Appendix A: Notes on Exercises

1.1 Solutions: (a) 1, (b) 3, (c) 2, (d) 4.

1.3 The rotations permute the coordinate axes: think what happens when you rotate a cube through 120° about a long diagonal. So the matrices are

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.8 It saves work to remember that $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$.

1.20 Put $\boldsymbol{r} = \alpha\boldsymbol{\omega} + \beta\,\tilde{\mathrm{D}}\boldsymbol{\omega} + \gamma\boldsymbol{\omega} \wedge \tilde{\mathrm{D}}\boldsymbol{\omega}$. Show that if $\tilde{\mathrm{D}}\boldsymbol{\omega} \wedge \boldsymbol{r} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{r}) = 0$, then $\alpha = \beta = \gamma = 0$.

1.23 Consider the motion relative to the frame $(O, (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}))$. The radius is $ab\Omega\big(a^2(n-\Omega)^2 + b^2\Omega^2\big)^{-1/2}$.

3.6 For the first part, multiply the equation of motion by v_a and sum over a.

3.12 Let O denote the midpoint of AB. Take the x-axis along OB and the y-axis vertically upwards. Then the coordinates of P are $x = b\cosh\varphi\cos\theta$, $y = -b\sinh\varphi\sin\theta$.

3.13 Write (2.36) in the form

$$\frac{\partial^2 L}{\partial v_a \partial q_b} v_b + \frac{\partial^2 L}{\partial v_a \partial v_b}\dot{v}_b + \frac{\partial^2 L}{\partial v_a \partial t} - \frac{\partial L}{\partial q_a} = 0. \tag{A.1}$$

This holds for all values of q_a, v_a, $\dot{v}_a$ and t. Deduce that $\partial^2 L/\partial v_a \partial v_b = 0$ and hence that $L = A_a(q,t)v_a + B(q,t)$. Substitute into (A.1) to get

$$\frac{\partial A_a}{\partial q_b} = \frac{\partial A_b}{\partial q_a}, \qquad \frac{\partial A_a}{\partial t} = \frac{\partial B}{\partial q_a}.$$

These imply that $A_a = \partial f/\partial q_a$ and $B = \partial f/\partial t$ for some function $f = f(q,t)$.

3.14 Use the polar angles θ and φ as coordinates. Without loss of generality, choose the polar axis so that $\theta = \pi/2$, $\dot{\theta} = 0$ initially.

4.7 For the second part, differentiate

$$u_a \frac{\partial L}{\partial q_a} + \left(\frac{\partial u_a}{\partial q_b} v_b + \frac{\partial u_a}{\partial t} \right) \frac{\partial L}{\partial v_a} = 0$$

with respect to v_b.

5.4 Write

$$T = \tfrac{1}{2} \int_0^1 m\big(t\boldsymbol{u} + (1-t)\boldsymbol{v}\big) \cdot \big(t\boldsymbol{u} + (1-t)\boldsymbol{v}\big)\, \mathrm{d}t.$$

5.6 For the last part, consider a circular disc of radius a and find a point of inertial symmetry on the axis of the disc.

5.8 Substitute $\omega_2 = K \tanh u$, where K is an appropriate constant.

5.9 Use (5.12) and (5.13) to write ω_1 and ω_3 in terms of T, J^2, and ω_2. Now substitute for ω_1 and ω_3 in $B^2\dot{\omega}_2^2 = (A-C)^2\omega_3^2\omega_1^2$.

5.10 The intersection of the instantaneous axis with the surface has coordinates $(\lambda\omega_1, \lambda\omega_2, \lambda\omega_3)$ where $\lambda = \pm k\sqrt{2T}$. The tangent planes at these points are given by

$$A\omega_1 x + B\omega_2 y + C\omega_3 z = \lambda(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$$

or, alternatively, by $\boldsymbol{J}_0 \cdot \boldsymbol{r} = \pm k\sqrt{2T}$. But $\boldsymbol{J}_0$ is fixed relative to the inertial frame and T is constant.

The shape of the surface is not important: the ellipsoid can be an imaginary surface in the body with equation $Ax^2+By^2+Cz^2 = k^2$. The motion is then such that the imaginary surface appears to be rolling between two fixed plane – a result due to Poinsot.

5.12 The angular momentum of the smaller sphere about its centre of mass is $\frac{2}{5}ma^2\boldsymbol{\omega}$ where m is its mass and $\boldsymbol{\omega}$ is its angular velocity. Hence if $\boldsymbol{R}$ is the force at the point of contact, then

$$ma\ddot{\boldsymbol{e}} = -mg\boldsymbol{k} + \boldsymbol{R}, \qquad \tfrac{2}{5}ma^2\dot{\boldsymbol{\omega}} = a\boldsymbol{e} \wedge \boldsymbol{R},$$

by the principles of linear and angular momentum. The rolling condition at the point of contact is $a\dot{\boldsymbol{e}} + \boldsymbol{\omega} \wedge (a\boldsymbol{e}) = 0$.

5.13 C. E. Easthope gives a full discussion in [4]. He also makes an interesting remark about golf.

Denote the mass of the sphere by m, its radius by a and let the radius of the cylinder be $a + c$. The centre of the sphere has position vector $\boldsymbol{r} = z\boldsymbol{k} + c\boldsymbol{e}$ from a fixed origin on the axis of the cylinder, where $\boldsymbol{k}$ and $\boldsymbol{e}$ are orthogonal unit vectors, with $\boldsymbol{k}$ pointing vertically upwards. Let$\boldsymbol{f} = \boldsymbol{k} \wedge \boldsymbol{e}$.

The triad $(\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{k})$ is orthonormal and has angular velocity $\Omega\boldsymbol{k}$ with respect to fixed axes, where Ω is some function of time.

Let $\boldsymbol{\omega}$ denote the angular velocity of the sphere. Put $n = \boldsymbol{\omega} \cdot \boldsymbol{e}$ and $N = \boldsymbol{\omega} \cdot \boldsymbol{f}$.

With the dot denoting the time derivative with respect to fixed axes, the equations of motion are

$$\tfrac{2}{5}ma^2\dot{\boldsymbol{\omega}} = a\boldsymbol{e} \wedge \boldsymbol{R}, \qquad m(\ddot{z}\boldsymbol{k} + c\ddot{\boldsymbol{e}}) = \boldsymbol{R} - mg\boldsymbol{k},$$

where $\boldsymbol{R}$ is the force at the point of contact. The rolling condition is

$$\dot{z}\boldsymbol{k} + c\dot{\boldsymbol{e}} + a\boldsymbol{\omega} \wedge \boldsymbol{e} = 0.$$

The following steps lead to the stated result.

(1) From all three equations

$$\tfrac{2}{5}a\dot{\boldsymbol{\omega}} = \boldsymbol{e} \wedge (-a\dot{\boldsymbol{\omega}} \wedge \boldsymbol{e} - a\boldsymbol{\omega} \wedge \dot{\boldsymbol{e}} + g\boldsymbol{k}).$$

Hence $\boldsymbol{e} \cdot \dot{\boldsymbol{\omega}} = 0$ and $\tfrac{7}{5}a\dot{\boldsymbol{\omega}} = (\Omega na - g)\boldsymbol{f}$.

(2) From the rolling condition

$$\dot{z} = aN \quad \text{and} \quad 0 = c\Omega + a\boldsymbol{\omega} \cdot \boldsymbol{k}.$$

(3) $\dot{\boldsymbol{\omega}} \cdot \boldsymbol{k} = 0$ and hence Ω is constant.

(4) By considering $\dot{\boldsymbol{\omega}} \cdot \boldsymbol{f}$,

$$7\ddot{N} + 2\Omega\dot{n} = 0.$$

(5) By considering $\dot{\boldsymbol{\omega}} \cdot \boldsymbol{e}$, $\dot{n} = \Omega N$.

(6) $7\dddot{z} + 2\Omega^2\dot{z} = 0$.

5.14 Part (b): remember that $A\Omega^2 2u_0 - Cn\Omega + mga = 0$ and that

$$\Omega = \frac{j - Cnu_0}{A(1 - u_0^2)}.$$

5.16 Use the fact that φ is cyclic to show that

$$\dot{\varphi} = \frac{V \sin\alpha}{a \sin^2\theta}.$$

Show that

$$\dot{\theta}^2(1 + 3\cos^2\theta) + 4\dot{\varphi}^2 \sin^2\theta$$

is constant. Write $u = \cos\theta$ and find an expression for $\mathrm{d}t/\mathrm{d}u$.

5.17 Let α denote the angle between the downward vertical and a line joining the centre of the smaller cylinder to a point on its rim and let x denote the horizontal distance of the centre of the larger cylinder from a fixed point. Let $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ be an orthonormal triad, with $\boldsymbol{k}$ vertical and $\boldsymbol{i}$ orthogonal to the axes of the cylinders. Then the angular velocities of the cylinders are $\dot{\theta}\boldsymbol{j}$ (large) and $\dot{\alpha}\boldsymbol{j}$ (small). The velocities of the centres are $\dot{x}\boldsymbol{i}$ (large) and

$$\dot{x}\boldsymbol{i} - (\tfrac{1}{2}a\cos\varphi)\dot{\varphi}\boldsymbol{i} + (\tfrac{1}{2}a\sin\varphi)\dot{\varphi}\boldsymbol{k}$$

(small). The vector $\boldsymbol{e} = -\cos\varphi\boldsymbol{k} - \sin\varphi\boldsymbol{i}$ is a unit vector pointing from the centre of the larger cylinder to the centre of the smaller cylinder. The rolling conditions are

$$\dot{x} - a\dot{\theta} = 0;$$

and

$$\dot{x}\boldsymbol{i} + \dot{\theta}\boldsymbol{j} \wedge (a\boldsymbol{e}) = \dot{x}\boldsymbol{i} - (\tfrac{1}{2}a\cos\varphi)\dot{\varphi}\boldsymbol{i} + (\tfrac{1}{2}a\sin\varphi)\dot{\varphi}\boldsymbol{k} + \dot{\alpha}\boldsymbol{j} \wedge (\tfrac{1}{2}a\boldsymbol{e}),$$

which gives $\dot{\alpha} = 2\dot{\theta} - \dot{\varphi}$ and hence $\alpha = 2\theta - \varphi + \text{constant}$.

Take θ and φ as generalized coordinates, and choose origins such that $x = a\theta$ and $\alpha = 2\theta - \varphi$. Then the total kinetic energy is

$$\tfrac{1}{2}ma^2\dot{\theta}^2 + \tfrac{1}{2}m(a\dot{\theta} - \tfrac{1}{2}a\dot{\varphi}\cos\varphi)^2 + \tfrac{1}{8}ma^2\dot{\varphi}^2\sin^2\varphi + \tfrac{1}{2}ma^2\dot{\theta}^2 + + \tfrac{1}{8}ma^2(2\dot{\theta} - \dot{\varphi})^2.$$

The result follows from conservation of energy ($\partial L/\partial t = 0$).

5.22 In the notation of (5.13) above: use as coordinates θ, φ, ψ, χ, and z, where θ, φ, and ψ are the Euler angles of a triad fixed relative in the sphere relative to the triad $(\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{k})$, χ is the angle between $\boldsymbol{e}$ and a fixed horizontal line, and z is the height of the centre of the sphere above a fixed origin on the axis of the cylinder.

Then

$$\begin{aligned} \boldsymbol{\omega} \cdot \boldsymbol{k} &= \dot{\chi} + \dot{\varphi} + \dot{\psi}\cos\theta \\ n &= -\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi \\ N &= \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi. \end{aligned}$$

The Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{5}ma^2\big(\dot\theta^2 + \dot\psi^2 + (\dot\varphi + \dot\chi)^2 + 2\dot\psi(\dot\varphi + \dot\chi)\cos\theta\big) \\ &\quad + \tfrac{1}{2}m(\dot z^2 + c^2\dot\chi^2) - mgz \end{aligned}$$

and the rolling conditions are

$$\begin{aligned} \dot z - a(\dot\theta\cos\varphi + \dot\psi\sin\theta\sin\varphi) &= 0 \\ c\dot\chi + a(\dot\varphi + \dot\chi + \dot\psi\cos\theta) &= 0, \end{aligned}$$

corresponding to which we have the Lagrange multipliers λ and μ.

The φ and χ equations give that $\lambda = 0$ and that $\dot\chi = \Omega$ is constant. After some manipulation, the ψ and θ equations give

$$\begin{aligned} \tfrac{2}{5}ma\ddot\psi\sin\theta + \tfrac{4}{5}ma\dot\psi\dot\theta\cos\theta + \tfrac{2}{5}mc\dot\chi\dot\theta &= -\mu\sin\varphi \\ \tfrac{2}{5}ma\ddot\theta - \tfrac{2}{5}ma\dot\psi^2\sin\theta\cos\theta - \tfrac{2}{5}mc\dot\chi\dot\psi\sin\theta &= -\mu\cos\varphi. \end{aligned}$$

The z equation is $m\ddot z + mg = \mu$. A little further work leads to $\dot n = \Omega N$ and $7\dot N + 2\Omega n + 5g/a = 0$ and hence $7\dddot z + 2\Omega^2\dot z = 0$, as before.

6.2 The roots are $\lambda = 1$, $\lambda = 1/2$, and $\lambda = 3/2$.

6.3 One might be tempted to begin by introducing four coordinates, the spherical polar angles θ_1 and φ_1 of A (with polar axis PA), and the two spherical polar angles θ_2 and φ_2 of B (with polar axis QB). This will not work, however. The polar coordinates are singular in the equilibrium configuration, as is reflected by the fact that large changes in φ_1 and φ_2 can correspond to small displacements in the system.

6.4 Note that the system has four degrees of freedom.

6.5 Take the origin to be the equilibrium position of the particle, with axes chosen so that the position vectors of A, B, C, and D are $\boldsymbol{a} = (1, 1, \sqrt{2})$, $\boldsymbol{b} = a(1, -1, \sqrt{2})$, $\boldsymbol{c} = a(-1, -1, \sqrt{2})$, and $\boldsymbol{d} = a(-1, 1, \sqrt{2})$. Suppose that the particle is at the point P with position vector $\boldsymbol{r} = (x, y, z)$. To the second order in x, y, z,

$$PA = 2a\left(1 - \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{4a^2} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}}{8a^2} - \frac{(\boldsymbol{a}\cdot\boldsymbol{r})^2}{32a^4}\right).$$

Therefore, also to the second order, the elastic potential energy of the string PA is

$$\frac{\lambda(PA - a)^2}{2a} = \frac{\lambda}{2a}\left(a^2 - \boldsymbol{a}\cdot\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}\cdot\boldsymbol{r} + \frac{1}{8a^2}(\boldsymbol{a}\cdot\boldsymbol{r})^2\right)$$

We have

$$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})\cdot\boldsymbol{r} = 4a\sqrt{2}z$$

and

$$(\boldsymbol{a}\cdot\boldsymbol{r})^2+(\boldsymbol{b}\cdot\boldsymbol{r})^2+(\boldsymbol{c}\cdot\boldsymbol{r})^2+(\boldsymbol{d}\cdot\boldsymbol{r})^2 = 4a^2(x^2+y^2+2z^2).$$

Hence

$$U = \frac{\lambda a}{2}\left(4-\frac{4\sqrt{2}z}{a}+\frac{5x^2}{2a^2}+\frac{5y^2}{2a^2}+\frac{3z^2}{a^2}\right)+mgz.$$

7.6 Consider

$$\frac{\partial(q,p,t)}{\partial(q',p',t')} \quad \text{and} \quad \frac{\partial(q'',p'',t'')}{\partial(q',p',t')}.$$

7.10 The vector field $\boldsymbol{x}$ is tangent to Σ if and only if

$$\boldsymbol{x}\cdot \text{ grad }\left(p-\frac{\partial S}{\partial q}\right)=0$$

on Σ, where grad is the gradient operator

$$\text{grad } = \boldsymbol{i}\frac{\partial}{\partial q}+\boldsymbol{j}\frac{\partial}{\partial p}+\boldsymbol{k}\frac{\partial}{\partial t}.$$

This is equivalent to

$$\begin{aligned} 0 &= -\frac{\partial h}{\partial q}-\frac{\partial^2 S}{\partial q^2}\frac{\partial h}{\partial p}-\frac{\partial^2 S}{\partial q\partial t} \\ &= -\frac{\partial}{\partial q}\left(\frac{\partial S}{\partial t}+h\left(q,\frac{\partial S}{\partial q},t\right)\right). \end{aligned}$$

7.12 See Exercise (3.12).

8.12 Suppose that $\gamma\subset C\times\mathbb{R}$ is a kinematic trajectory given by $q_a = q_a(t)$. Then γ determines a *lifted trajectory* $\hat{\gamma}\subset TC\times\mathbb{R}$, given by

$$q_a = q_a(t), \qquad v_a = \dot{q}_a(t).$$

The flow of ρ_s carries a kinematic trajectory in $\gamma\subset C\times\mathbb{R}$ into a family γ_s of trajectories labelled by s, while the flow

$$\hat{\rho}_s : TC\times\mathbb{R}\to TC\times\mathbb{R}$$

of $\hat{u}$ maps $\hat{\gamma}$ to the family of lifted trajectories corresponding to $\hat{\gamma}_s$. If one uses this property to characterize the flow $\hat{\rho}_s$, and takes $\hat{u}$ to be the generating vector field, then it is simple to see that the components of $\hat{u}$ are given by (8.21) and hence that they must transform as the components of a vector on $TC\times\mathbb{R}$ vector under change of coordinates on $C\times\mathbb{R}$.

8.10 Use the special coordinate system.

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