Appendix A

Optimization Methods

A.1 Dynamic Lagrangian Multipliers

The dynamic Lagrangian multipliers method is an extension of the Lagrangian multipliers method for dynamic optimization.

A dynamic optimization problem consists in finding the function $u(t)$ over the time interval T, eventually infinite, which minimizes the functional

$$J[u] = \int_T \mathcal{L}(x, u)\mathrm{d}t, \tag{A.1}$$

under the constraints of the differential equation defining the system dynamics and, optionally, prescribed initial and/or final states and additional inequality constraints.

The basic principle is to define using some additional variables, called Lagrangian multipliers, a new Lagrangian function $\mathcal{L}'$ that embeds the constraints. The solution of the new unconstrained optimization problem, called the *dual problem*, is, at worst, an upper bound of the solution of the original constrained problem and is actually coincident with that of the original problem if the so-called *strong duality* property is verified, as in the case of convex optimization problems.

The dual problem is solved by imposing the KKT (Karush Kuhn Tucker) optimality conditions. Since the constraints we are considering are linear, thanks to Abadie's constraint qualification, the KKT conditions are necessary and sufficient, *i.e.* every point satisfying the KKT conditions is a solution of the optimization problem and *vice versa*. The KKT conditions, on the other hand, are not always constructive meaning, that their equations do not always completely define the solution.

In the following we will detail the application of this method to a finite time dynamic optimization problem with prescribed initial and final states both with and without additional inequality conditions. In the first case the KKT conditions form a well-defined solution, while in the latter case the solution is not fully defined.

A.1.1 Inequality Constraints-free Optimization

The optimization problem under consideration is finding the function $u(t)$ over the time interval $T = [t_0, t_s]$ that minimizes the functional

$$J[u] = \int_{t_0}^{t_f} \mathcal{L}(x, u)\mathrm{d}t,$$

where

$$\mathcal{L}(x, u) = \frac{1}{2}\left(x^T \mathbf{Q} x + u^T \mathbf{R} u\right) \tag{A.2}$$

under the following constraints

$$\dot{x} = f(x, u) = \mathbf{A}x + \mathbf{B}u \tag{A.3}$$

$$x(t_0) = x_0 \quad x(t_f) = x_f.$$

As anticipated, a new Lagrangian function is defined

$$\mathcal{L}'(x, \dot{x}, u, \lambda) = \mathcal{L}(x, u) + \lambda^T \left(f(x, u) - \dot{x}\right),$$

defining a new unconstrained dynamic optimization problem: find u and λ that minimize the functional

$$\hat{J}[u, \lambda] = \int_{t_0}^{t_f} \mathcal{L}'(x, \dot{x}, u, \lambda)\mathrm{d}t$$

such that

$$x(t_0) = x_0 \quad x(t_f) = x_f.$$

Considering the functional variation between $\hat{J}[x(t), u(t)]$ and $\hat{J}[x(t)+h_x(t), u(t)+h_u(t)]$ we have

$$\Delta\hat{J} = \int_{t_0}^{t_f} \left[\mathcal{L}'(x + h_x, \dot{x} + \dot{h}_x, u + h_u, \lambda) + \mathcal{L}'(x, \dot{x}, u, \lambda)\right] \mathrm{d}t,$$

using a truncated Taylor series we can write

$$\delta\hat{J} = \int_{t_0}^{t_f} \left[\frac{\partial \mathcal{L}'}{\partial x} h_x + \frac{\partial \mathcal{L}'}{\partial \dot{x}} \dot{h}_x + \frac{\partial \mathcal{L}'}{\partial u} h_u\right] \mathrm{d}t.$$

Integration by parts of the second term gives

$$\delta\hat{J} = \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}'}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}'}{\partial \dot{x}} \right) h_x \mathrm{d}t + \int_{t_0}^{t_f} \frac{\partial \mathcal{L}'}{\partial u} h_u \mathrm{d}t + \left. \frac{\partial \mathcal{L}'}{\partial u} h_u \right|_{t_0}^{t_f} \tag{A.4}$$

where

$$\left. \left(\frac{\partial \mathcal{L}'}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}'}{\partial \dot{x}} \right) h_x \right|_{t_0}^{t_f} = 0$$

since $h_x(t_0) = h_x(t_s) = 0$ due to the final and initial states constraint.

As $h_x(t)$ and $h_u(t)$ are arbitrary

$$\frac{\partial \mathcal{L}'}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}'}{\partial \dot{x}} = 0 \tag{A.5}$$

$$\frac{\partial \mathcal{L}'}{\partial u} = 0.$$

By simple calculation

$$\frac{\partial \mathcal{L}'}{\partial x} = \frac{\partial \mathcal{L}'}{\partial x} + \lambda^T \frac{\partial f}{\partial x} = x^T \mathbf{Q} + \lambda^T \mathbf{A}$$

$$\frac{\partial \mathcal{L}'}{\partial \dot{x}} = -\lambda^T$$

$$\frac{\partial \mathcal{L}'}{\partial u} = \frac{\partial \mathcal{L}'}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = u^T \mathbf{R} + \lambda^T \mathbf{B}$$

we have a differential equation defining the evolution of the Lagrangian multipliers and the relationship between the optimal solution and the Lagrangian multipliers.

$$\dot{\lambda} = -\mathbf{Q}x - \mathbf{A}^T \lambda \tag{A.6}$$

$$u = -\mathbf{R}^{-1} \mathbf{B}^T \lambda. \tag{A.7}$$

These two equations, called *secondary KKT conditions*, together with the original system dynamic Equation A.3, called *primary KKT condition*, define a two-point boundary-value problem (TPBVP).

A.1.2 Optimization Under Inequality Constraints

We consider the problem of finding the function $u(t)$ over the time interval $T = [t_0, t_s]$ that minimizes

$$J[u] = \int_{t_0}^{t_f} \mathcal{L}(x, u) \mathrm{d}t,$$

where

$$\mathcal{L}(x,u) = \frac{1}{2}\left(x^T\mathbf{Q}x + u^T\mathbf{R}u\right)$$

under the constraints

$$\dot{x} = f(x,u) = \mathbf{A}x + \mathbf{B}u$$
$$x(t_0) = x_0 \quad x(t_f) = x_f$$
$$h(x,u) = \begin{bmatrix} u \\ -\mathbf{C}x \end{bmatrix} \leq 0.$$

As in the previous case we define a new Lagrangian function

$$\mathcal{L}'(x,\dot{x},u) = \mathcal{L}(x,u) + \lambda^T(f(x,u) - \dot{x}) + \mu^T(h(x,u) + s^2)$$

defining the dual problem. The embedding of the inequality constraints introduces a new series of Lagrange multipliers μ and ancillary variables s called *slack variables* that transform the inequality constraints in equality constraints.

Equations A.1.1, A.1.1 and A.4 still hold true and give Equation A.5. Calculating the partials

$$\frac{\partial\mathcal{L}'}{\partial x} = \frac{\partial\mathcal{L}'}{\partial x} + \lambda^T\frac{\partial f}{\partial x} + \mu^T\frac{\partial h}{\partial x} = x^T\mathbf{Q} + \lambda^T\mathbf{A} + \mu^T\begin{bmatrix} 0 \\ -\mathbf{C} \end{bmatrix}$$
$$\frac{\partial\mathcal{L}'}{\partial \dot{x}} = -\lambda^T$$
$$\frac{\partial\mathcal{L}'}{\partial u} = \frac{\partial\mathcal{L}'}{\partial u} + \lambda^T\frac{\partial f}{\partial u} + \mu^T\frac{\partial h}{\partial u} = u^T\mathbf{R} + \lambda^T\mathbf{B} + \mu^T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we have the *KKT secondary conditions*

$$\dot{\lambda} = -\mathbf{Q}x - \mathbf{A}^T\lambda - \begin{bmatrix} 0 & -\mathbf{C} \end{bmatrix}\mu$$
$$u = -\mathbf{R}^{-1}\mathbf{B}^T\lambda - \mathbf{R}^{-1}\begin{bmatrix} 1 & 0 \end{bmatrix}\mu$$
$$\mu \geq 0$$

which, together with the so-called *complementary slackness* condition

$$\mu^T\begin{bmatrix} u \\ -\mathbf{C}x \end{bmatrix} = 0$$

and the *KKT primary conditions*

$$\dot{x} = f(x,u) = \mathbf{A}x + \mathbf{B}u$$
$$h(x,u) = \begin{bmatrix} u \\ -\mathbf{C}x \end{bmatrix} \leq 0,$$

are the set of KKT conditions for the dual problem. The additional Lagrange multipliers μ are not completely defined by the previous equations. In this case, the KKT conditions are not constructive and do not define a solution to the optimization problem that has been solved using a quadratic programming formulation.

A.2 Alternative Solution of the Two-point Boundary-value Problem by Generating Functions

The optimal control $u(t)$ for the inequality constraint-free optimization problem is defined by the solution of the following TPBVP

$$\dot{x} = \mathbf{A}x + \mathbf{B}_e\Gamma_e + \mathbf{B}_c u \tag{A.8a}$$

$$\dot{\lambda} = -\mathbf{Q}x - \mathbf{A}^T\lambda \tag{A.8b}$$

$$u = -\mathbf{R}^{-1}\mathbf{B}_c^T\lambda \tag{A.8c}$$

given by the *primary KKT condition* (A.3) and the two *secondary KKT conditions.*

Two solutions to this problem have been presented in Chapter 3: the iterative shooting method and the analytical solution using a matrix exponential. An alternative solution of the TPBVP uses the properties of the Hamiltonian systems, namely the canonic transformations defined by a generating function, to obtain the initial co-state vector λ_0. This approach, quite complex on both the theoretical and practical planes, is the only available solution to finite-time optimal control problems over very long time intervals. The most frequent example of these systems found in the literature is the optimal orbit change and *rendezvous* planning for satellites having a propulsion system too weak to use the impulsive speed change approximation used in Boltzmann orbits.

The previously defined TPBVP can be written as an Hamiltonian system in homogeneous canonic form plus the forced reaction to the engine torque Γ_e

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial H(y,\lambda,u)}{\partial \lambda} \\ \frac{\partial H(y,\lambda,u)}{\partial y} \end{bmatrix} + \mathbf{B}_e\Gamma_e,$$

where $H(x,\lambda,u) = \mathcal{L}(x,u) + \lambda^T(\mathbf{A}x + \mathbf{B}_c u)$ is the Hamiltonian; $\mathcal{L}(x,u)$ is the Lagrangian defined by Equation A.2. $x \in \mathbb{R}^n$ is the state vector of the system subject to the optimal control and $\lambda \in \mathbb{R}^n$ the corresponding co-states or dynamic Lagrangian multipliers.

The TPBVP resolution method proposed in [28] and [18] is defined only for homogeneous Hamiltonian systems, *i.e.* for control problems where all the system inputs are controlled inputs.

By linearity, we can separate the free evolution of the system $z_{free}(t)$ from the forced one $z_{\Gamma_e}(t)$

$$z(t) = \mathrm{e}^{\mathbf{A}t}z_0 + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(\tau-t)}\mathbf{B}_e\Gamma_e(\tau)\mathrm{d}\tau = z_{free}(t) + z_{\Gamma_e}(t).$$

Since the engine torque Γ_e is assumed to be independent of the system evolution the forced evolution of the system can be simply calculated by forward integration before the solution of the optimal control problem.

The initial co-state vector $\lambda(t_0) = \lambda_0$, solution of the TPBVP

$$\begin{gathered}\dot{z} = \mathbf{A}z + \mathbf{B}_e \Gamma_e \\ x(t_0) = x_0 \quad x(t_s) = x_s\end{gathered}$$

is also the solution of the homogeneous TPBVP

$$\begin{gathered}\dot{z} = \mathbf{A}z \\ x(t_0) = x_0 \quad x(t_s) = x_s - x_{\Gamma_e}(t_s),\end{gathered}$$

where

$$z_{\Gamma_e}(t_s) = \int_{t_0}^{t_s} \mathrm{e}^{\mathbf{A}(\tau - t_s)} \mathbf{B}_e \Gamma_e(\tau) \mathrm{d}\tau = \left[x_{\Gamma_e}(t_s) \; \lambda_{\Gamma_e}(t_s) \right]^T .$$

For a better understanding of the TPBVP solution method by generating functions the Hamilton principle, also called the minimum effort principle, and the definition of a canonic transformation between extended phase spaces are briefly recalled.

Definition A.1 (Hamilton Principle). *The trajectory of a Hamiltonian system in the phase space makes the following integral extremal*

$$\int_{t_0}^{t_f} [\lambda \dot{y}^T - H(y, \lambda, t)]\, dt = \int_{t_0}^{t_f} \mathcal{L}(x, u)\, dt,$$

which implies

$$\delta \int_{t_0}^{t_f} \mathcal{L}(x, u)\, dt = 0.$$

Definition A.2 (Extended Phase Space). *Let $P \in \mathbb{R}^{2n}$ be a phase space, $P \times \mathbb{R}$ is called an extended phase space (by time).*

Definition A.3 (Canonic Transformation). *A map $f : P_1 \times \mathbb{R} \to P_2 \times \mathbb{R}$ is said to be a canonic transformation if:*

- *f is an isomorphism C^∞;*
- *f does not affect time,* i.e. $\exists\, g_T(z)$ *such that $f(z, t) = (g_T(z), t)$; and*
- *f preserves the canonic form of the Hamiltonian systems.*

The last point is equivalent to assuring that there exists a Hamiltonian K after the transformation $(X, \Lambda) = f(x, \lambda)$ such that the system dynamics can be written as

$$\begin{bmatrix} \dot{X} \\ \dot{\Lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial K}{\partial \Lambda} \\ \frac{\partial K}{\partial Y} \end{bmatrix} .$$

A.2.1 Generating Functions

The Hamilton principle, invariant for canonical transformations, implies that

$$\lambda \dot{x} - H = \Lambda \dot{X} - K + \frac{\mathrm{d}F}{\mathrm{d}t}, \tag{A.9}$$

where F is a function, called the generating function, defining the canonical transformation. In principle, this function depends on $4n+1$ parameters, *i.e.* x, λ, X, Λ and t but because of the $2n$ constraints imposed by the canonical transformation f, F is a function of just $2n+1$ parameters. Amongst all the possible choices for the set of independent parameters there are four classic formulations

$$F_1(x, X, t) \quad F_2(x, \Lambda, t) \quad F_3(\lambda, X, t) \quad F_4(\lambda, \Lambda, t).$$

Calculating the total derivative $\mathrm{d}F/\mathrm{d}t$ for the first two classical formulations and substituting the result in Equation A.9 we have, under the hypothesis of independent parameters

$$\begin{array}{ll} \lambda = \frac{\partial F_1(x,X,t)}{\partial x} & \lambda = \frac{\partial F_2(x,\Lambda,t)}{\partial x} \\ \Lambda = -\frac{\partial F_1(x,X,t)}{\partial X} & X = -\frac{\partial F_2(x,\Lambda,t)}{\partial \Lambda} \\ H(x,\lambda,t) + \frac{\partial F_1(x,X,t)}{\partial t} = K(X,\Lambda,t) & H(x,\lambda,t) + \frac{\partial F_2(x,\Lambda,t)}{\partial t} = K(X,\Lambda,t) \end{array}.$$

The equations in the last line are known as Hamilton-Jacobi PDEs; their solution allows the generating function to be obtained. Once the generating function has been obtained in one of the four classical formulations the other three can be calculated by applying a Legendre transformation.

A.2.2 Hamiltonian System Flow

The flow $\phi : (y(t_0), \lambda(t_0), t) \to (y(t), \lambda(t), t)$ of the Hamiltonian system is a canonical transformation. For a linear system this flow is usually expressed using a matrix exponential

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \mathrm{e}^{A_L(t-t0)} \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}.$$

The interest of the TPBVP solution by means of the generating functions is to obtain the same relationship without using the matrix exponential that causes some numerical difficulties. This method can also be used to obtain an approximated analytic solution of finite-time optimal control for non-linear systems.

Using the formalism introduced in the definition of a canonical transformation we have

$$X = x(t_0) = x_0 \quad \Lambda = \lambda(t_0) = \lambda_0$$

giving the corresponding Hamiltonian system

$$\begin{bmatrix} \dot{x}_0 = 0 \\ \dot{\lambda}_0 = 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial K(x_0,\lambda_0,t)}{\partial \lambda_0} \\ \frac{\partial K(x_0,\lambda_0,t)}{\partial x_0} \end{bmatrix}.$$

Since K is constant we can assume $K \equiv 0$ without any loss of generality.

A.2.3 Two-point Boundary-value Problem Solution

The generating function first classical formulation $F_1(x, x_0, t)$ is the most apt to solve the TPBVP since

$$\lambda_0 = \left.\frac{\partial F_1(x, x_0, t)}{\partial x}\right|_{x=x_f\ t=t_f}. \tag{A.10}$$

Unluckily this generating function cannot be directly obtained from the Hamilton Jacobi equation since in t_0 the assumption of independence of the parameters is not verified: $F_1(x_0, x_0, t_0)$.

The solution to this difficulty proposed in [18] consists in obtaining first the generating function in its second classical formulation $F_2(x, \lambda_0, t)$, which satisfies the independence assumption in t_0, from the Hamilton Jacobi equation

$$H(x, \lambda, t) + \frac{\partial F_2(x, \lambda_0, t)}{\partial t} = 0,$$

and then obtain $F_1(x, x_0, t_0)$ through a Legendre transformation

$$F_1(x, x_0, t) = F_2(x, \lambda_0, t) - x_0^T \lambda_0, \tag{A.11}$$

and finally obtain the initial co-states vector λ_0.

The finite time optimal control of a linear system

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$

with respect to the quadratic cost function

$$J[u] = \int_{t_0}^{t_s} \left[x^T \mathbf{Q} x + u^T \mathbf{R} u\right] \mathrm{d}t$$

induces a quadratic Hamiltonian

$$H(x, \lambda, t) = \frac{1}{2} \begin{bmatrix} x \\ \lambda \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{A}^T \\ \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix},$$

which allows F_2 to be written as

$$F_2(x, \lambda_0, t) = \frac{1}{2} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} \mathbf{F}_{xx} & \mathbf{F}_{x\lambda_0} \\ \mathbf{F}_{x\lambda_0}^T & \mathbf{F}_{\lambda_0\lambda_0} \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}. \tag{A.12}$$

Substituting Equation A.12 in the PDE Hamilton Jacobi equation relative to F_2 we obtain a system of matrix differential equations

$$\dot{\mathbf{F}}_{xx} + \mathbf{Q} + \mathbf{F}_{xx}\mathbf{A} + \mathbf{A}^T\mathbf{F}_{xx} - \mathbf{F}_{xx}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{F}_{xx} = 0 \tag{A.13a}$$

$$\dot{\mathbf{F}}_{x\lambda_0} + \mathbf{A}^T\mathbf{F}_{x\lambda_0} - \mathbf{F}_{xx}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{F}_{x\lambda_0} = 0 \tag{A.13b}$$

$$\dot{\mathbf{F}}_{\lambda_0\lambda_0} - \mathbf{F}_{x\lambda_0}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{F}_{x\lambda_0} = 0, \tag{A.13c}$$

having as initial conditions

$$\mathbf{F}_{xx}(t_0) = 0_{n\times n} \quad \mathbf{F}_{x\lambda_0}(t_0) = I_{n\times n} \quad \mathbf{F}_{\lambda_0\lambda_0}(t_0) = 0_{n\times n}.$$

Once the matrix differential system (A.13) is resolved the F_1 formulation of the generating function is obtained by the Legendre transformation (A.11). Finally, thanks to the relation (A.10), we have

$$\lambda_0 = \mathbf{F}_{\lambda_0\lambda_0}^{-1}(t_s)\left[x_0 - \mathbf{F}_{\lambda_0 x}(t_s)x_s\right].$$

This solution is numerically stable but quite complex since it requires the integration of a system of 48 differential equations. In the case of a clutch optimal engagement control the solution by quadratic programming formulation is still feasible thanks to a relatively short optimization horizon. This solution, which is both simpler and more powerful since it allows the inclusion of additional inequality constraints, has been chosen as the standard solution of the optimization problem.

A.3 Reconduction to a Quadratic Programming Formulation

The optimization program posed by the optimal engagement control is to find the function $u(t)$ over the interval $T = [t_0, t_s]$ minimizing the functional

$$J[u] = \int_{t_0}^{t_f} \mathcal{L}(x, u)\mathrm{d}t,$$

where

$$\mathcal{L}(x, u) = \frac{1}{2}\left(x^T\mathbf{Q}x + u^T\mathbf{R}u\right)$$

under the constraints

$$\dot{x} = f(x, u) = \mathbf{A}x + \mathbf{B}u \tag{A.14}$$

$$x(t_0) = x_0 \quad x(t_f) = x_f \qquad (A.15)$$
$$\mathbf{A}_c x \leq \mathbf{B}_c.$$

The sampling of the dynamic of the system subject to the optimal control is used to reduce the dynamic optimization to a quadratic program, *i.e.* the optimization of a vector composed by the samples u_k of the function $u(t)$ taken at the sampling instants t_k. The solution to the optimization problem is, thus, the vector

$$\bar{u} = \left[u_0 \; u_1 \; \ldots \; u_{N-1} \right]^T,$$

where N is the number of samples over the optimization horizon T.

Iterating the finite-difference equation of the sampled system

$$x_{k+1} = A_d x_k + B_d u_k,$$

we have the following relation

$$x_k = A_d^k x_0 + A_d^{k-1} B_d u_0 + A_d^{k-2} B_d u_1 + \ldots + A_d B_d u_{k-2} + B_d u_{k-1}$$

defining the sample x_k as a function of the initial state x_0 and the input samples u_i with $i \in [0, k-1]$.

This relation can be put in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} B_d & 0 & \cdots & 0 \\ A_d B_d & B_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N-1} B_d & A_d^{N-2} B_d & \cdots & B_d \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} + \begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^N \end{bmatrix} x_0$$

which, in a more compact form, is written as

$$\bar{x} = E\bar{u} + F x_0. \qquad (A.16)$$

The previous equation expresses the vector $\bar{x}$ formed by the sampled state vectors as a function of the initial state vector x_0 and the vector $\bar{u}$.

Due to the sampling the integral functional is simplified in a sum

$$J[\bar{u}] = \sum_{k=1}^{N} x_k^T \mathbf{Q} x_k + \sum_{k=0}^{N-1} u_k^T \mathbf{R} u_k,$$

which can be expressed using the vectors $\bar{x}$ and $\bar{u}$

$$\begin{aligned} J[\bar{u}] &= \left[x_1 \; \ldots \; x_N \right] \begin{bmatrix} \mathbf{Q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{Q} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_N \end{bmatrix} \\ &+ \left[u_0 \; \ldots \; u_{N-1} \right] \begin{bmatrix} \mathbf{R} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{R} \end{bmatrix} \begin{bmatrix} u_0 \\ \cdots \\ u_{N-1} \end{bmatrix} \qquad (A.17) \\ &= \bar{x}^T \bar{\mathbf{Q}} \bar{x} + \bar{u}^T \bar{\mathbf{R}} \bar{u}. \end{aligned}$$

Substituting Equation A.16 in Equation A.17 we have, finally, a cost function in the standard QP formulation

$$J = \bar{u}^T \left(E^T \bar{Q} E + \bar{R} \right) \bar{u} + x_0 F^T \bar{Q} E \bar{u}.$$

The equality constraint Equation A.14 due to the system dynamic has been embedded, through substitution, in the cost function. We still have to include the initial and final states constraints together with the inequality constraints.

From the last line of the Equation A.16 we have

$$x_s = \left[\mathbf{A}_d^{N-1} \mathbf{B}_d \cdots \ \mathbf{B}_d \right] \bar{u} + \mathbf{A}^N x_0,$$

which in standard representation gives

$$\begin{aligned} \left[\mathbf{A}_d^{N-1} \mathbf{B}_d \cdots \ \mathbf{B}_d \right] \bar{u} &= x_s - \mathbf{A}^N x_0 \\ \mathbf{A}_{eq} \bar{u} &= b_{eq}. \end{aligned}$$

The inequality constraints in matrix form become

$$\begin{bmatrix} \mathbf{A}_c & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_N \end{bmatrix} \le \begin{bmatrix} b_c \\ \cdots \\ b_c \end{bmatrix}$$

$$\bar{\mathbf{A}}_c \bar{x} \le \bar{b}_c$$

substituting Equation A.16 we finally have

$$\bar{\mathbf{A}}_c \mathbf{E} \bar{u} \le \bar{b}_c$$

$$\mathbf{A}_{in} \bar{u} \le b_{in}.$$

The sampling thus reduces the dynamic optimization into a static optimization that can be written in the standard QP formulation:

Find the vector $\bar{u}$ *minimizing*

$$J = \bar{u}^T \left(E^T \bar{Q} E + \bar{R} \right) \bar{u} + x_0 F^T \bar{Q} E \bar{u},$$

under the constraints

$$\boldsymbol{A}_{eq} \bar{u} = b_{eq}$$

$$\boldsymbol{A}_{in} \bar{u} \le b_{in}.$$

Appendix B

Proof of Theorem 6.1

Lemma B.1. *The linear time-invariant system:*

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u \tag{B.1a}$$

$$y = \boldsymbol{C}x + \boldsymbol{D}u \tag{B.1b}$$

is finite gain $\mathcal{L}_p$ stable for every $p \in [1, \infty]$ if $\boldsymbol{A}$ is Hurwitz. Furthermore, the inequality relation

$$\|y\|_{\mathcal{L}_p} \leq \gamma \|u\|_{\mathcal{L}_p} + \beta$$

is verified for:

$$\gamma = \|\boldsymbol{D}\|_2 + \frac{2\lambda^2_{max}(\boldsymbol{P})\|\boldsymbol{B}\|_2\|\boldsymbol{C}\|_2}{\lambda_{min}(\boldsymbol{P})} \tag{B.2}$$

$$\beta = \rho\|\boldsymbol{C}\|_2\|x_0\|\sqrt{\frac{\lambda_{max}(\boldsymbol{P})}{\lambda_{min}(\boldsymbol{P})}} \tag{B.3}$$

$$\rho = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{2\lambda_{max}(\boldsymbol{P})}{p}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases} \}$$

where $\boldsymbol{P}$ is the solution of the Riccati equation $\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} = -\boldsymbol{I}$.

Proof. This Lemma is the Corollary 5.2 of Theorem 5.1, the interested reader can find its proof on page 202 of [20].

Proof. By simple substitution

$$\dot{\tilde{x}} = (\mathbf{A} - \mathbf{K}\mathbf{C})\tilde{x} + \mathbf{W}_1\epsilon_1 - \mathbf{K}\mathbf{W}_2\epsilon_2.$$

By hypothesis, $\mathbf{A} - \mathbf{KC}$ is diagonalizable, *i.e.* a base change exists

$$\tilde{x} = \mathbf{T}z \tag{B.4}$$

such that

$$\dot{z} = \mathbf{D}z + \bar{\mathbf{B}}_1\epsilon_1 + \bar{\mathbf{B}}_2\epsilon_2, \tag{B.5}$$

where $\mathbf{D} = \mathbf{T}^{-1}(\mathbf{A} - \mathbf{KC})\mathbf{T}$ is a diagonal matrix, $\bar{\mathbf{B}}_1 = \mathbf{T}^{-1}\mathbf{W}_1$ and $\bar{\mathbf{B}}_2 = \mathbf{T}^{-1}\mathbf{KW}_2$. Furthermore, $\mathbf{T}$ has as its columns the eigenvectors of $A - KC$ that are defined but for a multiplicative constant. This degree of liberty allows $\|\mathbf{T}\|_2 = 1$ to be assumed without any loss of generality. Since $\mathbf{P} = -2\mathbf{D}^{-1}$ is the solution of the Lyapunov equation $\mathbf{D}^T\mathbf{P} + \mathbf{PD} = -\mathbf{I}$ we have the following relation

$$\lambda_{max}(P) = -1/(2\lambda_{min}(A - KC)) \tag{B.6a}$$

$$\lambda_{min}(P) = -1/(2\lambda_{max}(A - KC)). \tag{B.6b}$$

Using the superposition principle, the Lemma B.1 and the relation between the eigenvalues (B.6) to the system (B.5) we have:

$$\|z\|_{\mathcal{L}_p} \leq \gamma_1\|\epsilon_1\|_{\mathcal{L}_p} + \gamma_2\|\epsilon_2\|_{\mathcal{L}_p} + \beta \tag{B.7}$$

$$\gamma_1 = -\frac{\lambda_{max}}{\lambda_{min}^2}\|\bar{B}_1\|_{\mathcal{L}_p} \quad \gamma_2 = -\frac{\lambda_{max}}{\lambda_{min}^2}\|\bar{B}_2\|_{\mathcal{L}_p} \tag{B.8}$$

$$\beta = \rho\|z(0)\|\sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \tag{B.9}$$

$$\lambda_{max} = \max\{\lambda(D)\} \tag{B.10}$$

$$\lambda_{min} = \min\{\lambda(D)\}. \tag{B.11}$$

Since $\lambda(D) = \lambda(A - KC)$ and $\|\tilde{x}\|_{\mathcal{L}_p} = \|Tz\|_{\mathcal{L}_p} \leq \|T\|_2\|z\|_{\mathcal{L}_p} = \|z\|_{\mathcal{L}_p}$ we have the thesis.

□

Appendix C

Brief Description of the LuGre Model

The LuGre model is a dynamic friction model presented in [9]. Friction is modeled as the average deflection force of elastic springs. When a tangential force is applied the bristles will deflect like springs. If the deflection is sufficiently large the bristles start to slip. The average bristle deflection for a steady state motion is determined by the velocity. It is lower at low velocities, which implies that the steady state deflection decreases with increasing velocity. This models the phenomenon that the surfaces are pushed apart by the lubricant, and models the Stribeck effect. The model also includes rate dependent friction phenomena such as varying break-away force and frictional lag. The model has the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v - \sigma_0 \frac{|v|}{g(v)} z$$

$$F = \sigma_0 z + \sigma_1(v) \frac{\mathrm{d}z}{\mathrm{d}t} + F(v)$$

where z denotes the average bristle deflection. The model behaves like a spring for small displacements. The parameter σ_0 is the stiffness of the bristles, and $\sigma_1(v)$ the damping. The function $g(v)$ models the Stribeck effect, and $f(v)$ is the viscous friction. A reasonable choice of $g(v)$ which gives a good approximation of the Stribeck effect is

$$g(v) = \alpha_0 + \alpha_1 e^{-(v-v_0)^2}$$

The sum $\alpha_0 + \alpha_1$ then corresponds to stiction force and α_0 to Coulomb friction force. The parameter v_0 determines how $g(v)$ varies within its bounds $\alpha_0 \leq g(v) \leq \alpha_0 + \alpha_1$. A common choice of $f(v)$ is linear viscous friction $f(v) = \alpha_2 v$.

For a more advanced analysis of this model please see [9] and [27].

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