How to Proceed Further?

Miroslav Josipović

M. Josipović, *Geometric Multiplication of Vectors*,
Compact Textbooks in Mathematics,
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This question does not have the same answer for everyone; however, we can give a few ideas. Apart from studying the literature, it would be good to apply your knowledge to specific problems. You can also write to the author and point out errors in this book. However, several guidelines can be of value in general.

Geometric algebra should begin to live in high schools, and we need to invest considerable effort to achieve that goal. If you want to take part, you can join the group *Pre-University Geometric Algebra* on Linkedin (the owner is James Smith). Games with oriented objects could be introduced at an early age.

It would be a good idea to learn about *spacetime algebra* (STA). A good place to begin is the book *Space-Time Algebra*, by David Hestenes, Birkhäuser, 2015. We will outline two reasons for learning this powerful algebra. First, many authors use this algebra, and second, there is a beautiful formulation of the general theory of relativity in a flat space and in the language of this algebra (see Ref. [20]).

Geometric calculus is an inevitable and difficult bite, and one of the best sources for this topic is Ref. [25].

Thank you for reading and good luck!

Quotes

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There are two versions of math in the lives of many Americans: the strange and boring subject that they encountered in classrooms and an interesting set of ideas that is the math of the world, and is curiously different and surprisingly engaging. Our task is to introduce this second version to today's students, get them excited about math, and prepare them for the future.

(Jo Boaler in *What's Math Got to Do with It?*, Penguin 2008)

We all agree that your theory is crazy, but is it crazy enough?

(Niels Bohr)

A mathematician is a blind man in a dark room looking for a black cat which isn't there.

(Charles Darwin)

It is a striking (and not commonplace) fact that Clifford algebras and their representations play an important role in many fundamental aspects of differential geometry. These algebras emerge repeatedly at the very core of an astonishing variety of problems in geometry and topology. Even in discussing Riemannian geometry, the formalism of Clifford multiplication will be used in place of the more conventional exterior tensor calculus. The Clifford multiplication is strictly richer than exterior multiplication; it reflects the inner symmetries and basic identities of the Riemannian structure. The effort invested in becoming comfortable with this algebraic formalism is well worthwhile.

(H. Lawson Jr. and M.L. Michelson, 1989)

The geometric algebra is a conceptually appealing and mathematically powerful formalism. If you want to understand rotations, Lorentz transformations, spin-1/2 particles, and supersymmetry, and you want to do actual calculations elegantly and (relatively) easily, then the geometric algebra is the thing to learn.

(Andrew J.S. Hamilton)

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

(Emil Artin in Geometric Algebra, p. 14)

It appears as one of the fundamental principles of Nature that the equations expressing basic laws should be invariant under the widest possible group of transformations.

(Dirac, P.A.M. 1973, Proceedings of the Royal Society of London Series A, 333, 403)

Credits

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Figure	Credits
Figure in Sect. 1.6.9	The image *adapted* with kind permission of Kamenko Ćulap, the man in the adapted image (photo by Držislav Korade)
Fig. 1.29, Reflection on water	*Image reprinted with kind permission from* https://www.flickr.com/photos/stephi2006/12725183683/
Fig. 1.29, Total reflection in optical fibers	*The image adapted with kind permission from* https://www.flickr.com/photos/borshop/14869592682/
Fig. 1.29, Andromeda galaxy rotates	*The image adapted with kind permission from* https://apod.nasa.gov/apod/ap150830.html Image Credit & Copyright: Robert Gendler
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G. Sobczyk, Sect. 1.15.1	The image *adapted* with kind permission of Garret Sobczyk
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Fig. 5.1	The used Möbius Ring by the artist David Weitzman (in image)
Argon laser test, Sect. 5.6	The image *adapted* with kind permission from https://www.flickr.com/photos/mightyohm/3938307729/

(continued)

Figure	Credits
Golden spiral, Sect. 6.3	The image *adapted* with kind permission of Ken Pratt http://kenpratt.net/portfolio/raytracer/
A cat, Sect. 6.4	The image *adapted* with kind permission of Sanja Josipović, original photo of the cat Luna by Vesna Josipović

The other figures are made by the author, using Wolfram *Mathematica* and the Package that can be found at L1

Glossary

A

algebra of physical space (APS) $Cl3$, Pauli algebra
antiautomorphism $\mathrm{f}(MN) = \mathrm{f}(N)\mathrm{f}(M)$
automorphism $\mathrm{f}(MN) = \mathrm{f}(M)\mathrm{f}(N)$

B

binary star a system of two stars bound by their mutual gravitational forces
Bondi factor (coefficient) $k = \sqrt{(1+v)/(1-v)}$; v is the velocity of an inertial reference frame

C

center of an algebra subset of the algebra whose elements commute with all elements of the algebra
comoving frame a frame of reference that is not attached to a moving object and can be treated as an inertial reference frame in which the moving object has the velocity zero at some instant of time
contraction an operation that generally lowers the grade (or rank) of an element of an algebra

D

dual number number of the form $\alpha + \beta h$, $h^2 = 0$, $\alpha, \beta \in \mathbb{R}$

E

expectation value the probabilistic *expected value* of the result of measurement of an experiment
Euclidean vector space a vector space in which the square of nonzero vectors is a positive real number

F

Faraday in $Cl3$, a common name for a complex vector of an electromagnetic field, $\mathbf{F} = \mathrm{E} + j\mathbf{B}$

G

geometric calculus a calculus in geometric algebra
Gibbs products standard scalar and cross products
group a set of objects with a binary operation (closure) that is associative and has a unit element and inverses

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H

Hermitian adjoint (of a matrix) transposition followed by complex conjugation

Hermitian matrix a matrix equal to the Hermitian adjoint, like the Pauli matrices

hyperbolic number (split complex, perplex; see double number)

I

ideal (left) of an algebra for an algebra A, it is a subset L of the algebra with the property $a \in A, l \in L \Rightarrow al \in L$

J

Jacobi identity a property of a binary operation that describes how the order of evaluation (the placement of parentheses in a multiple product) affects the result of the operation: $a \otimes (b \otimes c) + b \otimes (c \otimes a) + c \otimes (a \otimes b) = 0$

K

Kronecker delta a special symbol, δ_{ij}, equal to 1 for $i = j$, zero otherwise

L

left ideal see *ideal*

Levi-Civita symbol ε_{ijk}, represents a collection of numbers; defined from the *sign of a permutation* of the natural numbers 1, 2, 3 (it can be defined with any number of indices)

M

Minkowski vector space a vector space with signature $(n - 1, 1)$

N

null-vector $a^2 = 0, \quad a \neq 0$ ($a \cdot a = 0$ in CGA model)

O

observable in quantum mechanics, a quantity that can be measured

operator in linear algebra, an object that operates on vectors, usually represented by matrices (in geometric algebra, “operators” are elements of the algebra)

P

paravector a sum of a real number and a vector, $\alpha + \mathbf{x}$, $\alpha \in \mathbb{R}$

Q

quantum information theory a quantum mechanical information theory contained in quantum systems

quaternion discovered by Hamilton, a number of the form $q^0 + q^i I_i$ (summation), $i = 1, 2, 3$, $I_i^2 = -1$, $I_i I_j = -I_j I_i$, $I_1 I_2 = I_3$, etc. (cyclic), $I_1 I_2 I_3 = -1$

R

reduced mass for a two-body problem we define $\mu = m_1 m_2 / (m_1 + m_2)$

S

spacetime algebra (STA) a geometric algebra based on $\Re(1, 3)$

spinor elements of a (complex) vector space that can be associated with Euclidean space; in *Cl*3, they are elements of the even part of the algebra

standard model the theory of particle physics, describing three of the four known fundamental forces in the universe (the electromagnetic, weak, and strong interactions)

symmetry a transformation that leaves a property of a system unchanged

T

Taylor expansion (series) a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point

tensor a geometric object that describes linear relations between geometric vectors, scalars, and other tensors

U

unimodular $M\bar{M} = 1$, where $\bar{M}$ is the Clifford conjugation of the multivector M

unipodal number $z_1 + z_2 h$, $z_1, z_2 \in \mathbb{C}$, $h^2 = 1$

unitary $MM^\dagger = 1$, where $M^\dagger$ is the reverse involution of the multivector M (a complex square matrix U is unitary if its conjugate transpose $U^\dagger$ is also its inverse)

V

vector space a collection of objects called vectors, objects with special transformation rules, which may be added together and multiplied ("scaled") by numbers, called scalars; usually, scalars are real (a real vector space) or complex (a complex vector space)

W

wedge the character $\wedge$, common in GA to designate the *outer product*

whirl a complex vector with the property $\mathbf{F}^2 \in \mathbb{R}$ (author's suggestion)

X

x-rays a form of electromagnetic radiation, Röntgen rays (there are indications that Tesla discovered x-rays before Röntgen)

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References and Links to Specific Subjects

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R12: J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999)

Links to Supplementary Materials

L1: http://extras.springer.com/2019/978-3-030-01755-2
L2: https://ocw.mit.edu/resources/res-8-001-applied-geometric-algebra-spring-2009/

3D Geometry

http://geocalc.clas.asu.edu/GA_Primer/GA_Primer/index.html

Some Web Resources

https://gaupdate.wordpress.com
http://geocalc.clas.asu.edu/html/Evolution.html (the diagram at the end of the book is taken from this page and adapted by the author with the kind permission of David Hestenes)
https://staff.science.uva.nl/l.dorst/clifford/

Software

Cinderella. https://www.cinderella.de/tiki-index.php
M Clifford. Mathematica package. https://arxiv.org/abs/0810.2412, 2018
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GA20 and GA30 (Formerly called *pauliGA*). https://github.com/peeterjoot/gapauli
Gaalet. https://sourceforge.net/projects/gaalet/
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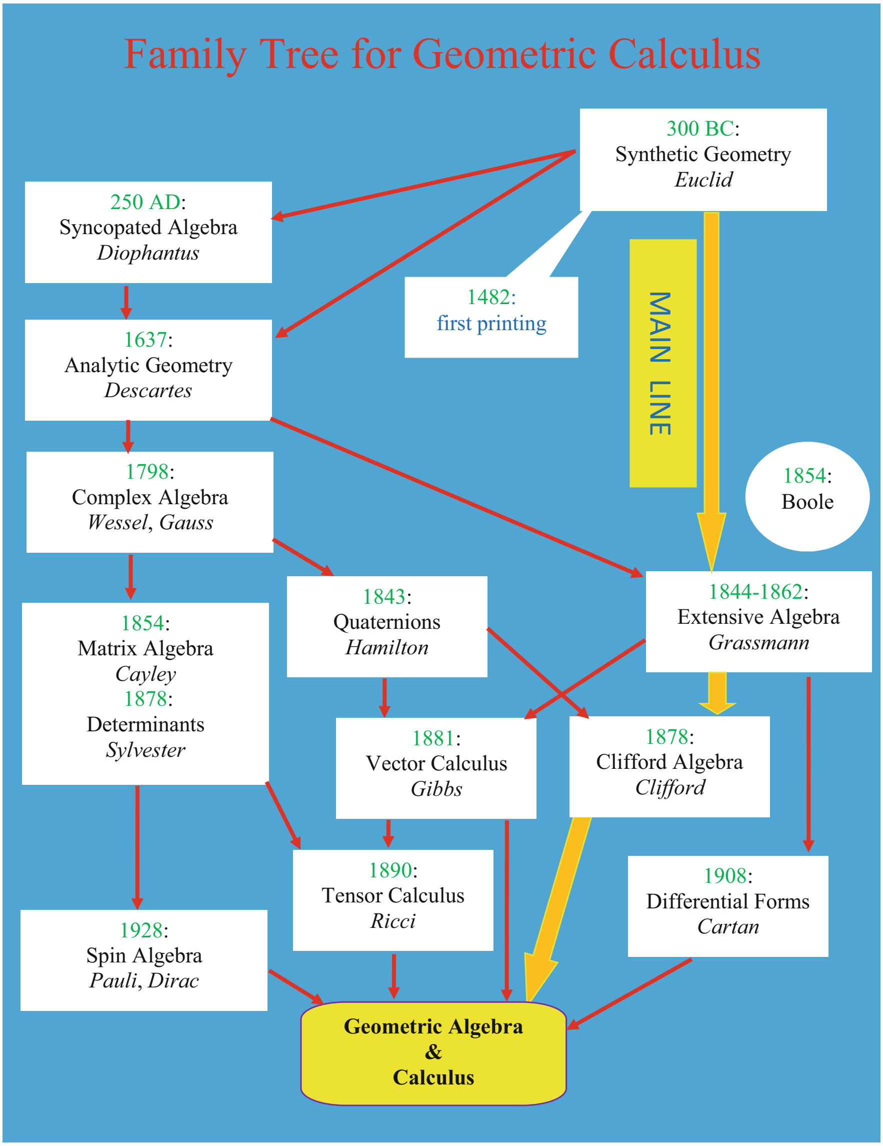
GABLE. https://staff.fnwi.uva.nl/l.dorst/GABLE/index.html
Gaigen. https://sourceforge.net/projects/g25/
GAlgebra. https://github.com/brombo/galgebra
GA Sandbox. https://sourceforge.net/projects/gasandbox/
GAViewer. http://www.geometricalgebra.net/downloads.html; this nice tool is recommended with the text. You can manipulate the images to some extent
GluCat. https://sourceforge.net/projects/glucat/
GMac. https://gacomputing.info/gmac-info/
SpaceGroupVisualizer. http://spacegroup.info/
GrassmannAlgebra package. https://sites.google.com/site/grassmannalgebra/thegrassmannalgebrapackage
Versor. http://versor.mat.ucsb.edu/
Some important details can be found at https://gacomputing.info/ga-software/
https://ga-explorer.netlify.com/index.php/ga-online-resources/



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http://geocalc.clas.asu.edu/html/Evolution.html