Solutions to Selected Exercises

Problems of Chapter 1

1.2 Choosing an appropriate Cartesian reference, the points are $A(0, 0)$ and $B(3, 5)$, whereas the mountain range is $[1, 2] \times \mathbb{R}$. We have to decide the points $C(1, x)$ and $D(2, y)$ such that the cost of the road $[A, C, D, B]$ is minimized. Denoting $\alpha = \sqrt{1.6}$, the problem is

$$P: \text{ Min } f(x, y) = \sqrt{1 + x^2} + \alpha\sqrt{1 + (y - x)^2} + \sqrt{1 + (5 - y)^2}.$$

As this function is differentiable and coercive, the global minimum will be the best of its critical points, that is $(2, 3)$. The optimal solution is $[A, C, D, B]$, with $C(1, 2)$ and $D(3, 3)$.

1.4 Denoting by x_1, x_2, and x_3 the length, the width, and the height of the box (in meters), the optimal solution to the approximating continuous model is $(2a, 2a, a/2)^T$, where $a = \sqrt[5]{5/2}$. The optimal choice is to take 115 trips, using a box with a square base of side 2.4052 and 0.6013 m height. The total cost of the operation will be 2885.41 c.u.

1.6 Let us denote by T the temperature and by t the time.
(a) $T = 25 - 20e^{-\alpha t}$.
(b) Defining $u = \ln\left(\frac{25-T}{20}\right)$, the least squares fitting of the line $u = -\alpha t$ to the data provides $\alpha = -0.1552$, which should be replaced in (a).

1.7 The model is

$$\begin{aligned} P : \text{Min } & \sum_{i,j} w_{ij} \left\| \begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} a_j \\ b_j \end{pmatrix} \right\|_1 \\ \text{s.t. } & \sum_{j=1}^{n} w_{ij} \leq c_i, i = 1, \ldots, m, \\ & \sum_{i=1}^{m} w_{ij} \geq r_j, j = 1, \ldots, n, \\ & w_{ij} \geq 0, \forall i \forall j. \end{aligned}$$

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The objective function can be linearized by means of the technique introduced in Subsubsection 1.1.5.6 ("ℓ_1 linear regression via linear optimization").

1.10 After doing the variable transformation

$$\left.\begin{array}{l} x_1 = \sin y_1 \sin y_2 \ldots \sin y_n \\ x_2 = \cos y_1 \sin y_2 \ldots \sin y_n \\ x_3 = \cos y_2 \ldots \sin y_n \\ \ldots \\ x_n = \cos y_n \end{array}\right\},$$

one obtains an equivalent unconstrained problem with one less variable.

1.15 (a) By the first-order necessary condition, the unique candidate to local minima is, in all cases, $(1, -2)^T$.
(i) $(\overline{x}, \overline{y}) := (1, -2)$ satisfies the first-order sufficient condition because

$$\nabla f(x, y)^T \begin{pmatrix} x-1 \\ y+2 \end{pmatrix} = 2\big(((x-1)+(y+2))^2 + 2(y+2)^2\big) > 0 \qquad \text{(A.1)}$$

when $y \neq -2$ and also when $y = -2$, by the hypothesis that $(x, y) \neq (1, -2)$ (there is always a positive term in (A.1)). It cannot be asserted that $(1, -2)^T$ is a global minimum.

(ii) As $\nabla^2 f(x, y) = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$ is positive definite, the unique singular point of f, $(1, -2)^T$, is a strict local minimum, but it cannot be guaranteed that it is a global minimum.

(iii) Since $(1, -2)^T$ is the unique candidate to be a local minimum, it will necessarily be a global minimum if one proves that f is coercive. Let us do the change of variables $x = u + 1$, $y = v - 2$. Then, we have

$$f(x, y) = g(u, v) = u^2 + 3v^2 + 2uv = (u+v)^2 + 2v^2 \geq 0, \quad \forall (u, v)^T \in \mathbb{R}^2.$$

Therefore, $S_\lambda(g) \neq \emptyset \Leftrightarrow \lambda \geq 0$. Obviously, $S_0(g) = \{0_2\}$. Let $\lambda > 0$. Then, we have $(u, v)^T \in S_\lambda(g) \Rightarrow |u+v| \leq \sqrt{\lambda}$ and $|v| \leq \sqrt{\frac{\lambda}{2}} \Rightarrow |u| \leq \sqrt{\lambda} + \sqrt{\frac{\lambda}{2}}$. Therefore, $S_\lambda(g)$ is bounded. As g is coercive on $\mathbb{R}^2$, so is f.

(b) The point $(1, 1)^T$ is a strict local minimum, and it can only be classified by means of the first-order condition.

1.17 The key here is the coercivity of f, consequence of

$$S_\lambda(f) \subset \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x, y, z \leq \ln \lambda, x + y + z \geq -\ln\left(\frac{\lambda}{2}\right) \right\},$$

for all $\lambda > 0$.

1.20 There is no global minimum, for any value of a, because $\lim_{x_1 \to -\infty} f(x_1, 0) = -\infty$; for $a \neq 0$, $(a, a)^T$ is a (strict) local minimum of f whereas there is no local minimum when $a = 0$.

1.22 Let x be the distance traveled by Bond, running along the beach shore, until he starts swimming. The time spent will be

$$f(x) = \frac{100 - x}{5} + \frac{1}{2}\sqrt{x^2 + 2500}$$

and we have to see if $\inf\{f(x) : x \in \mathbb{R}\} \leq 44$. As f is coercive and differentiable, the solutions will be critical points of f. The unique critical point is $\overline{x} = 21.82$ and as $f(\overline{x}) = 42.913 < 44$, Bond has enough time to disable the bomb and save the world.

1.23 If the position of a bather in $[a, b]$ is $Z \sim U(a, b)$ and the bar is on a, the average distance (round trip) of the bather to the bar is

$$E(2(Z - a)) = 2\int_a^b \left(\frac{z - a}{b - a}\right) dz = b - a.$$

The expression is the same if the bar is on b. Therefore, the average distance of a bather who is served at an end point of the interval is its length. This observation will help us to express the objective functions.
(a) Let x be the distance from the bar to one of the ends of the beach, located, for example, at the point 0. The average sum of the distances of the bathers that are located on the left side of the bar is the average distance, x, multiplied by the number of bathers at that side of the beach, which is also x, as it is proportional to the length of the beach. Analogously, the total average distance walked around by the bathers that are located on the right side to the bar is $(1 - x)^2$. Therefore, we have to minimize the function $f(x) = x^2 + (1 - x)^2$ on $[0, 1]$. The unique optimal solution is $\frac{1}{2}$, with average distance walked around by the bathers of $\frac{1}{2}$ km, which is the same for both sides of the beach.
(b) It is sufficient to consider the bathers located in $\left[0, \frac{1}{2}\right]$, who will need to walk around an average distance of $\frac{1}{2}$ km.
(c) If x is the distance from the first bar to 0 (the nearest end of the beach) and y is the distance between both bars, the total distance is proportional to

$$f(x, y) = x^2 + \frac{y^2}{2} + (1 - x - y)^2,$$

whose minimum on the feasible set of the system $\{x + y \leq 1; x \geq 0; y \geq 0\}$ is $\left(\frac{1}{4}, \frac{1}{2}\right)^T$, with $f\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4}$ km (average distance). Bathers will need to walk around half of the distance than in (b).

Problems of Chapter 2

2.5 (a) f is convex and continuous, but it is not strongly convex, coercive, or Lipschitz continuous on $\mathbb{R}$ (although it is on any bounded interval).
(b) f' is convex and Lipschitz continuous on $\mathbb{R}$ (with constant 1), but it is not strongly convex or coercive.

2.6 (a) f_p is convex for $p = 1$ and when p is even. It is strongly convex for $p = 2$, and it is Lipschitz continuous at 0 for $p = 1$.
(b) f_p is convex for $p = 1$ and when p is even. It is strongly convex for $p = 2$, and it is Lipschitz continuous at 0, with Lipschitz constant $p\alpha^{p-1}$, for all $p \in \mathbb{N}$.

2.7 f_r satisfies the three properties for all $r \in \mathbb{N}$; the same happens with $\inf f_r = f_1$; finally, $\lim_r f_r = \sup_r f_r = |\cdot|$ is continuous and convex, but not differentiable.

2.9 C is convex, and $\nabla^2 f$ is positive semidefinite on $\operatorname{int} C$. Moreover, f is continuous on C, and thus, it is convex on C.

2.12 (a) $C = \mathbb{R} \times \mathbb{R}_{--}$, where $\mathbb{R}_{--} = -\mathbb{R}_{++}$.
(b) As $f(0, y) = 0$ for all $y \in \mathbb{R}_{--}$, $\lim_{y\to-\infty} f(0, y) = 0$ and f is not coercive. Therefore, f is not strongly convex.
(c) If we take $y = -1$, we have

$$\lim_{x\to+\infty} \frac{\partial f(x, -1)}{\partial x} = \lim_{x\to+\infty} 2x\left(\frac{1}{e} + 1\right) = +\infty,$$

so we suspect f is not Lipschitz continuous on horizontal directions. We shall use this idea for doing the proof: As

$$\begin{aligned}\lim_{r\to+\infty} \frac{f(r, -1) - f(1, -1)}{\| (r, -1) - (1, -1) \|} &= \lim_{r\to+\infty} \frac{r^2\left(\frac{1}{e} + 1\right) - \left(\frac{1}{e} + 1\right)}{r - 1} \\ &= \left(\frac{1}{e} + 1\right) \lim_{r\to+\infty} (r + 1) = +\infty,\end{aligned}$$

it does not exist $L \geq 0$ such that $|f(x, y) - f(u, v)| \leq L\|(x, y) - (u, v)\|$ for all $\binom{x}{y}, \binom{u}{v} \in C$. Therefore, f cannot be Lipschitz continuous on C.

2.14 (a) f is continuous, convex, and coercive, but, in general, it is neither strictly convex nor strongly convex (it is easy to prove it for $n = 1$).
(b) $F^* = [1, 3]$.
(c) 1.

2.17 $f \circ g$ is the composition of two convex functions and the second one is nondecreasing, so $f \circ g$ is convex.

2.19 (a) It is false: For instance, if $f(x, y) = x^2y^2$, one has that $\nabla^2 f(0, 0)$ is the null matrix, which is positive semidefinite, but $\nabla^2 f(x, y)$ is not positive semidefinite at points $\binom{x}{y}$ that are arbitrarily close to the origin.
(b) It is true: If $\nabla^2 f(\bar{x})$ is positive definite, all the director principal minors are positive. Any principal minor is a sum of products of second-order partial derivatives of f, so it is a continuous function on $\mathbb{R}^n$. Therefore, each director principal minor is positive on an open ball with center $\bar{x}$. As there are n director principal minors, the intersection of the n balls is the ball with least radio, on which $\nabla^2 f(x)$ is positive definite and f is convex.

2.20 The argument would be correct if (a) was true, as well as the successive implications (a)⇒(b),..., (d)⇒(e). This is not the case because (d)⇏(e).

Problems of Chapter 3

3.5 (a) $\left\{x \in \mathbb{R}^2 : x_1 + x_2 = 0\right\}$.
(b) $\overline{x} = \left(\frac{2617}{2439}, \frac{763}{813}, \frac{2792}{2439}\right)^T = (1.073, 0.9385, 1.1447)^T$.

3.6 We have to minimize the function

$$h(x) := \|g - f\|_2^2 = \frac{1}{3}x_1^2 + x_1x_2 + x_2^2 - (\ln 2)x_1 - \frac{\pi}{2}x_2 + \int_0^1 \frac{dt}{\left(1 + t^2\right)^2},$$

that is, $h(x) = \frac{1}{2}x^T Ax + a^T x + b$, with

$$A = \begin{pmatrix} \frac{2}{3} & 1 \\ 1 & 2 \end{pmatrix},$$

that is positive definite, $a = \left(-\ln 2, -\frac{\pi}{2}\right)^T$ and $b = \int_0^1 \frac{dt}{(1+t^2)^2}$. Therefore, the unique optimal solution of P is the one of the linear system $\{Ax = -a\}$, that is, $\overline{x} = \left(6 \ln 2 - \frac{3}{2}\pi, \pi - 3 \ln 2\right)^T$. In conclusion, the affine function we are looking for is

$$g(t) = \left[6 \ln 2 - \frac{3}{2}\pi\right]t + (\pi - 3 \ln 2) \simeq 1.062 - 0.554t.$$

3.7 The point we are looking for is $\overline{x} = \frac{1}{m} \sum_{i=1}^{m} x^i$. When $m = 4$ and the points form a quadrilateral, $\overline{x}$ is the midpoint of the midpoints of any pair of opposite edges. When the points form a parallelogram, there exist vectors $a, b, c \in \mathbb{R}^3$ such that $\left\{x^1, \ldots, x^4\right\} = \{a, a + b, a + c, a + b + c\}$, with

$$\overline{x} = a + \frac{b + c}{2} = \frac{a + b}{2} + \frac{a + c}{2} = \frac{a}{2} + \frac{a + b + c}{2},$$

so $\overline{x}$ is the intersection of the diagonals.

3.9 Denoting by S and p the area and the perimeter of the triangle, and by x, y, z the distances from P to the three sides, one has $S = \frac{1}{2}p(x+y+z)$, so that $x+y+z = \frac{2S}{p}$. Since $(xyz)^{\frac{1}{3}} \leq \frac{x+y+z}{3} = \frac{2S}{3p}$, the maximum is attained whenever $x = y = z$, which corresponds to the intersection of the three internal angle bisectors; that is, the incenter.

3.12 By the geometric-arithmetic inequality,

$$\begin{aligned} 1 &= 3\left(\frac{3x + 4y + 12z}{3}\right) \\ &\geq 3(3x)^{\frac{1}{3}}(4y)^{\frac{1}{3}}(12z)^{\frac{1}{3}} \\ &= 3\sqrt[3]{144}(xyz)^{\frac{1}{3}}, \end{aligned}$$

so the maximum is attained at $(\overline{x}, \overline{y}, \overline{z})^T = \left(\frac{1}{9}, \frac{1}{12}, \frac{1}{36}\right)^T$ and the optimal value is $v(P) = \frac{1}{3^5 2^4} = \frac{1}{3888} = 2.572 \times 10^{-4}$.

3.13 As the volume and the height of both cans coincide, both cans have the same base area. So, it is sufficient to compare the lateral areas, S_c and S_r. We denote the volume and the height by V and h, while x, y are the dimensions of the rectangular base and z is the radio of the circular base. Taking into account that $V = \pi z^2 h$ and $S_c = 2\pi z h$ for the can with circular base, whereas $V = xyh$ and $S_r = 2(x+y)h$ for the one with rectangular base, and that $\sqrt{xy} \leq \frac{x+y}{2}$ by the geometric-arithmetic inequality, we have

$$\begin{aligned} S_r - S_c &= 2h(x + y - \pi z) \geq 2h\left(2\sqrt{xy} - \pi z\right) \\ &= 2h\left(2\sqrt{\frac{V}{h}} - \sqrt{\pi}\sqrt{\frac{V}{h}}\right) = 2\sqrt{hV}\left(2 - \sqrt{\pi}\right) > 0. \end{aligned}$$

3.15 We denote the base dimensions by x_1 and x_2 and the height by x_3.
(a) The problem to solve is

$$\begin{aligned} P_1 : \text{Max } & f_1(x) = x_1 x_2 x_3 \\ \text{s.t. } & h_1(x) = x_1 x_2 + 2x_1 x_3 + 2x_2 x_3 = S \\ & x \in \mathbb{R}^n_{++}. \end{aligned}$$

By the geometric-arithmetic inequality, one has

$$\begin{aligned} S &= 3\left(\frac{x_1x_2 + 2x_1x_3 + 2x_2x_3}{3}\right) \geq 3(x_1x_2)^{\frac{1}{3}}(2x_1x_3)^{\frac{1}{3}}(2x_2x_3)^{\frac{1}{3}} \\ &= 3 \times 4^{\frac{1}{3}}(x_1x_2x_3)^{\frac{2}{3}} = 3 \times 4^{\frac{1}{3}}(f_1(x))^{\frac{2}{3}}. \end{aligned} \tag{A.2}$$

The maximum of $f_1(x)$ is attained when the inequality (A.2) is an equality, that is, when $x_1x_2 = 2x_1x_3 = 2x_2x_3 = \frac{S}{3}$. So, the optimal solution is $\overline{x}=\left(\sqrt{\frac{S}{3}}, \sqrt{\frac{S}{3}}, \frac{1}{2}\sqrt{\frac{S}{3}}\right)^T$.
(b) Now, the problem is

$$\begin{aligned} P_2 : \text{Min } & f_2(x) = x_1x_2 + 2x_1x_3 + 2x_2x_3 \\ \text{s.t. } & h_2(x) = x_1x_2x_3 = V \\ & x \in \mathbb{R}^n_{++}. \end{aligned}$$

By the geometric-arithmetic inequality, we have

$$f_2(x) = 3\left(\frac{x_1x_2 + 2x_1x_3 + 2x_2x_3}{3}\right) \geq 3 \times 4^{\frac{1}{3}} V^{\frac{2}{3}}.$$

The optimal solution $\overline{x} = \left(\sqrt[3]{2V}, \sqrt[3]{2V}, \frac{\sqrt[3]{2V}}{2}\right)^T$ is obtained by solving $x_1x_2 = 2x_1x_3 = 2x_2x_3 = 4^{\frac{1}{3}} V^{\frac{2}{3}}$.

3.16 We denote the base radio and the height by x_1 and x_2, respectively.
(a) The problem to solve is

$$\begin{aligned} P_1 : \text{Max } & f_1(x) = x_1^2x_2 \\ \text{s.t. } & h_1(x) = c_1x_1^2 + c_2x_1x_2 = k_0, \\ & x \in \mathbb{R}^2_{++}, \end{aligned}$$

with $k_0 = \frac{c_0}{2\pi}$. By the geometric-arithmetic inequality, we have

$$\begin{aligned} h_1(x) &= 3\left(\frac{1}{3}c_1x_1^2 + \frac{1}{3}\frac{c_2x_1x_2}{2} + \frac{1}{3}\frac{c_2x_1x_2}{2}\right) \geq 3\left(c_1x_1^2\right)^{\frac{1}{3}}\left(\frac{c_2x_1x_2}{2}\right)^{\frac{2}{3}} \\ &= 3 \times 2^{-\frac{2}{3}} c_1^{\frac{1}{3}} c_2^{\frac{2}{3}} (f_1(x))^{\frac{2}{3}}, \end{aligned} \tag{A.3}$$

so the maximum value for $f_1(x)$ is obtained when $c_1x_1^2 = \frac{c_2x_1x_2}{2}$, that is, at

$$\overline{x} = \left(\sqrt{\frac{k_0}{3c_1}}, 2\sqrt{\frac{k_0c_1}{3c_2}}\right)^T.$$

(b) We have to solve the problem

$$\begin{aligned} P_2 : \text{Min } & f_2(x) = c_1x_1^2 + c_2x_1x_2 \\ \text{s.t. } & h_2(x) = x_1^2x_2 = \frac{V_0}{\pi} \\ & x \in \mathbb{R}^2_{++}. \end{aligned}$$

Now, inequality (A.3) can be rewritten as

$$f_2(x) \geq 3c_1^{\frac{1}{3}} \left(\frac{c_2}{2}\right)^{\frac{2}{3}} (h_2(x))^{\frac{2}{3}},$$

and the minimum cost is attained at

$$\overline{x} = \left(\sqrt[3]{\frac{c_2 V_0}{2c_1 \pi}}, \sqrt[3]{\frac{4c_1^2 V_0}{c_2 \pi}} \right)^T.$$

3.20 By applying the geometric-arithmetic inequality, we deduce that

$$f(x) = 5\left(\frac{1}{5}\left(\frac{500}{x_1 x_2}\right) + \frac{1}{5}\left(\frac{500}{x_1 x_2}\right) + \frac{1}{5}(2x_1) + \frac{1}{5}(2x_2) + \frac{1}{5}(x_1 x_2)\right) \geq 5 \times 1000^{\frac{2}{5}},$$

where the equality is obtained if and only if

$$\frac{500}{x_1 x_2} = 2x_1 = 2x_2 = x_1 x_2. \tag{A.4}$$

Since the system (A.4) is inconsistent, the lower bound given by the geometric-arithmetic inequality is never attained and we cannot solve the problem by means of the geometric-arithmetic inequality. Now, we consider the dual problem D in order to find the optimal solution of P. Since the dual feasible set G is formed by the solutions of the linear system

$$\left\{ \begin{array}{rrrrl} y_1 & +y_2 & +y_3 & +y_4 & = 1 \\ -y_1 & +y_2 & & +y_4 & = 0 \\ -y_1 & & +y_3 & +y_4 & = 0 \end{array} \right\},$$

with $y \in \mathbb{R}^4_{++}$, after some algebra, we obtain

$$G = \left\{ (y_1, 1 - 2y_1, 1 - 2y_1, 3y_1 - 1)^T : \frac{1}{3} < y_1 < \frac{1}{2} \right\}.$$

Therefore, D consists in maximizing

$$g(y) = \left(\frac{1}{3y_1 - 1}\right)^{3y_1 - 1} \left(\frac{2}{1 - 2y_1}\right)^{2(1-2y_1)} \left(\frac{1000}{y_1}\right)^{y_1}.$$

Taking $s = y_1$ to simplify and using the logarithmic transformation, D is equivalent to maximize

$$h(s) = (1 - 3s) \ln(3s - 1) + 2(1 - 2s)(\ln 2 - \ln(1 - 2s)) + s(\ln 1000 - \ln s)$$

on $\left]\frac{1}{3}, \frac{1}{2}\right[$. The maximum of h is attained at the unique point of the interval where the derivative vanishes, that is, approximately, the point $s = 0.438897$ corresponding to the point

$$\bar{y} = (0.4389, 0.1222, 0.1222, 0.3167)^T,$$

with $g(\bar{y}) = 84.82073$. After solving the system (3.22), we obtain the solution $\bar{x} = (5.18284, 5.18284)^T$.

3.23 The feasible set G of the dual problem is formed by the solutions on $\mathbb{R}^4_{++}$ of the system

$$\left\{\begin{array}{rrrrl} y_1 & +y_2 & +y_3 & +y_4 & = 1 \\ & & 3y_3 & -y_4 & = 0 \\ & -6y_2 & & +4y_4 & = 0 \\ -y_1 & & & +2y_4 & = 0 \\ & 2y_2 & +2y_3 & +2y_4 & = 0 \end{array}\right\}.$$

As this system is inconsistent, $G = \emptyset$ and $F^* = \emptyset$.

Problems of Chapter 4

4.2 (a) The problem is

$$\begin{array}{ll} P_1 : \text{Min} & f(x, y) = x^3 + y^3 + (a - x - y)^3 \\ \text{s.t.} & (x, y)^T \in \mathbb{R}^2. \end{array}$$

As $f(r, r) \to -\infty$ when $r \to \infty$, $v(P_1) = -\infty$.
(b) Now the problem is

$$\begin{array}{ll} P_2 : \text{Min} & f(x, y) = x^3 + y^3 + (a - x - y)^3 \\ \text{s.t.} & (x, y)^T \in F_2, \end{array}$$

where

$$F_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x > 0, y > 0, x + y < a \right\}$$

is an open convex set. As

$$\nabla^2 f(x, y) = 6 \begin{bmatrix} a - y & a - x - y \\ a - x - y & a - x \end{bmatrix}$$

is positive definite on F_2, f is strictly convex on F_2. Then, F_2^* is formed by the critical points of f on F, that is, $F_2^* = \left\{ \left(\frac{a}{3}, \frac{a}{3}\right)^T \right\}$. The optimal solution consists in decomposing a into three equal parts.
(c) In this case, the problem is a convex optimization problem with linear constraints:

$$\begin{aligned} P_3 : \text{Min } & f(x, y) = x^3 + y^3 + (a - x - y)^3 \\ \text{s.t. } & x + y \leq a \\ & -x \leq 0, -y \leq 0. \end{aligned}$$

Any $\binom{x}{y} \in F_3 = \text{cl}\, F_2$ can be written as $\binom{x}{y} = \lim_{r\to\infty}\binom{x_r}{y_r}$, with $\binom{x_r}{y_r} \in F_2$ for all $r \in \mathbb{N}$. Therefore, $f\left(\frac{a}{3}, \frac{a}{3}\right) \leq f(x_r, y_r)$ and taking limits when $r \to \infty$ we obtain $f\left(\frac{a}{3}, \frac{a}{3}\right) \leq f(x, y)$. The uniqueness is a consequence of that f is strictly convex.

4.4 Taking into account that the function $y \mapsto \frac{1}{y}$ is decreasing on $\mathbb{R}_{++}$, P is equivalent to the problem P_1 obtained by replacing the objective function f by $f_1(x_1, x_2) = -x_1x_2$. Consider the relaxed problem P_2 obtained by replacing the strict inequalities $x_1 > 0$ and $x_2 > 0$ by the weak ones $-x_1 \leq 0$ and $-x_2 \leq 0$, whose unique minimal solution is $\overline{x} = (1, 1)^T$. Denoting by F_1 and F_2 the feasible sets of P_1 and P_2, one has $\overline{x} \in F_1 \subset F_2$, so that $\overline{x}$ is unique minimal solution of P_1 and, consequently, the unique minimal solution of P.

4.6 We want to know whether $\overline{x}$ is the unique optimal solution of the problem

$$\begin{aligned} P : \text{ Min } & f(x) = \|x\|^2 \\ \text{s.t. } & x \in F, \end{aligned}$$

where F represents the solution set of the system. P is a convex quadratic optimization problem with convex quadratic objective function f and linear inequality constraints, written in the form $g_i(x) \leq 0, i = 1, \ldots, 6$ (with "$\leq$" instead of "$\geq$"). As f is strongly convex, P has a unique optimal solution. We check that $\overline{x} \in F$. Moreover, $I(\overline{x}) = \{2, 3\}$, with $\nabla f(\overline{x}) + \frac{3}{2}\nabla g_2(\overline{x}) + \frac{31}{6}\nabla g_3(\overline{x}) = 0_4$. Therefore, $F^* = \{\overline{x}\}$.

4.7 (a) It is obtained by doing the variable change $y_1 = 2x_1$ and $y_2 = x_2$; (b) $\overline{y} = (2, 2)^T$, that is, $\overline{x} = (1, 2)^T$; (d) $\overline{y} = (2, 3)^T$, that is, $\overline{x} = (1, 3)^T$.

4.11 The problem P is bounded because $f(x) = (x_1 - 1)^2 + x_2^2 - 1 \geq -1$. Moreover, P satisfies SCQ. As f is strongly convex (Proposition 3.1), $\overline{x} = \left(\frac{1}{2}, -\frac{1}{2}\right)^T$ is the unique optimal solution and $\overline{\lambda} = (1, 0)^T$ is the unique KKT vector. Therefore, the variation of the optimal value satisfies

$$\triangle\vartheta = \vartheta(z) - \vartheta(0_2) = \vartheta(z) - f\left(\frac{1}{2}, -\frac{1}{2}\right) = \vartheta(z) + \frac{1}{2} \geq -\overline{\lambda}^T z = -z_1,$$

for all $z \in \text{dom}\,\vartheta$. Thus, small perturbations of the right-hand side on the second constraint do not affect the optimal value (because it is not active at $\overline{x}$), whereas small perturbations of the right-hand side on the first constraint produce a minimum decreasing of the optimal value of one unit per each unit increased by the right-hand side.

4.15 We can consider any rectangular tetrahedron as the intersection of $\mathbb{R}^3_+$ with a half-space of the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$. Its vertices are $O(0, 0, 0)$, $A(a, 0, 0)$,

$B(0, b, 0)$, and $C(0, 0, c)$. The hypotenuse area is a half of the norm of the cross product of $AB = (-a, b, 0)$ and $AC = (-a, 0, c)$, that is, $\frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$. The pyramid height is the distance from the origin to the plane of the hypotenuse, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$, that is, $\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = h$, so we have that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{h^2} =: k$. The problem we need to solve is

$$\begin{aligned} P_1 : \text{Min } & f_1(a, b, c) = a^2b^2 + a^2c^2 + b^2c^2 \\ \text{s.t. } & \tfrac{1}{a^2} + \tfrac{1}{b^2} + \tfrac{1}{c^2} = k, \\ & (a, b, c)^T \in \mathbb{R}^3_{++}. \end{aligned}$$

Taking $x_1 := k^{-1}a^{-2}$, $x_2 = k^{-1}b^{-2}$, and $x_3 = k^{-1}c^{-2}$, we obtain the following geometric optimization problem which is equivalent to P_1:

$$\begin{aligned} P_2 : \text{Min } & f_2(x_1, x_2, x_3) = \tfrac{1}{x_1x_2} + \tfrac{1}{x_1x_3} + \tfrac{1}{x_2x_3} \\ \text{s.t. } & x_1 + x_2 + x_3 = 1, \\ & x \in \mathbb{R}^3_{++}. \end{aligned}$$

As we cannot solve P_2 by means of the geometric-arithmetic inequality, we consider the relaxed problem P_3 which is the result of eliminating the constraint set $x \in \mathbb{R}^3_{++}$. Applying KKT conditions, we obtain that the unique possible local minimum of P_3 is $\overline{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$. As $\overline{x} \in \mathbb{R}^3_{++}$, this point is also the unique possible local (global) minimum of P_2. Consequently, the unique possible global minimum of P_1 is obtained by taking $\overline{a} = \overline{b} = \overline{c} = \sqrt{\frac{3}{k}} = h\sqrt{3}$. Finally, we have to prove the existence of global minimum of P_1. We shall see that F_1 is closed (so the sets $S_\lambda(f_1)$ are also closed) and that $S_\lambda(f_1)$ is bounded for all $\lambda > 0$.
Let $\left\{(a_r, b_r, c_r)^T\right\}_{r\in\mathbb{N}}$ be a sequence in F_1 such that $(a_r, b_r, c_r) \to (a, b, c)$. As $\frac{1}{a_r^2} + \frac{1}{b_r^2} + \frac{1}{c_r^2} = k$, $\frac{1}{a_r^2} \le k$ and $a_r \ge \frac{1}{\sqrt{k}}$, so we have that $a \ge \frac{1}{\sqrt{k}} > 0$. Similarly, $b > 0$ and $c > 0$. Moreover, $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = k$ by a simple argument of continuity, so $(a, b, c)^T \in F_1$.
Let $(a, b, c)^T \in F_1$ be such that $f_1(a, b, c) \le \lambda$, with λ a positive sufficiently large number. Then, one has $a, b, c \ge \frac{1}{\sqrt{k}}$ and $ab, ac, bc \le \lambda$. Therefore, $\frac{1}{\sqrt{k}} \le a \le \frac{\lambda}{b} \le \lambda\sqrt{k}$, inequalities also valid for b and c. So, the set $S_\lambda(f_1)$ is contained in the cube $\left[\frac{1}{\sqrt{k}}, \lambda\sqrt{k}\right]^3$.

4.16 We prove that P does not satisfy SCQ because $\|x\| - x_1 = 0$ for all $x \in \mathcal{F}(0) = F$.
(a) One has

$$\mathcal{F}(z) = \left\{ \begin{array}{ll} \emptyset, & z < 0 \\ \mathbb{R}_+ \times \{0\}, & z = 0 \\ \left\{x \in \mathbb{R}^2 : x_1 \ge \frac{x_2^2}{2z} - \frac{z}{2}\right\}, & z > 0 \end{array} \right\} \text{ and } \vartheta(z) = \left\{ \begin{array}{ll} +\infty, & z < 0 \\ 1, & z = 0 \\ 0, & z > 0 \end{array} \right\}.$$

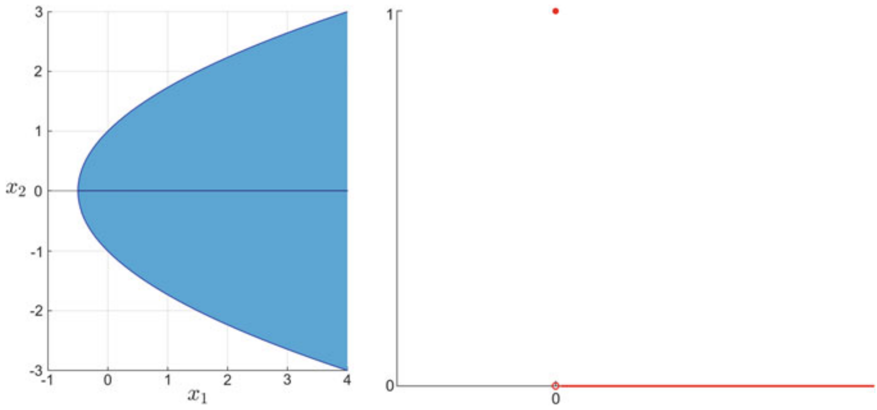

Fig. A.1 $\mathcal{F}(1)$ (left) y gph ϑ (right)

(b) ϑ is convex and differentiable on $\mathbb{R}_{++}$, but it is not continuous at 0, as it is shown in Fig. A.1. Also, observe that $0 \notin \operatorname{int} \operatorname{dom} \vartheta =]0, +\infty[$.
(c) There are no sensitivity vectors.
(d) As $L(x, 0) = e^{-x_2}$, $\inf_{x \in \mathbb{R}^2} L(x, 0) = 0 < 1 = v(P)$. If $\lambda > 0$,

$$\inf_{x \in \mathbb{R}^2} L(x, \lambda) \leq \inf_{z \in \mathbb{R}} L\big((z, 1)^T, \lambda\big) = \frac{1}{e} + \lambda \inf_{z \in \mathbb{R}} \left\{ \sqrt{z^2 + 1} - z \right\} = \frac{1}{e} < 1 = v(P).$$

Therefore, $v(D) \leq \frac{1}{e} < 1 = v(P)$.

4.17 It is easy to see that P satisfies SCQ.
(a) As the feasible set of $P(z)$ is $\mathcal{F}(z) = \left\{ x \in \mathbb{R}^2 : x_1 + x_2 \leq z \right\}$, dom $\mathcal{F} = \mathbb{R}$,

$$\operatorname{gph} \mathcal{F} = \left\{ \begin{pmatrix} z \\ x \end{pmatrix} \in \mathbb{R}^3 : x_1 + x_2 - z \leq 0 \right\} \text{ and } \vartheta(z) = \max\left\{ 0, -\frac{z}{\sqrt{2}} \right\};$$

see Fig. A.2.
(b) ϑ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$.
(c) The sensitivity vectors for P are the elements of the interval $\left[0, 1/\sqrt{2}\right]$.
(d) $F^* = \{0\}$.
(e) By Cauchy–Schwarz inequality, $|x_1 + x_2| \leq \sqrt{2}\|x\|$. Therefore, $x_1 + x_2 \geq -\sqrt{2}\|x\|$ and we have

$$L\left(x, \frac{1}{2}\right) = \|x\| + \frac{1}{2}(x_1 + x_2) \geq \|x\| - \frac{\sqrt{2}}{2}\|x\| = \left(1 - \frac{\sqrt{2}}{2}\right)\|x\| \to +\infty$$

when $\|x\| \to +\infty$. Thus, $L\left(x, \frac{1}{2}\right)$ attains its global minimum on $\mathbb{R}^2$. As $L\left(x, \frac{1}{2}\right)$ has no critical points on $\mathbb{R}^2 \setminus \{0_2\}$ (where it is differentiable), its unique global minimum

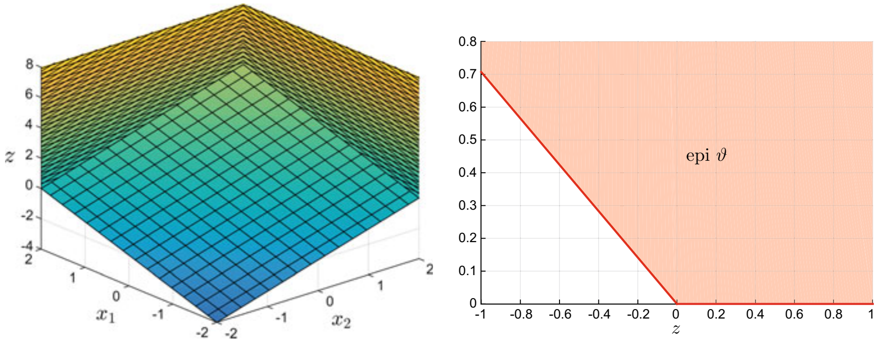

Fig. A.2 gph $\mathcal{F}$ (left) and gph ϑ (right)

is 0_2 and, effectively, $v(P) = \min_{x \in \mathbb{R}^2} L\left(x, \frac{1}{2}\right) = 0$ holds. Observe that $\left(0, 0, \frac{1}{2}\right)$ is a saddle point.

(f) If $\lambda > \frac{1}{\sqrt{2}}$, then $L(x_1, x_1, \lambda) = \sqrt{2}\left(|x_1| + \sqrt{2}\lambda x_1\right) \to -\infty$ when $x_1 \to -\infty$. Therefore,

$$h(y) = \begin{cases} 0, & 0 \leq y \leq \frac{1}{\sqrt{2}}, \\ -\infty, & y > \frac{1}{\sqrt{2}}. \end{cases}$$

Then, we have $v(D) = v(P) = 0$, with $G^* = \left[0, \frac{1}{\sqrt{2}}\right]$.

4.18 (a) $F^* = \left\{(-1, -1, -1)^T\right\}$ (the metric projection of 0_3 onto the boundary of F; (b) The value function is

$$\vartheta(z) = \begin{cases} \frac{(z-3)^2}{3}, & z < 3, \\ 0, & z \geq 3, \end{cases}$$

and the unique sensitivity vector (a scalar here) is $\overline{\lambda} = -\vartheta'(0) = 2$; (c) Here $Q = 2I_3$, $c = 0_3$, $A = (1, 1, 1)$, and $b = -3$.

Regarding the Lagrange dual of P, $L_Q(x, y) = \|x\|^2 + (x_1 + x_2 + x_3 + 3)y$, $h(y) = -\frac{3y^2}{4} + 3y$, and $v\left(D^L\right) = 3$, with unique optimal solution 2.

The Wolfe dual problem of P is

$$\begin{aligned} D_Q^W : \text{Max } & -\|u\|^2 + 3y \\ & u_i = -\tfrac{y}{2}, i = 1, 2, 3, \\ \text{s.t. } & y \geq 0, \end{aligned}$$

whose unique optimal solution is $(-1, -1, -1, 2)$, with $v\left(D_Q^W\right) = 3$.

Thus, strong duality holds for both dual pairs.

4.19 (a) The analytical solution cannot be obtained via KKT because P is convex but it does not satisfy SCQ. Nevertheless, $F^* = \{0\}$ because $F = \{0\}$.
(b) The feasible set multifunction is

$$\mathcal{F}(z) = \begin{cases} \emptyset, & z < 0, \\ \{0\}, & z = 0, \\ \left[-\sqrt{z}, \sqrt{z}\right], & z > 0, \end{cases}$$

with

$$\operatorname{gph} \mathcal{F} = \left\{(z, x) \in \mathbb{R}_+ \times \mathbb{R} : x^2 \leq z\right\} = \operatorname{conv}\left\{\left(x^2, x\right) : x \in \mathbb{R}_+\right\}.$$

(c) The value function is

$$\vartheta(z) = \begin{cases} +\infty, & z < 0, \\ -\sqrt{z}, & z \geq 0. \end{cases}$$

(d) ϑ is differentiable on $\operatorname{dom} \vartheta = \mathbb{R}_{++}$.
(e) There does not exist any sensitivity vector (note that P is bounded, but it does not satisfy SCQ).
(f) The Lagrange function is $L(x, \lambda) = x + \lambda x^2$, whose graph shown in Fig. A.3 suggests that there exist no saddle points. Indeed, let us suppose that $\left(\overline{x}, \overline{\lambda}\right) \in \mathbb{R} \times \mathbb{R}_+$ is a saddle point of L, that is,

$$\overline{x} + \lambda \overline{x}^2 \leq \overline{x} + \overline{\lambda}\overline{x}^2 \leq x + \overline{\lambda} x^2 \quad \forall x \in \mathbb{R}, \lambda \in \mathbb{R}_+.$$

From the first inequality, we deduce that $\overline{x} = 0$. Replacing it in the second one we have

$$0 \leq x + \overline{\lambda} x^2 \quad \forall x \in \mathbb{R},$$

which provides a contradiction, whether $\overline{\lambda} = 0$ or $\overline{\lambda} > 0$. Therefore, L has no saddle points.
(g) There exist no KKT vectors.
(h) Since SCQ fails, strong duality may fail too for both dual pairs.
On the one side, the Lagrange dual of P is

$$\begin{aligned} D^L : &\operatorname{Max} h(y) \\ &\text{s.t.} \quad y \geq 0, \end{aligned}$$

where

$$h(y) = \begin{cases} -\frac{1}{4y}, & y > 0, \\ -\infty, & \text{else}, \end{cases}$$

so that $v\left(D^L\right) = 0$, but the supremum of h is not attained on $\mathbb{R}_+$.
On the other side, the Wolfe dual of P is

$$\begin{aligned} D^W : &\operatorname{Max} L(u, y) = u + yu^2 \\ &\text{s.t.} \quad y \geq 0, 1 + 2yu = 0, \end{aligned}$$

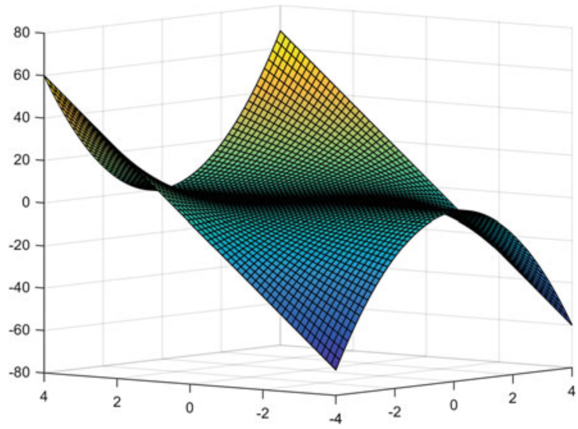

Fig. A.3 Graph of the Lagrange function is $L(x, \lambda) = x + \lambda x^2$

whose feasible set is the branch of hyperbola

$$G = \left\{ \left(u, -\frac{1}{2u} \right) \in \mathbb{R}^2 : u < 0 \right\}.$$

Since $L\left(u, -\frac{1}{2u}\right) = \frac{u}{2}$, $v\left(D^W\right) = 0$, but the supremum of L on G is not attained.

We conclude that the strong duality fails for both dual pairs, even though the duality gap is zero in both cases.

Problems of Chapter 5

5.2 (a) According with the computations in the proof of Proposition 5.17, one has that the stepsize when performing an exact line search at the point x_k in the direction $-g_k \equiv -\nabla f(x_k) = -Qx_k$ is

$$\alpha_k = \frac{g_k^T g_k}{g_k^T Q g_k}.$$

Thus, we obtain

$$g_0 = Q\left(\frac{1}{\lambda_{min}} u_{min} + \frac{1}{\lambda_{max}} u_{max}\right) = u_{min} + u_{max},$$
$$g_0^T g_0 = (u_{min} + u_{max})^T (u_{min} + u_{max}) = \|u_{min}\|^2 + \|u_{max}\|^2 = 2,$$
$$g_0^T Q g_0 = (u_{min} + u_{max})^T (\lambda_{min} u_{min} + \lambda_{max} u_{max}) = \lambda_{min} + \lambda_{max},$$

Therefore,

$$\alpha_0 = \frac{2}{\lambda_{min} + \lambda_{max}},$$

and

$$\begin{aligned} x_1 = x_0 - \alpha_0 g_0 &= \frac{1}{\lambda_{min}} u_{min} + \frac{1}{\lambda_{max}} u_{max} - \frac{2}{\lambda_{min} + \lambda_{max}} (u_{min} + u_{max}) \\ &= \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \left(\frac{1}{\lambda_{min}} u_{min} - \frac{1}{\lambda_{max}} u_{max} \right). \end{aligned}$$

Hence, we deduce

$$\begin{aligned} f(x_1) &= \frac{1}{2} \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 \left(\frac{1}{\lambda_{min}} u_{min} - \frac{1}{\lambda_{max}} u_{max} \right)^T Q \left(\frac{1}{\lambda_{min}} u_{min} - \frac{1}{\lambda_{max}} u_{max} \right) \\ &= \frac{1}{2} \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 \left(\frac{1}{\lambda_{min}} u_{min} - \frac{1}{\lambda_{max}} u_{max} \right)^T (u_{min} - u_{max}) \\ &= \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 \frac{1}{2} \left(\frac{1}{\lambda_{min}} + \frac{1}{\lambda_{max}} \right) = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 f(x_0). \end{aligned}$$

Similar computations produce the following results

$$x_2 = x_1 - \alpha_1 g_1 = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 \left(\frac{1}{\lambda_{min}} u_{min} + \frac{1}{\lambda_{max}} u_{max} \right) = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 x_0,$$

and

$$f(x_2) = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 f(x_1) = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^4 f(x_0).$$

Completing the reasoning (by finite induction), it is possible to check that

$$x_k = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^k \left(\frac{1}{\lambda_{min}} u_{min} + \frac{(-1)^k}{\lambda_{max}} u_{max} \right), \quad k = 0, 1, 2, \ldots,$$

deducing that

$$f(x_{k+1}) = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2 f(x_k) = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^{2k+2} f(x_0).$$

(b) Since

$$\frac{f(x_{k+1})}{f(x_k)} = \left(\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \right)^2,$$

the upper bound for the quotient $\frac{f(x_{k+1})}{f(x_k)}$ established in Proposition 5.17 is attained in this case.

(c) On one side the even points $x_0, x_2, \ldots, x_{2p}, \ldots$, and on the other side the odd points $x_1, x_3, \ldots, x_{2p+1}, \ldots$, are aligned with the origin, which is the unique global minimum of the function f.

5.5 (a) On one side, if $x \neq 0_n$, we have

$$\nabla f(x) = \frac{3}{2}\|x\|^{1/2}\nabla(\|x\|) = \frac{3}{2}\|x\|^{1/2}\frac{x}{\|x\|} = \frac{3}{2}\frac{x}{\|x\|^{1/2}},$$

and on the other side, computing the partial derivatives of f in $x = 0_n$, we obtain

$$\nabla f(0_n) = 0_n.$$

Therefore,

$$\|\nabla f(x)\| = \frac{3}{2}\|x\|^{1/2}, \quad \text{for all } x \in \mathbb{R}^n.$$

Any open set U containing the sublevel set $\{x \in \mathbb{R}^n : f(x) \leq K\}$ would contain certain ball $\rho\mathbb{B}$, since 0_n belongs to this sublevel set. If ∇f is Lipschitz continuous on U, there exists some constant $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \forall x, y \in U. \tag{A.5}$$

If we choose some x such that $\|x\| \leq \rho$ and $y = 0_n$, by (A.5), we get

$$\frac{3}{2}\|x\|^{1/2} \leq L\|x\|,$$

that is,

$$\frac{3}{2L} \leq \|x\|^{1/2},$$

but this inequality cannot hold if $\|x\|$ is sufficiently small.

(b) One has that

$$x_{k+1} = x_k - \alpha\nabla f(x_k) = \left(1 - \frac{3\alpha}{2}\frac{1}{\|x_k\|^{1/2}}\right)x_k, \quad k = 0, 1, \ldots.$$

Case 1.- Observe that if

$$1 - \frac{3\alpha}{2}\frac{1}{\|x_k\|^{1/2}} = 0 \;\Leftrightarrow\; \|x_k\| = \frac{9\alpha^2}{4},$$

it must be $x_{k+1} = 0_n$, and the method converges to the unique global minimum $\overline{x} = 0_n$ in a finite number of iterations.

Case 2.- If

$$1 - \frac{3\alpha}{2}\frac{1}{\|x_k\|^{1/2}} < -1 \;\Leftrightarrow\; \|x_k\| < \frac{9\alpha^2}{16},$$

then

$$\|x_{k+1}\| = \|x_k\| \left| 1 - \frac{3\alpha}{2} \frac{1}{\|x_k\|^{1/2}} \right| > \|x_k\|.$$

Case 3.- If

$$1 - \frac{3\alpha}{2} \frac{1}{\|x_k\|^{1/2}} > -1 \;\Leftrightarrow\; \|x_k\| > \frac{9\alpha^2}{16},$$

then

$$\|x_{k+1}\| < \|x_k\|.$$

Case 4.- Finally, if

$$1 - \frac{3\alpha}{2} \frac{1}{\|x_k\|^{1/2}} = -1 \;\Leftrightarrow\; \|x_k\| = \frac{9\alpha^2}{16},$$

then

$$x_{k+1} = -x_k,$$

and the algorithm infinitely oscillates between the two points x_k and $-x_k$, starting from the iteration k.

5.8 Since the matrix Q is invertible, all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are different than zero, and we can consider the partition of the set of indices

$$N := \{i \mid \lambda_i < 0\} \quad \text{and} \quad P := \{i \mid \lambda_i > 0\}.$$

By hypothesis $N \neq \emptyset$.

Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of associated eigenvectors. If x_0 is a starting point for the algorithm, we can express

$$x_0 = \sum_{i \in N} \xi_i u_i + \sum_{i \in P} \xi_i u_i.$$

We have

$$\begin{aligned} x_{k+1} &= x_k - \alpha Q x_k = (I - \alpha Q) x_k = \ldots = (I - \alpha Q)^k x_0 \\ &= \sum_{i \in N} \xi_i (1 - \alpha \lambda_i)^k u_i + \sum_{i \in P} \xi_i (1 - \alpha \lambda_i)^k u_i. \end{aligned}$$

Taking norms

$$\|x_{k+1}\|^2 = \sum_{i \in N} (\xi_i)^2 (1 - \alpha \lambda_i)^{2k} + \sum_{i \in P} (\xi_i)^2 (1 - \alpha \lambda_i)^{2k}.$$

Case 1.- If there exists $i_0 \in N$ such that $\xi_{i_0} \neq 0$, we have

$$\lim_{k\to\infty} \|x_{k+1}\|^2 \geq \lim_{k\to\infty} (\xi_{i_0})^2(1-\alpha\lambda_{i_0})^{2k} = +\infty,$$

since $1-\alpha\lambda_{i_0} > 1$, and the method diverges (neither the sequence $\{x_k\}$ nor any of its subsequences converge).

Case 2.- Otherwise, if we assume that $\xi_i = 0$ for all $i \in N$ and also $P \neq \emptyset$, and one takes α satisfying

$$\alpha < \min\{2/\lambda_i : i \in P\},$$

one has

$$|1-\alpha\lambda_i| < 1, \quad \text{for all } i \in P,$$

and consequently

$$\lim_{k\to\infty} \|x_{k+1}\|^2 = \lim_{k\to\infty} \sum_{i\in P} \xi_i^2(1-\alpha\lambda_i)^{2k} = 0.$$

Thus, the sequence $\{x_k\}$ converges to 0_n, which is a saddle point of the function f.

5.9 Obviously, $\overline{x} = 0$ is the (strict global) minimum of f over $\mathbb{R}$.

(a) We have that

$$f'(x) = 2x^2 \operatorname{sign}(x) + x.$$

The function f is convex, since $f''(x) = 4|x| + 1 > 0$ for all $x \in \mathbb{R}$.

If we apply the steepest descent method with $\alpha_k = \frac{\gamma}{k+1}$, we obtain

$$x_{k+1} = x_k\left(1 - \frac{\gamma(2|x_k|+1)}{k+1}\right), \quad \text{for } k = 0, 1, \ldots. \tag{A.6}$$

Let $\gamma = 1$. If $x_k \geq k+1$, from the expression above we deduce

$$x_{k+1} = x_k\left(1 - \frac{2x_k+1}{k+1}\right) = \frac{x_k(k-2x_k)}{k+1} \leq -(k+2).$$

If $x_k \leq -(k+1)$, we have

$$x_{k+1} = x_k\left(1 - \frac{-2x_k+1}{k+1}\right) = \frac{x_k(k+2x_k)}{k+1} \geq k+2.$$

We conclude that, for all k,

$$|x_k| \geq k+1 \Rightarrow |x_{k+1}| \geq k+2.$$

Since $|x_0| \geq 1$, by induction, we obtain that $|x_k| \geq k+1$, for all k, and the sequence $\{x_k\}$ diverges.

(b) Define

$$y_k := |x_k|, \quad k = 1, 2, \ldots.$$

Then, condition (5.101) becomes

$$\gamma(2y_0 + 1) < 2, \tag{A.7}$$

and (A.6) implies

$$y_{k+1} = y_k \left| 1 - \frac{\gamma(2y_k + 1)}{k+1} \right|, \quad k = 0, 1, \ldots. \tag{A.8}$$

Based on (A.8), we will prove by induction that

$$\gamma(2y_k + 1) < 2, \quad k = 0, 1, \ldots. \tag{A.9}$$

Obviously (A.9) holds for $k = 0$ by the choice of x_0 (see (A.7)). Let us check that if (A.9) holds for some k, then it also holds for $k + 1$. In fact, (A.9) implies

$$1 - \frac{\gamma(2y_k + 1)}{k+1} > 1 - \frac{2}{k+1}. \tag{A.10}$$

If $k = 0$, we deduce from (A.7) that

$$1 > 1 - \gamma(2y_0 + 1) > -1,$$

and by (A.8)

$$y_1 < y_0.$$

If $k > 0$, by (A.10) we get

$$1 > 1 - \frac{\gamma(2y_k + 1)}{k+1} \geq 0, \tag{A.11}$$

and by (A.8),

$$y_{k+1} < y_k,$$

and also

$$\gamma(2y_{k+1} + 1) < \gamma(2y_k + 1) < 2.$$

Therefore, (A.9) holds for all k.

Finally, let us prove that $\{y_k\}$ converges to 0. By (A.8) and (A.11), we deduce that, for $k = 0, 1, \ldots$,

$$y_{k+1} = y_k \left(1 - \frac{\gamma(2y_k + 1)}{k+1} \right) < y_k \left(1 - \frac{\gamma}{k+1} \right) < y_k \exp\left(-\frac{\gamma}{k+1} \right),$$

where we have used the inequality

$$1 - x < \exp(-x), \quad \text{for all } x > 0.$$

Thus,

$$\begin{aligned} y_{k+1} &< y_k \exp\left(-\frac{\gamma}{k+1}\right) < y_{k-1} \exp\left(-\frac{\gamma}{k+1} - \frac{\gamma}{k}\right) \\ &< y_0 \exp\left\{-\gamma\left(1 + \frac{1}{2} + \ldots + \frac{1}{k+1}\right)\right\}, \end{aligned}$$

and then,

$$0 \leq \lim_{k\to\infty} y_{k+1} \leq y_0 \exp\left\{-\gamma \lim_{k\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{k+1}\right)\right\} = 0.$$

5.11 Having in mind that $\nabla f(x_k) = Qx_k$, we can write

$$x_{k+1} = (I - \alpha Q)x_k - \alpha e_k.$$

Hence,

$$\begin{aligned} \|x_{k+1}\| &\leq \|(I - \alpha Q)x_k\| + \|-\alpha e_k\| \leq \max\{|1 - \alpha\lambda_{min}|, |1 - \alpha\lambda_{max}|\}\|x_k\| + \alpha\delta \\ &= q\|x_k\| + \alpha\delta \leq \ldots \leq q^{k+1}\|x_0\| + \alpha\delta(1 + q + \ldots + q^k). \end{aligned}$$

If $q < 1$, we obtain (5.102).

5.14 (a) We begin by plotting the graph of the function f over the interval [2, 6], see Fig. A.4.

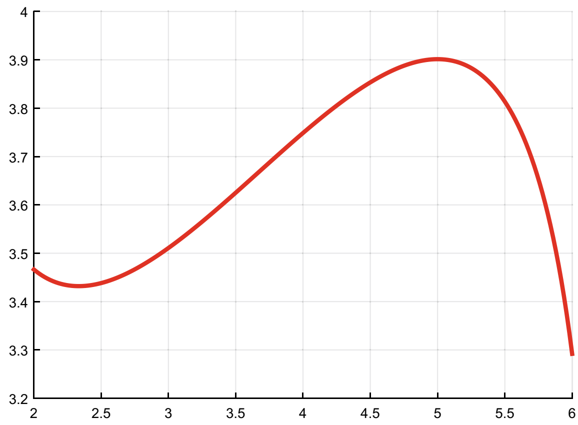

Fig. A.4 Graph of the function f

By making the derivative of f equal to zero,

$$f'(x) = 1 - \frac{1}{x-1} - \frac{1}{\frac{19}{3} - x} = 0$$

we obtain two solutions: $\overline{x} = \frac{7}{3}, \widehat{x} = 5$. Since

$$f''(x) = \frac{1}{(x-1)^2} - \frac{1}{\left(\frac{19}{3} - x\right)^2},$$

one has that $f''(\overline{x}) = 0.5$, i.e., $\overline{x} = 7/3$ is a local minimum, and $f''(\widehat{x}) = -0.5$, i.e., $\widehat{x} = 5$ is a local maximum.

Further, $f''(\widetilde{x}) = 0 \Rightarrow \widetilde{x} = 11/3$. Therefore, the function f is convex on the interval $]1, 11/3[$.

(b) To apply the hint, we should check that $\varphi = f''$ is convex on $[2, 3] \subset]1, 11/3[$. Since

$$\varphi''(x) = f''''(x) = \frac{6}{(x-1)^4} - \frac{6}{(19/3 - x)^4} \geq 0, \quad \forall x \in]1, 11/3[,$$

the convexity of φ is proved. Then, by (2.14), the Lipschitz constant of f'' on $[2, 3]$ is

$$L = \max\left\{\left|f'''(2)\right|, \left|f'''(3)\right|\right\}.$$

Since

$$f'''(x) = \frac{2}{\left(x - \frac{19}{3}\right)^3} - \frac{2}{(x-1)^3},$$

we obtain $f'''(2) = -2.0246$ and $f'''(3) = -0.304$, and consequently, $L = 2.0246$.

(c) On one hand, it must be

$$\beta \leq \frac{7}{3} - 2 = \frac{1}{3}.$$

On the other hand, one has

$$\widetilde{L} := L\left|1/f''(\overline{x})\right| = 2.0246 \times 2 = 4.0492,$$

and

$$\beta\widetilde{L} < 1 \text{ if and only if } \beta < 1/(4.0492) = 0.24696.$$

Taking β as the minimum value between these two ($1/3$ and 0.24696), the convergence of the pure Newton's method with quadratic convergence rate is guaranteed if

$$x_0 \in [7/3 - 0.24696, 7/3 + 0.24696] = [2.0864, 2.5803].$$

(d) The pure Newton's method gives us the recursion formula

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} = x_k - \left(1 - \frac{1}{x_k - 1} - \frac{1}{\frac{19}{3} - x_k}\right) \frac{(x_k - 1)^2\left(\frac{19}{3} - x_k\right)^2}{\left(\frac{22}{3} - 2x_k\right)\frac{16}{3}},$$

and if we take as initial point $x_0 = 2.5$, we obtain $x_1 = 2.3075$, $x_2 = 2.3328$, and $x_3 = 2.3333$.

5.16 (a) We have that

$$\begin{aligned} \varphi(\lambda_1, \ldots, \lambda_n) := f\left(\sum_{i=1}^n \lambda_i d_i\right) &= \frac{1}{2}\left(\sum_{i=1}^n \lambda_i d_i\right)^T Q \left(\sum_{i=1}^n \lambda_i d_i\right) - b^T\left(\sum_{i=1}^n \lambda_i d_i\right) \\ &= \sum_{i=1}^n \left(\frac{1}{2}(\lambda_i d_i)^T Q(\lambda_i d_i) - b^T(\lambda_i d_i)\right) \\ &= \sum_{i=1}^n \left(\frac{1}{2}\lambda_i^2 {d_i}^T Q d_i - \lambda_i b^T d_i\right). \end{aligned}$$

(b) Since this function is separable with respect to the λ_i, we can separately solve (i.e., using parallel computing) the problems

$$(P_i) \qquad \min_{\lambda \in \mathbb{R}} \left\{\frac{1}{2}\left(d_i^T Q d_i\right)\lambda^2 - \left(b^T d_i\right)\lambda\right\}.$$

(c) The objective function (of one variable λ) of the problem (P_i) is quadratic and strictly convex. Its minimum can be obtained by making the first derivative equal to zero:

$$\left(d_i^T Q d_i\right)\lambda_i^* = b^T d_i,$$

that is,

$$\lambda_i^* = \frac{b^T d_i}{d_i^T Q d_i}, \; i = 1, 2, \ldots, n,$$

and the optimal value is

$$v(P) = \sum_{i=1}^n \left(\frac{1}{2}(\lambda_i^*)^2\left(d_i^T Q d_i\right) - \lambda_i^*\left(b^T d_i\right)\right) = -\frac{1}{2}\sum_{i=1}^n \frac{\left(b^T d_i\right)^2}{d_i^T Q d_i}.$$

5.19 The set $\{d_1, \ldots, d_r\}$ is necessarily linearly dependent (otherwise, there would exist a basis of $\mathbb{R}^n$ that spans $\mathbb{R}^n$ positively). Hence, there are scalars $\lambda_1, \ldots, \lambda_r$ (not all zero) such that $\lambda_1 d_1 + \ldots + \lambda_r d_r = 0_n$. Take $i_0 \in \{1, \ldots, r\}$ for which $\lambda_{i_0} \neq 0$; so

$$d_{i_0} = -\sum_{i=1, i\neq i_0}^{r} \frac{\lambda_i}{\lambda_{i_0}} d_i. \tag{A.12}$$

Now let d be an arbitrary vector in $\mathbb{R}^n$. Since $\{d_1, \ldots, d_r\}$ spans $\mathbb{R}^n$ positively, there exist nonnegative scalars $\alpha_1, \ldots, \alpha_r$ such that $d = \alpha_1 d_1 + \ldots + \alpha_r d_r$. Then, using (A.12), one gets

$$d = \sum_{i=1, i\neq i_0}^{r} \alpha_i d_i + \alpha_{i_0} d_{i_0} = \sum_{i=1, i\neq i_0}^{r} \left(\alpha_i - \lambda_i \frac{\alpha_{i_0}}{\lambda_{i_0}}\right) d_i$$

Since d is arbitrary, we have proved that $\{d_1, \ldots, d_r\} \setminus \{d_{i_0}\}$ spans $\mathbb{R}^n$.

Problems of Chapter 6

6.2 (a) Through the change of variables (6.62), we reformulate our optimization problem as follows:

$$\begin{aligned} \widetilde{P} : \text{Min } & \textstyle\sum_{i=1}^{n} \exp(y_i) \\ \text{s.t. } & \textstyle\sum_{i=1}^{n} y_i = 0. \end{aligned}$$

The Lagrangian function of problem $\widetilde{P}$ is

$$L(y, \lambda) = \sum_{i=1}^{n} (\exp(y_i) + \lambda y_i).$$

The first-order optimality conditions are: There exist $\overline{y} \in \mathbb{R}^n$ and $\overline{\lambda} \in \mathbb{R}$ such that

$$\nabla_y L(\overline{y}, \overline{\lambda}) = 0_n \Leftrightarrow \exp(\overline{y}_i) + \overline{\lambda} = 0,\ i = 1, 2, \ldots, n, \tag{A.13}$$

and

$$\nabla_\lambda L(\overline{y}, \overline{\lambda}) = 0 \Leftrightarrow \sum_{i=1}^{n} \overline{y}_i = 0. \tag{A.14}$$

The system of $n+1$ equations with $n+1$ unknowns (A.13)–(A.14) has a *unique* solution

$$\overline{y}_i = 0,\ i = 1, 2, \ldots, n,\ \overline{\lambda} = -1.$$

Since the objective function $\sum_{i=1}^{n} \exp(y_i)$ is continuous and coercive over the feasible set F of points y such that $\sum_{i=1}^{n} y_i = 0$ (if $y \in F$ and $\|y\| \to \infty$, one must have $\|y\|_\infty = \max_{i=1,2,\ldots,n} |y_i| = \max_{i=1,2,\ldots,n} y_i \to \infty$), the problem $\widetilde{P}$ must have a global minimum (which is unique, why?). Moreover, since every feasible point is regular (if $h(y) := \sum_{i=1}^{n} y_i$, $\nabla h(y) = (1, \ldots, 1)^T$), the Lagrange necessary optimality conditions must be satisfied in such a minimum, which has to be $\overline{y} = 0_n$. Obviously, this minimum corresponds to the unique global minimum of P, which is

$$\overline{x}_i = \exp(\overline{y}_i) = 1, \quad i = 1, 2, \ldots, n.$$

(b) Let $x_1, \ldots, x_n$ be some arbitrary positive numbers, and consider the product

$$a := x_1 x_2 \ldots x_n.$$

Then

$$\left(\frac{x_1}{a^{1/n}}\right)\left(\frac{x_2}{a^{1/n}}\right)\ldots\left(\frac{x_n}{a^{1/n}}\right) = 1,$$

and, by (a),

$$\sum_{i=1}^{n} \frac{x_i}{a^{1/n}} \geq n \text{ (optimal value of problem } P).$$

Consequently,

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq a^{1/n} = (x_1 x_2 \ldots x_n)^{1/n}.$$

6.3 Because the feasible set is compact, there exists a global maximum. Since the gradient of the function in the unique constraint is never zero on the feasible set, the Lagrange conditions must hold at $\overline{x}$. The Lagrange function is

$$L(x, \lambda) = y^T x + \lambda\left(\|x\|^2 - 1\right),$$

and the Lagrange conditions

$$y + 2\overline{\lambda}\overline{x} = 0_n, \ \|\overline{x}\| = 1.$$

Hence, since $2\overline{\lambda}\overline{x} = -y$, taking norms on both sides, we get

$$2\left|\overline{\lambda}\right| \|\overline{x}\| = 2\left|\overline{\lambda}\right| = \|-y\| = 1,$$

whence, $\overline{\lambda} = \pm 1/2$. Since

$$y^T \overline{x} = -2\overline{\lambda}(\overline{x})^T \overline{x} = -2\overline{\lambda},$$

if $\overline{x}$ is a maximum, it must be $\overline{\lambda} = -1/2$, $y^T \overline{x} = 1$, and $\overline{x} = y$, leading to the well-known Cauchy–Schwarz inequality for the Euclidean norm.

6.5 We denote

$$x(t) := (t \cos(1/t), t \sin(1/t))^T, \quad \text{for } t > 0.$$

Obviously, $\|x(t)\| = t$, so every point of F can be identified by its norm, and $x(t) \to 0_2$ if and only if $t \to 0$.

(i) Let us prove first that $\mathcal{T}_{0_2} = \mathbb{R}^2$.

Since $\mathcal{T}_{0_2}$ is a cone, it is sufficient to prove that if $d \in \mathbb{R}^2$ is a vector with norm one, then $d \in \mathcal{T}_{0_2}$. If d has norm one, there must exist some $\theta \in [0, 2\pi[$ such that $d \in (\cos\theta, \sin\theta)^T$. Consider now

$$t_r := \frac{1}{\theta + 2\pi r}, \quad r = 1, 2, \ldots,$$

and define, for $r = 1, 2, \ldots$,

$$\begin{aligned} x^r := x(t_r) &:= \frac{1}{\theta + 2\pi r}(\cos(\theta + 2\pi r), \sin(\theta + 2\pi r))^T \\ &= \frac{1}{\theta + 2\pi r}(\cos\theta, \sin\theta)^T = \frac{1}{\theta + 2\pi r} d^T. \end{aligned}$$

It is obvious that $F \ni x^r \to 0_2$ when $r \to \infty$, and that

$$d = \lim_{r\to\infty} \frac{1}{t_r}(x^r - 0_2),$$

that is, $d \in \mathcal{T}_{0_2}$.

(ii) Let us see now that $T_{0_2} = \{0_2\}$. In fact, if $d \in T_{0_2} \setminus \{0_2\}$, there must be some $\varepsilon > 0$ and some function $\alpha : [0, \varepsilon] \to F$ that is differentiable on $[0, \varepsilon]$ and such that $\alpha(0) = 0_2$ and $\alpha'(0) = d$.

Since $\alpha(\lambda) \in F$, the condition $\alpha'(0) = d$ implies that for $\lambda > 0$ sufficiently small, one has $\alpha(\lambda) \neq 0_2$; that is, there exists some $t_\lambda > 0$ such that $\alpha(\lambda) = x(t_\lambda)$. Since

$$\alpha'(0) = d = \lim_{\lambda\searrow 0} \frac{\alpha(\lambda)}{\lambda}, \tag{A.15}$$

and because differentiability implies continuity, one must have

$$\alpha(\lambda) = x(t_\lambda) \to \alpha(0) = 0_2 \quad \text{when } \lambda \to 0.$$

Thus, one must have $t_\lambda \to 0$ for $\lambda \to 0$, and by (A.15),

$$d = \lim_{\lambda\searrow 0} \frac{\alpha(\lambda)}{\lambda} = \lim_{\lambda\searrow 0} \frac{t_\lambda}{\lambda}\frac{x(t_\lambda)}{t_\lambda} = \lim_{\lambda\searrow 0} \frac{t_\lambda}{\lambda}(\cos(1/t_\lambda), \sin(1/t_\lambda))^T. \tag{A.16}$$

Taking norms, we obtain

$$\lim_{\lambda\searrow 0} \frac{t_\lambda}{\lambda} = \|d\| \neq 0,$$

and therefore, we deduce that (A.16) cannot hold because of the oscillating character of $(\cos(1/t_\lambda), \sin(1/t_\lambda))^T$ when $\lambda \searrow 0$.

6.6 Let $x \in F$ and let us define $d := x - \overline{x}$. If we consider the sequences

$$x^r := \overline{x} + (1/r)d \quad \text{and} \quad \lambda_r = r,\ r = 1, 2, \ldots,$$

by convexity of F, it is clear that $d \in \mathcal{T}_{\overline{x}}$. Since $x \in F$ has been arbitrarily chosen, one has $F - \overline{x} \subset \mathcal{T}_{\overline{x}}$. On the other hand, because $\mathcal{T}_{\overline{x}}$ is a cone, we have

$$\operatorname{cone}(F - \overline{x}) = \mathbb{R}_+(F - \overline{x}) \subset \mathcal{T}_{\overline{x}},$$

and since $\mathcal{T}_{\overline{x}}$ is closed,

$$\operatorname{cl}\operatorname{cone}(F - \overline{x}) \subset \mathcal{T}_{\overline{x}}.$$

Let us prove the opposite inclusion. If $d \in \mathcal{T}_{\overline{x}}$, by definition of $\mathcal{T}_{\overline{x}}$, there exist sequences $\{x^r\} \subset F$ and $\{\lambda_r\} \subset]0, +\infty[$ such that

$$\lim_{r\to\infty} x^r = \overline{x} \quad \text{and} \quad \lim_{r\to\infty} \lambda_r(x^r - \overline{x}) = d.$$

Then, $x^r - \overline{x} \in F - \overline{x}$, and hence

$$\lambda_r(x^r - \overline{x}) \in \mathbb{R}_+(F - \overline{x}) = \operatorname{cone}(F - \overline{x}),$$

which implies

$$d = \lim_{r\to\infty} \lambda_r(x^r - \overline{x}) \in \operatorname{cl}\operatorname{cone}(F - \overline{x});$$

that is, we have $d \in \operatorname{cl}\operatorname{cone}(F - \overline{x})$.

6.9 The Mangasarian–Fromovitz constraint qualification holds, that is, $\widetilde{G}_{\overline{x}} \neq \emptyset$. Let $d \in \widetilde{G}_{\overline{x}}$, i.e.,

$$\nabla g_i(\overline{x})^T d < 0, \quad i \in I(\overline{x}).$$

Then, for $i \in I(\overline{x})$, we have

$$g_i(\overline{x} + td) = g_i(\overline{x}) + t\nabla g_i(\overline{x})^T d + o(|t|) = t\left(\nabla g_i(\overline{x})^T d + \frac{o(|t|)}{t}\right),$$

and if one takes $t > 0$ sufficiently small

$$\nabla g_i(\overline{x})^T d + \frac{o(|t|)}{t} < 0,$$

which implies

$$g_i(\overline{x} + td) < 0. \tag{A.17}$$

Since $I(\overline{x})$ is finite, if $t_0 > 0$ is sufficiently small and we define $\widehat{x} = \overline{x} + t_0 d$, by (A.17),

$$g_i(\widehat{x}) < 0, \ i \in I(\overline{x}),$$

that is, the Slater qualification holds.

6.10 The Slater qualification holds at every $\overline{x} \in F$, which implies the Abadie qualification, that is,

$$\mathcal{T}_{\overline{x}} = G_{\overline{x}}.$$

Since F is convex, we have by Exercise 6.6,

$$G_{\overline{x}} = \mathcal{T}_{\overline{x}} = \operatorname{cl}\operatorname{cone}(F - \overline{x}).$$

6.11 (a) Reasoning by contradiction, suppose that $\overline{d}$ is a solution of (6.65). Then, for small t and thanks to the differentiability assumption, we can write

$$\begin{cases} f\left(\overline{x} + t\overline{d}\right) - f(\overline{x}) = t\left(\nabla f(\overline{x})^T \overline{d} + \frac{o(t)}{t}\right), \\ g_i\left(\overline{x} + t\overline{d}\right) - g_i(\overline{x}) = t\left(\nabla g_i(\overline{x})^T \overline{d} + \frac{o(t)}{t}\right), \ i \in I(\overline{x}), \\ A\left(\overline{x} + t\overline{d}\right) - b = tA\overline{d} = 0_m, \end{cases}$$

and for t positive and small enough, taking into account the continuity of g_i, $i \in \{1, 2, \ldots, p\} \setminus I(\overline{x})$ at $\overline{x}$, we see that $\overline{x} + t\overline{d}$ is feasible and $f\left(\overline{x} + t\overline{d}\right) < f(\overline{x})$, contradicting the local optimality of $\overline{x}$.

(b) Since the system (6.65) has no solution d, we apply the extended Gordan theorem (see Section 6.4) to conclude the existence of $\lambda_0 \geq 0$, $\widehat{\lambda} \in \mathbb{R}_+^{I(\overline{x})}$, $\widehat{\mu} \in \mathbb{R}^m$ such that $\left(\lambda_0, \widehat{\lambda}\right) \neq 0_{1+|I(\overline{x})|}$ and

$$\lambda_0 \nabla f(\overline{x}) + \sum_{i \in I(\overline{x})} \widehat{\lambda}_i \nabla g_i(\overline{x}) + A^T \widehat{\mu} = 0_n. \tag{A.18}$$

The possibility $\lambda_0 = 0$ is precluded because then $\widehat{\lambda} \neq 0_{|I(\overline{x})|}$ and (A.18) would lead to a contradiction with LICQ. Hence, dividing (A.18) by $\lambda_0 > 0$ and defining

$$\overline{\lambda}_i := \begin{cases} \widehat{\lambda}_i / \lambda_0, & \text{if } i \in I(\overline{x}), \\ 0, & \text{if } i \in \{1, 2, \ldots, p\} \setminus I(\overline{x}), \end{cases} \quad \text{and} \quad \overline{\mu} := \frac{1}{\lambda_0} \widehat{\mu},$$

we obtain

$$\nabla f(\overline{x}) + \sum_{i=1}^{p} \overline{\lambda}_i \nabla g_i(\overline{x}) + A^T \overline{\mu} = 0_n,$$

i.e. $\overline{x}$ is a KKT point for problem P.

6.13 Consider the problem

$$\begin{aligned} P(y) : \text{Min } & f(x) = -y^T x \\ \text{s.t. } & g(x) = x^T Q x - 1 \leq 0, \end{aligned}$$

whose optimal value, that we represent by $v(y)$, is the opposite of the optimal value of problem (6.66), having both problems the same optimal solutions.

As the feasible set of $P(y)$ is compact and the objective function is linear, there is a global minimum. We will distinguish two cases:

(i) If $y = 0_n$, $v(0_n) = 0 = -\sqrt{0_n^T Q^{-1} 0_n}$ and (6.67) trivially holds for $y = 0_n$ and any x.

(ii) If $y \neq 0_n$, $\nabla f(x) = -y$ and $\nabla g(x) = 2Qx$.

Since Q is symmetric and positive definite, g is convex, $g(0_n) = -1$, and $\widehat{x} = 0_n$ is a Slater point. Thus, if $\overline{x}$ is a minimum of $P(y)$, there exists $\overline{\lambda} \geq 0$ satisfying the KKT conditions:

$$-y + 2\overline{\lambda} Q\overline{x} = 0_n \tag{A.19}$$

and

$$\overline{\lambda}(\overline{x}^T Q\overline{x} - 1) = 0. \tag{A.20}$$

As $y \neq 0_n$, obviously by (A.19) it cannot be $\overline{\lambda} = 0$, and the complementarity condition (A.20) implies

$$\overline{x}^T Q\overline{x} = 1. \tag{A.21}$$

Then, we deduce from (A.19)

$$\overline{x} = \frac{1}{2\overline{\lambda}} Q^{-1} y. \tag{A.22}$$

Premultiplying (A.19) by $\overline{x}^T$ we obtain, having (A.21) into account,

$$-v(y) = y^T\overline{x} = 2\overline{\lambda}\overline{x}^T Q\overline{x} = 2\overline{\lambda},$$

and substituting into (A.22)

$$\overline{x} = \frac{1}{y^T\overline{x}} Q^{-1} y.$$

From this last relation, we have

$$y^T\overline{x} = \frac{1}{y^T\overline{x}} y^T Q^{-1} y,$$

and hence

$$(y^T\overline{x})^2 = y^T Q^{-1} y,$$

i.e.,

$$y^T\overline{x} = \pm\sqrt{y^T Q^{-1} y}.$$

Then, the optimal value of the original problem (maximization problem) is

$$-v(y) = y^T\overline{x} = \sqrt{y^T Q^{-1} y}.$$

If $x \neq 0_n$, it is obvious that

$$\widetilde{x} := \frac{1}{\sqrt{x^T Q x}} x$$

is a feasible point, so

$$y^T\left(\frac{1}{\sqrt{x^T Q x}} x\right) \leq \sqrt{y^T Q^{-1} y},$$

which is the same as (6.67).

6.14 The optimization problem (6.68) is equivalent to the problem

$$\begin{aligned} P: \text{Min } & f(x) = -\tfrac{1}{2}\|Ax\|^2 = -\tfrac{1}{2}x^T A^T A x, \\ \text{s.t. } & g(x) = \tfrac{1}{2}x^T x - \tfrac{1}{2} \leq 0. \end{aligned}$$

This problem has an optimal solution because we are minimizing a continuous function over a compact set. Let us see that if $\overline{x}$ is an optimal solution of P, one has $\|\overline{x}\| = 1$:

(a) If $\overline{x} = 0_n$, by (6.68), we have $\|A\| = 0$, and because the norm is zero, A must be the null matrix, which can be dismissed by the standing assumptions.

(b) If $\|A\| > 0$ and $0 < \|\overline{x}\| < 1$, it is clear that $\overline{z} := \|\overline{x}\|^{-1}\overline{x}$ satisfies

$$-\frac{1}{2}\overline{z}^T A^T A\overline{z} = -\frac{1}{2\|\overline{x}\|^2}\overline{x}^T A^T A\overline{x} < -\frac{1}{2}\overline{x}^T A^T A\overline{x},$$

which contradicts that $\overline{x}$ is an optimal solution of P.

Therefore, $\|\overline{x}\| = 1$ and $\nabla g(\overline{x}) = \overline{x} \neq 0_n$; that is, the linear independence qualification is satisfied at $\overline{x}$. Then, the KKT conditions hold, and there exists $\overline{\lambda} \geq 0$ such that

$$-A^T A\overline{x} + \overline{\lambda}\overline{x} = 0_n.$$

Obviously, one must have $\overline{\lambda} > 0$, because $\overline{\lambda} = 0$ implies $-A^T A\overline{x} = 0_n$, and then $f(\overline{x}) = -\frac{1}{2}\|A\|^2 = 0$, a possibility which is excluded.

From $A^T A\overline{x} = \overline{\lambda}\overline{x}$, we deduce that $\overline{x}$ is an eigenvector associated with the eigenvalue $\overline{\lambda} > 0$. Now, let $\widetilde{\lambda}$ be another eigenvalue (which is real, since $A^T A$ is symmetric) and let $\widetilde{x}$ be an associated eigenvector of norm one. By optimality of $\overline{x}$, one has

$$\widetilde{\lambda} = \widetilde{x}^T\left(\widetilde{\lambda}\widetilde{x}\right) = \widetilde{x}^T A^T A\widetilde{x} \leq \overline{x}^T A^T A\overline{x} = \overline{x}^T\left(\overline{\lambda}\overline{x}\right) = \overline{\lambda},$$

i.e., $\overline{\lambda}$ is the largest eigenvalue of $A^T A$.

Thus,

$$\|A\|^2 = \overline{x}^T A^T A\overline{x} = \overline{\lambda},$$

and

$$\|A\| = \sqrt{\rho(A^T A)},$$

whence, $\rho(A^T A)$ is the spectral radius of the matrix $A^T A$, that is, the maximum eigenvalue of the matrix $A^T A$.

If A is symmetric and $\mu_{min} := \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n := \mu_{max}$ are its eigenvalues, it must hold

$$\|A\| = \sqrt{\rho(A^2)} = \sqrt{\max\{\mu_{min}^2, \mu_{max}^2\}} = \max\{|\mu_{min}|, |\mu_{max}|\}.$$

6.19 (a) Simple computations show that there are three KKT points:

$$\overline{x} = (1/2, 1/2)^T, \; I(\overline{x}) = \{1\}, \; \overline{\lambda} = 1/2,$$
$$\widetilde{x} = (0, 1)^T, \; I(\widetilde{x}) = \{1, 2\}, \; \widetilde{\lambda}_1 = \widetilde{\lambda}_2 = 1,$$
$$\widehat{x} = (1, 0)^T, \; I(\widehat{x}) = \{1, 3\}, \; \widehat{\lambda}_1 = \widehat{\lambda}_3 = 1.$$

(b) As the constraints are linear, the KTCQ must hold (see Exercise 6.4), and then every local optimum of P will be a KKT point. Therefore, the local optima can be found among all the KKT points, but not every KKT point will be a local optimum.

We have

$$L(x, \lambda) = (x_1 - 1)(x_2 - 1) + \lambda_1(x_1 + x_2 - 1) + \lambda_2(-x_1) + \lambda_3(-x_2),$$
$$\nabla_x L(x, \lambda) = (x_2 - 1 + \lambda_1 - \lambda_2, \; x_1 - 1 + \lambda_1 - \lambda_3)^T,$$

and

$$\nabla_{xx}^2 L(x, \lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b1) For $\overline{x} = (1/2, 1/2)^T$, one has

$$M(\overline{x}, \overline{\lambda}) = \left\{d \in \mathbb{R}^2 : \; \nabla g_1(\overline{x})^T d = 0\right\} = \{(d_1, d_2)^T : d_1 + d_2 = 0\},$$

and for $d_0 = (-1, 1)^T \in M(\overline{x}, \overline{\lambda})$ it holds

$$d_0^T \nabla_{xx}^2 L(\overline{x}, \overline{\lambda}) d_0 = -2,$$

and the second-order necessary optimality condition fails; that is, $\overline{x} = (1/2, 1/2)^T$ is not a local minimum.

(b2) For $\widetilde{x} = (0, 1)^T$, one has

$$M(\widetilde{x}, \widetilde{\lambda}) = \left\{d \in \mathbb{R}^2 : \; \nabla g_1(\widetilde{x})^T d = 0, \; \nabla g_2(\widetilde{x})^T d = 0\right\}$$
$$= \left\{(d_1, d_2)^T : d_1 + d_2 = 0 \text{ and } d_2 = 0\right\} = \{0_2\},$$

and the second-order necessary optimality condition trivially holds. Thus, $\widetilde{x}$ is a strict local minimum, according to Theorem 6.37.

(b3) The analysis for the point $\widehat{x} = (1, 0)^T$ is identical to the one made in (b2). We can thus conclude that $\widehat{x}$ is another strict local minimum. In fact, $\widehat{x}$ and $\widetilde{x}$ are both global minima.

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