frac{hA}{mc_p}\Delta T = -\frac{\Delta T}{\tau}. \tag{B.68}$$

The constant $hA/(mC_p)$ will have unit [1/s] and is therefore an inverse time constant. This is the form of Newton's law of cooling as stated in 1701 (Cheng and Fujii 1998; Winterton 2001), where the rate of loss of temperature is proportional to the temperature difference. The importance of the connection with the Fourier heat law is that it makes it clear that Newton's law of cooling also implies an infinite propagation speed.

References

C. Bai, A. Lavine, On hyperbolic heat conduction and the second law of thermodynamics. J. Heat Transf. **117**(2), 256–263 (1995)

J. Beaudoin, A. Sade, B. Schulze, J.K. Hall, Dead sea multi-beam echo sounder survey. Hydro Int. **15**, 21–23 (2011)

J. Bercoff, M. Tanter, M. Muller, M. Fink, The role of viscosity in the impulse diffraction field of elastic waves induced by the acoustic radiation force. IEEE Trans. Ultrason. Ferroelectr., Freq. Control **51**(11), 1523–1536 (2004)

A.B. Bhatia, *Ultrasonic Absorption: An Introduction to the Theory of Sound Absorption and Dispersion in Gases, Liquids, and Solids* (Courier Dover Publications, New York, 1967)

L.C. Burmeister, *Convective Heat Transfer*, 2. edn. (Wiley, New Jersey, 1993)

C. Cattaneo, Sulla conduzione de calore (On the conduction of heat). Atti. Sem. Mat. Fis. Univ. Modena **3**, 83–101 (1948)

S. Chandrasekhar, *Newton's Principia for the Common Reader* (Clarendon Press, Oxford, 1995)

K. Cheng, T. Fujii, Heat in history Isaac Newton and heat transfer. Heat Transf. Eng. **19**(4), 9–21 (1998)

N.P. Chotiros, *Acoustics of the Seabed as a Poroelastic Medium* (Springer and ASA Press, Switzerland, 2017)

A. Compte, R. Metzler, The generalized Cattaneo equation for the description of anomalous transport processes. J. Phys. A Math. Gen. **30**(21), 7277–7289 (1997)

A. Einstein, Zur Elektrodynamik bewegter Körper, (On the electrodynamics of moving bodies). Ann. Phys. **17**(10), 891–921 (1905)

M.E. Gurtin, A.C. Pipkin, A general theory of heat conduction with finite wave speeds. Arch. Ration. Mech. Anal. **31**(2), 113–126 (1968)

M.F. Hamilton, D.T. Blackstock, *Nonlinear Acoustics* (Acoustical Society of America, New York, 2008)

S. Karim, L. Rosenhead, The second coefficient of viscosity of liquids and gases. Rev. Mod. Phys. **24**(2), 108 (1952)

L.E. Kinsler, A.R. Frey, A.B. Coppens, J,V. Sanders, Fundamentals of Acoustics, 4th edn. (Wiley-VCH, New York, 1999)

J.C. Kolecki, *An Introduction to Tensors for Students of Physics and Engineering* (National Aeronautics and Space Administration, Glenn Research Center, 2002)

H. Kolsky, *Stress Waves in Solids*, vol. 1098 (Courier Corporation, Massachusetts, 1963)

L.D. Landau, E.M. Lifshitz, *Fluid Mechanics, 3rd Edition: Vol. 6 of Course of Theoretical Physics* (Elsevier, Amsterdam, 1987)

L.D. Landau, E.M. Lifshitz, *Mechanics, 3rd Edition: Vol. 1 of Course of Theoretical Physics* (Elsevier, Amesterdam, 1976)

L.D. Landau, E.M. Lifshitz, L.P. Pitaevskii, A.M. Kosevich, *Theory of Elasticity, 2nd Edition: Vol. 7 of Course of Theoretical Physics* (Elsevier, Amsterdam, 1986)

L. Liebermann, The second viscosity of liquids. Phys. Rev. **75**(9), 1415 (1949)

K.V. Mackenzie, Nine-term equation for sound speed in the oceans. J. Acoust. Soc. Am. **70**(3), 807–812 (1981)

I. Müller, T. Ruggeri, *Rational Extended Thermodynamics* (Springer, Berlin, 1998)

I. Newton, Philosophiæ naturalis principia mathematica ("Mathematical principles of natural philosophy") (London 1687)

A.N. Norris, An inequality for longitudinal and transverse wave attenuation coefficients. JASA **141**(1), 475–479 (2017)

K.J. Parker, L.S. Taylor, S. Gracewski, D.J. Rubens, A unified view of imaging the elastic properties of tissue. J. Acoust. Soc. Am. **117**(5), 2705–2712 (2005)

Y.Z. Povstenko, Thermoelasticity that uses fractional heat conduction equation. J. Math. Sci. **162**(2), 296–305 (2009)

D. Royer, E. Dieulesaint, *Elastic Waves in Solids*, vol. I (Springer, Berlin, 2000)

M. Rubin, Hyperbolic heat conduction and the second law. Int. J. Eng. Sci. **30**(11), 1665–1676 (1992)

H. Whiteside, Newton's derivation of the velocity of sound. Am. J. Phys. **32**, 384–384 (1964)

R.H. Winterton, Heat in history. Heat Transf. Eng. **22**(5), 3–11 (2001)

W. Zhang, X. Cai, S. Holm, Time-fractional heat equations and negative absolute temperatures. Comput. Math. Appl. **67**(1), 164–171 (2014)

Index

S. Holm, *Waves with Power-Law Attenuation*,
https://doi.org/10.1007/978-3-030-14927-7