A
Von Staudt and his Influence

A.1 Von Staudt

The fundamental criticism of the work of Chasles and Möbius is that in it cross-ratio is defined as a product of two ratios, and so as an expression involving four lengths. This makes projective geometry, in their formulation, dependent on Euclidean geometry, and yet projective geometry is claimed to be more fundamental, because it does not involve the concept of distance at all. The way out of this apparent contradiction was pioneered by von Staudt, taken up by Felix Klein, and gradually made its way into the mainstream, culminating in the axiomatic treatments of projective geometry between 1890 and 1914.

That a contradiction was perceived is apparent from remarks Klein quotes in his *Zur Nicht-Euklidische Geometrie* [136] from Cayley and Ball.[1] Thus, from Cayley: "It must however be admitted that, in applying this theory of v. Staudt's to the theory of distance, there is at least the appearance of arguing in a circle." And from Ball: "In that theory [the non-Euclidean geometry] it seems as if we try to replace our ordinary notion of distance between two points by the logarithm of a certain anharmonic ratio. But this ratio itself involves the notion of distance measured in the ordinary way. How then can we supersede the old notion of distance by the non-Euclidean one, inasmuch as the very definition of the latter involves the former?"

The way forward was to define projective concepts entirely independently of Euclidean geometry. The way this was done was inevitably confused at first,

[1] In Klein, *Gesammelte mathematische Werke*, I [135, pp. 353–383], the quotations are on p. 354.

because it is a complicated process. An investigator has to decide what can be assumed, and what indeed is to be proved. Initially, the understanding was that the subject matter was that of real projective geometry in two and three dimensions – ideas about complex projective geometry were not at all those one would expect today. Then one has to decide how coordinates enter the picture: are they given in advance or to be derived from some logically antecedent structure? How are constructions related to transformations? What projective transformations are there? Two ideas in particular were to cause problems. One was continuity, the other the connection between coordinates and transformations. If the coordinates are to be real numbers, recall that Dedekind's rigorous ideas about them were published only in 1872, and if the transformations are to form a group, note that Jordan's major book on group theory came out only in 1870.

A.1.1 Von Staudt's *Geometrie der Lage*

The first mathematician to advance the study of projective geometry in its own right, independent of metrical considerations, was Karl Georg Christian von Staudt, who lived a quiet life working in the small university of Erlangen. It was a backwater, with few students, and his two major books crept almost unobserved onto the shelves, where they remained until after his death in 1867 and it became gradually clear that he had gone a long way to solve the problem of giving independent foundations to real projective geometry. Among the first to rescue him from obscurity was the young Felix Klein, at the time a student at Berlin, who was alerted by his friend Otto Stolz to the significance of von Staudt's work for questions he was interested in.

Von Staudt's first book, his *Geometrie der Lage* [225], is based on the idea that there are entities called points, lines and planes. Lines in the same plane may meet or be parallel – the presence of parallelism in his geometry is a complication that Klein was later to show can be written out of the theory. So one might say that von Staudt took over from Euclid all and only the non-metrical concepts of the *Elements*. He began by observing that if three points lie on a line then one is between the other two, and that if four points lie on a line then they form two separated pairs. Both statements are reasonable because he had not yet introduced points "at infinity". He noted that there is a unique line joining any two points. He could now define points at infinity in terms of a pencil of parallel lines in a plane, and he showed how to extend his earlier ideas to the new setting. He invoked the idea of figures in perspective as a typical transformation of figures, noting that a pencil of lines through a point can correspond to a pencil of parallel lines.

Now he introduced the idea of a reciprocity (his word) or duality between points and planes in space. Quite generally von Staudt preferred to work on the geometry of three dimensions, deducing results about plane geometry as a consequence. So he stated his version of Desargues' theorem as a theorem about figures in two different planes, and used it to show [225, ch. 8] that given three distinct points on a line there is a point which is the fourth harmonic point with respect to these three, and moreover it is unique. Such a set of four points he called a harmonic set of points, and he showed that a harmonic set of points is mapped to another harmonic set of points by a perspectivity.

Von Staudt then introduced projective transformations, which Möbius and some later writers called collineations, as those 1–1 maps which send lines to lines (and, in three dimensions, planes to planes) and send sets of four harmonic points to sets of four harmonic points. A reciprocity may also be a projective transformation, if it sends a harmonic set of points to a harmonic set of planes (with the obvious definition). He showed by exhibiting a suitable sequence of perspectivities that any three distinct collinear points may be mapped to any three distinct collinear points by a collineation. Next he produced a peculiar argument [225, §106], much criticised by later writers, in support of the claim that a map sending three points on a line to three points on the same line extends to a map of the whole line. He argued that the claim is trivial if the point A is mapped to the point A', the point B to the point B', and the whole segment between A and B to the whole segment between A' and B', because then every point outside the segment is the harmonic conjugate of a point inside the segment and the map extends in an obvious way. If on the other hand the segment AB is not mapped in this fashion, then, he said, exactly one of the interior and the exterior of AB contains a point that is mapped to itself, but, by the theorem on the fourth harmonic point, this leads to a contradiction (the details are perforce omitted here). Klein was to argue that this requires a discussion of continuity.

Subsequent generations of mathematicians and historians of mathematics have been most impressed by von Staudt's insistence on duality. Von Staudt insisted on speaking of a figure and its dual simultaneously. For him, a duality (which he called a correlation) was a 1–1 correspondence between points and lines in a plane which sends harmonic sets of points to harmonic sets of lines and vice versa. Such a transformation he called a polarity. For example, one might have a self-polar triangle in which, for each vertex, the line that corresponds to a vertex of the triangle is the corresponding side of the triangle. Given a polarity, it might be that a point P lies on the line ℓ to which it corresponds. This led von Staudt to his remarkable definition of a conic section as a locus of points each of which lies on its corresponding line. Indeed, for von Staudt, a conic was both its locus as a set of points and the corresponding dual locus of lines (such a conic, as he noted, may well be an empty set, and here is an example: $x^2+y^2+z^2=0$). Möbius had noted that in a space of even dimension the self-dual figures are conics, but in spaces of odd dimension there are self-dual figures that are not conics – the so-called null systems – and von Staudt did the same.

The upshot of all this work is that von Staudt showed in his *Geometrie der Lage* [225] that the familiar, real projective geometry can be built up from the non-metrical concepts of Euclidean geometry – or rather, and more precisely, he had mapped out a way in which that might be done. However, many details remained to be established properly.

He showed how one could as it were measure the cross-ratio of four points (at least if they lie in a chain) by moving three of them into a standard position and noting the coordinate of the fourth point. This shows that cross-ratios may be used as lengths are in Euclidean geometry to give necessary and sufficient conditions for one set of four points to be equivalent to another.[2]

He then showed how one could iterate the construction of the fourth harmonic point, to obtain what he called a chain of harmonic points on a line, and to obtain a Möbius net from any four coplanar points (no three on a line). The Möbius net permits the introduction of coordinates which are rational multiples of an arbitrary constant. He then assumed without discussion that a map from a Möbius net in one plane onto a Möbius net in another plane extended to a unique map of the one plane onto the other.

The difficulties with this work all lie beneath the surface. Some may even strike the reader as artificial, and so they are if the aim is to establish real projective geometry on its own terms, as von Staudt's was. But artificial or not, the incidence axioms for plane projective geometry say things like this: through any two distinct points there passes exactly one line; any two distinct lines meet in exactly one point. They do not say that there are infinitely many points on a line, or infinitely many lines through a point. They do not, for example, guarantee that there are even four points on a line, and if there were to be only three then the whole construction of the fourth harmonic point would of course fail. (As we saw when discussing Fano's work, see page 264, it is entirely possible to have a projective geometry with only three points on each line, so there is something to do here.) Understandably, on occasions like this, von Staudt assumed things that eventually later mathematicians felt the need to prove, or to dispatch with an axiom. The same is true of Desargues' theorem. Von Staudt was operating in a context, not all of which he explicitly recognised, which permitted him to prove Desargues' theorem in the plane. We shall return to this point later.

[2] See also the discussion in Part II of the *Beiträge* [226] on sums, products, and powers of transformations.

A.1.2 Klein's response to von Staudt

It is rather more understandable that von Staudt would slip into imprecision over the passage from Möbius nets in the projective plane to collineations of the whole projective plane. Given a proper set of definitions, it is elementary to show that a continuous function defined on a dense set of points on the line extends to a continuous map on the closure of the dense set, but none of that body of theory was available to von Staudt, and even a rigorous definition of the real numbers had still to be given. Klein saw early on that it was not only possible to develop von Staudt's ideas without introducing the idea of parallel lines, but it was advisable to do so, because this opened the way to connections between non-Euclidean geometry and projective geometry. He was also of the opinion that something had to be done to establish the claim that the projective map sending three given distinct collinear points to three given distinct collinear points is unique (there is no problem, he agreed, in establishing its existence). Von Staudt had also shown that the sequence of fourth harmonic points established by a triple of points cannot suddenly stop (by closing up). But he did not show that it necessarily had points in every interval in the line. Klein therefore proposed in his article of 1973 [129] to insist that it did, whereupon Lüroth and Zeuthen wrote to him to say that his some of his worries were unnecessary.

Klein replied in an article of 1874 [131]. He accepted Zeuthen's argument completely, even quoting it in his paper word for word in the original French. Zeuthen took four harmonic points A, B, C, D, where A and B separate C and D, and supposed there was a maximal segment FG on the line which the succession of fourth harmonic points obtained from A, B and C never entered. So if F is not a point of this interval, it is a limit of points in a chain of fourth harmonic points. He now argued by contradiction, as follows.

Let H be the fourth harmonic point of the points A, F and G, and let J be the point such that A and G harmonically separate F and J. Let B be a point of the chain suitably close to F and let K be the point such that A and H harmonically separate B and K. It is possible to chose B so that K is in the segment GJ and so close to G that KJ contains a point of the chain. Call this point C. Let L be the point such that A and L harmonically separate B and J, so L will be in HG. Now let D be the point such that A and D harmonically separate B and C. The point D lies not only in the segment HL but also in the segment FG, thus establishing the requisite contradiction.

This shows that a harmonic chain is a dense set of points on a projective line. Does it follow that a projective map defined on such a set extends to a unique map on the remaining points? Klein was now able to say that whatever it meant for a set of points on a line to be "continuous", the same applied to points on a projective line, because the matter had recently been clarified by Heine, Cantor and Dedekind. Since we now apply the adjective "continuous" to functions rather than sets of points, we must interpret this as concerning sets of points which are connected. But even so, he said, the answer was self-evidently "no", just as it was clear there was no way a function defined on the rational numbers could be extended to a continuous function on the whole real line. It was necessary, he insisted, to add to von Staudt's definition of a collineation that it be continuous.

There the matter rested until 1880, when Darboux wrote to Klein (who promptly published the relevant part of the letter in *Mathematische Annalen*, a journal he now edited) [133]. Darboux said that while everyone had agreed with Klein, the only flaw in von Staudt's original presentation was with the method of proof, not the claim itself. In other words, collineations as defined by von Staudt were automatically continuous. Darboux's argument was very elegant. It was required to show that a map ϕ which maps three points to themselves is the identity map on all points. First, he said, suppose we are allowed metrical arguments. Then a simple argument from the information that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(\infty) = \infty$ shows that ϕ satisfies the functional equation $\phi(x) + \phi(y) = \phi(x + y)$. (There is no problem with the use of ∞, which merely simplifies the formulae.) Now, he said, this conclusion on its own would not show that $\phi(x) = x$ and therefore is continuous, as Cauchy had been the first to notice. But the conclusion would follow if ϕ satisfied some extra conditions, and in fact the functional equation had been derived without using all the properties of ϕ. A little more work showed that $\phi(x)$ was positive when x was positive, and this was enough to rule out pathological behaviour and show that indeed the function ϕ was continuous.

He then gave a non-metrical argument to the same conclusion, which invoked Zeuthen's result discussed above, and for good measure showed that some other theorems of a similar kind are true without the need for assumptions of continuity. For example, Möbius had shown that a continuous map of the plane sending circles to circles is an inversion or a sequence of inversions, but the assumption that the map be continuous was unnecessary.

The proof of Desargues' theorem that von Staudt offered also worried Klein. He noted that it was essentially an incidence proof, in which the key ingredients were that two "points" lie on a unique "line", two "planes" meet in a "line", and so forth, where the quotation marks are to indicate that it is the incidence properties that make the argument work, not any other properties of lines or planes. So one could imagine the theorem being true of figures drawn with the appropriate kinds of curved lines and planes, to be precise curved surfaces which are determined by three distinct points and have the property that if two such surfaces meet in a curve, then any surface through two points on that curve contains the whole curve. Indeed, recall that the incidence proof of Desargues' theorem goes as follows. The lines OAA' and OBB' lie in a plane, and the lines AB and $A'B'$ in that plane meet in a point N. Similarly, the lines BC and $B'C'$ meet at the point L and the lines CA and $C'A'$ meet at the point M. The points L, M and N lie in the planes of the triangles ABC and $A'B'C'$, and so lie on the line common to these two planes. It is easy to see that the proof works for points, curves and surfaces subject to suitable restrictions; straightness and flatness are not involved.

What worried Klein was that Desargues' theorem in von Staudt's hands was the key to introducing coordinates in such a way that the surfaces involved had linear equations. This, Klein saw, invited an obvious generalisation down to two dimensions. One would discuss curves with the property that any two curves met in a point, and any two points determined a unique curve, and presumably deduce that Desargues' theorem allowed one to introduce coordinates in such a way that the curves were given by linear equations. But Klein knew that this could not be done, because Beltrami had shown that among the curves with that property in a disc-like region were geodesics with respect to a metric, and they could only be given linear equations if the metric had constant curvature.

This meant that von Staudt's trick of proving theorems in the projective geometry of two dimensions by passing to three dimensions could not be used. This suggested to Klein that projective geometry in three dimensions could be established more directly than projective geometry in only two dimensions, but he did not, as Enriques was later to suggest that Klein had done, conjecture that Desargues' theorem might even be false in two dimensions.

A.2 Non-orientability

In the course of all this work, a novel and unexpected topic emerged onto the mathematical scene: orientability. Both Möbius, who is usually credited with the discovery of non-orientable surfaces, because of the eponymous Möbius band, and Listing, seem to have been thinking of the band in 1858 – indeed Listing's unpublished note of that year [148] pre-dates Möbius's unpublished note [163] by a few months. Both men were connected to Gauss, who had died in 1855, and it might even be that the concept goes back to him. Be that as it may, the simple idea of orienting a surface is to imagine each point of the surface surrounded by a small disc. The boundary of each disc is a circle, and we can order the points on it by choosing three distinct points A, B, C say, and stating that they occur in that order. We say that the surface is orientable if all the discs can be oriented in a compatible way, and non-orientable otherwise.

The cylinder is an example of an orientable surface, and the Möbius band an example of a non-orientable surface.[3]

The relationship of the real projective plane to the usual Euclidean plane was understood in many ways. For example, the projective plane can be thought of as the Euclidean plane with the addition of a line at infinity. By the early 1870s the work of several authors had promoted another consideration, that of algebraic topology. Möbius, Listing, Jordan and Riemann in various ways had produced an analysis of surfaces, including something like a classification of what, with later terminology and ideas could be called compact surfaces. These include the surfaces defined by complex algebraic curves, such as the sphere, the torus (defined, for example, by the equation $w^2 = z\,(z-1)\,(z-2)\,(z-3)$ and in this form familiar from the theory of elliptic functions) and others. Central to this approach was what Riemann called the order of connectivity of the surface, and which he defined, impressionistically, as the smallest number of closed curves that can be drawn on the surface without it falling into two pieces. The connectivity of the sphere is 0, of the torus 2, and so on.

In the early 1870s, Schläfli and Klein were independently interested in the surfaces that arise in projective geometry, and they noticed that more complicated behaviour can occur, and this imperilled the intuitive enumeration. In 1874 Schläfli wrote to Klein to say that order could be restored if one regarded the usual plane as a double plane or as the limiting case of a family of two-sheeted hyperboloids. Klein published his version of these ideas in the *Mathematische Annalen* in 1874 [131], but it must be said that they are a little obscure, which shows how unfamiliar this point must have been and how difficult to grasp. It seems better, therefore, to explain it without staying too close to the text of his paper.

Klein observed that Riemann's treatment of what happened out towards infinity had the effect of making infinity a point, and that this could be seen by stereographic projection. We might add that, topologically, this is the one-point compactification of the plane. However, in (three-dimensional) projective geometry one thought of there being a plane at infinity, and in plane projective geometry one supposed there was a line at infinity. The way forward was unclear to him, but in an article of 1879 [131] he can be seen groping for ideas like these. Consider the projective plane as the space of all lines through the origin in Euclidean three-space, and the Euclidean plane as the plane with equation $z = -1$. Each Euclidean point gives rise to a sloping line through the origin (the line through the origin and the given point). The projective points correspond to the horizontal lines through the origin. Now, the space of all lines through

[3] Listing published his account of the band in 1861, Möbius only in 1865. What is at stake is the recognition of the mathematical significance of non-orientability; pictures of the band have been traced as far back as the 3rd century CE.

the origin is an unpleasant thing to visualise, so represent each line through the origin by the two antipodal points it marks out on the unit sphere with centre the origin. We then immediately have a 2–1 map from the sphere to the real projective plane. The map sends a point on the sphere to the line through the origin and the given point. The map is 2–1 because antipodal points on the sphere define the same projective point.

Now, the sphere is also a picture of the plane under stereographic projection. So one may think of the plane as a double cover of the projective plane. This is what Schläfli urged upon Klein. The double or Euclidean plane has an unexpected property: a line drawn upon one plane does not disconnect the double plane. This is easier to see if we switch over to the modern picture. The projective plane is, as noted, the image of the sphere under a 2–1 map. This allows us to see the projective plane as the northern hemisphere with antipodal points identified. Consider the effect of passing a plane through the origin. It cuts the sphere in a great circle, of course, but what can we say about the plane and the projective plane? We have a choice. It might be that we say that the great circle is mapped 2–1 onto its image, or we might merely look at the image. If we take the second alternative, we see a curve γ in the northern hemisphere that meets the equator at two antipodal points. This curve does not disconnect the projective plane, because of the identifications on the equator, and it lifts to a semicircle on the sphere which does not disconnect the plane. If we double the curve, however, we do disconnect the projective plane, and the image of the doubled curve is a whole line disconnecting the plane.

Klein went on to note that this strange property (there are curves which close up on doubling) was already visible in the Möbius band (which he did not call by that name). He did not observe that the cylinder is in the same relation to the Möbius band as the sphere is to the projective plane, and more tantalisingly he did not observe that the strange connectivity of the projective plane is connected to the fact that it contains a Möbius band. (Indeed, a thickened neighbourhood of the curve γ is a Möbius band.) In fact, the Möbius band is present in every drawing of a hyperbola and its asymptotes, once one knows to look for it. For example, consider the hyperbola with equation $x^2 - y^2 = 1$, and its asymptotes $x = y$, $x = -y$. For definiteness, consider the asymptote $x = y$. It goes off to infinity as it were north-east with the hyperbola on its right, and comes back (from the south-west) with the hyperbola on its left, showing that a thickened neighbourhood of the asymptote in the projective plane is a Möbius band, and that the asymptote has not cut the projective plane into two pieces (a north-west and a south-east part).

In terms of projective geometry, a straight (Euclidean) line extends to a closed curve, and a conic is also a closed curve in projective geometry. The difference between them is precisely that the straight projective line does not disconnect the projective plane, and the conic of course does. This observation also rippled through the community of projective geometers (it is visible in Zeuthen's article of 1876 [246], for example).

That Klein took these topological considerations to heart is noticeable in his little book of 1882, *Riemann's Theory of Algebraic Functions and their Integrals* [134], where in §23 the Klein bottle seems to make its first appearance. It is an amusing exercise to see the torus as a double cover of the Klein bottle.

A.3 Axiomatics – independence

In the years between 1899 and 1914 a number of mathematicians gave more or less definitive versions of axiomatic projective geometry. The Italians Pieri, Fano and Enriques were the first, followed in Germany by Hilbert and later Vahlen, in America by Veblen and Young, and then in England by Russell and Whitehead. In these years the Italians were widely appreciated, but for a variety of reasons they were eclipsed by Hilbert in the years after 1918, to the point where their achievements were almost forgotten, and they have had to be rediscovered by historians of mathematics.

What these mathematicians accomplished in various ways was the identification of projective geometry conceived analytically with a synthetic presentation given by axioms. By an analytic presentation is meant an account like this: projective space of dimension n consists of all the lines through the origin in an $n+1$-dimensional space over the real numbers, the allowed transformations form the group $PSL(n+1;\mathbb{R})$ and so forth. The question for all these investigators was: what should an appropriate axioms system be? Rather than pursue the historical development, let us jump to the end of the story and consider a set of suitable axioms for projective geometry. The treatment that follows is taken from Hartshorne's *Foundations of Projective Geometry* [106].

Four are entirely unproblematic:

A1. Two distinct points lie on exactly one line.

A2. Two distinct lines meet in at most one point.

A3. There are three non-collinear points.

A4. Every line contains at least three points.

It is clear that axiom A2 is equivalent to the assumption that two distinct lines meet in exactly one point, which is more obviously the dual version of A1. Axiom A3 says that the geometry is at least two-dimensional. Axiom A4 is needed to rule out the space consisting of three points and the three lines joining them in pairs as a projective space.

The next axioms are more substantial:

A5. Desargues theorem holds.

A6. Pappus's theorem holds.

A7. (Fano's axiom): the diagonal points of a complete quadrilateral are not collinear.

It is striking that Desargues' theorem must be assumed. It is not a consequence of the first four axioms of projective geometry. This is all the more remarkable when one considers the incidence proof of it, and indeed if one writes down the obvious axioms for projective geometry in three or more dimensions then Desargues' theorem is a consequence of those axioms. But it is not a consequence of the axioms of plane projective geometry, and there are projective planes in which it is false.

It is also the case that Pappus's theorem implies Desargues' – a result known as Hessenberg's theorem after its discoverer, see Hessenberg [114]. So in any (necessarily plane) projective geometry in which Desargues' theorem does not hold, Pappus's theorem also fails. On the other hand, if Pappus's theorem (and therefore Desargues') is true and Fano's axiom holds, then one can prove the fundamental theorem of plane projective geometry: that there is a unique projective transformation taking any four points, no three of which are collinear, to any four points, no three of which are collinear. Conversely, given axioms A1–A4, Desargues' theorem and Fano's axiom, one can prove Pappus's theorem.

What about the uniqueness of the fourth harmonic point? It doesn't hold in the Moulton plane. Moulton implies no fourth harmonic point, so the theorem of the fourth harmonic point implies that the plane is not a Moulton plane. What about Desargues' theorem in general?

It may be helpful to note that, in the presence of A1–A4, the only implication between axioms A5, A6 and A7 is that A6 implies A5 (Pappus implies Desargues). To establish the independence of the remaining axioms, examples must be given of geometries satisfying all the remaining possible combinations of axioms:

1. None of A5, A6 and A7 holds;
2. A5 holds, but not A6 or A7;
3. A6 holds (and therefore A5 holds) but not A7;
4. A7 holds, but not A5 or A6;
5. A5 and A6 hold, but not A7;

6. A5 and A7 hold, but not A6;
7. A5, A6, and A7 all hold.

These are all duly given in Hartshorne's book [106, ch. 6].

Another route, more in keeping with the Kleinian approach to geometry, is to accept the first four axioms and then to specify the existence of enough transformations. This is the approach of Artin in his *Geometric Algebra* [4]. Artin preferred to study projective geometry via affine geometry, so he allowed himself the concept of parallel lines. (An affine plane is obtained from a projective plane by singling out a line (to be called the line at infinity) and restricting attention to transformations that map this line to itself. Two lines are said to be parallel if they meet in the line at infinity.) He defined a map to be a dilatation if it maps the points P and Q, say, to P' and Q' respectively, in such a way that the line through P' parallel to PQ passes through Q'. Degenerate cases aside, a dilatation maps a line to a parallel line. Artin called a dilatation a translation if it is either the identity map or has no fixed points.

Artin's axiom 4a asserts that given any two distinct points P and Q, there is a translation taking P to Q. His axiom 4b asserts that given three distinct collinear points P, Q and R, there is a dilation mapping P to itself and Q to R. The existence of translations imposes conditions on the group of projective transformations, and so, ultimately, on the coordinates (if any can be admitted) of points. For example, the group of all translations is a commutative group if translations exist with different directions. Artin observed that it can be the case that translations might be confined to a single direction, in which case it was not known if the corresponding group had to be commutative. The Moulton plane is such a space because the axis can only be mapped to itself, but here the corresponding group is commutative.

Artin confined his attention to what he called the "good" case, in which axioms 4a and 4b were satisfied, and he showed that in this case one can introduce coordinates for points and linear equations for lines. Naively, the idea is that one picks an origin O arbitrarily, and then picks distinct translations in different directions, say τ_1 and τ_2, and uses $\tau_1(O)$ and $\tau_2(O)$ as the units of length in these directions.

Artin then worked backwards, starting with a division ring, taking pairs of elements from the division ring as coordinates of points, thus obtaining an affine plane, and thence a coordinatised projective plane. He now assumed that the first three of his axioms applied in this setting, but not the fourth, and instead postulated either D_a or DP, that is, either Desargues' theorem when the centre of perspective is at infinity (D_a) or at a finite point (DP). He then established that D_a is true if and only if axiom 4a is true, and DP is true if and only if DP is true.

If coordinates can be introduced into projective geometry, they form a division ring. Facts about division rings (from Artin's *Geometric Algebra* [4]) include:

A finite division ring is a field (Wedderburn's theorem).

A weakly ordered division ring (other than 0, 1) is ordered (ordered means the additive subgroup is a union of three sets of the form $-P \cup \{0\} \cup P$, where P has the property that $P + P \subset P$, $P.P \subset P$).

There are ordered non-commutative division rings (one was constructed by Hilbert).

All Archimedean fields are subfields of the real numbers.

There is a unique ordering on the real numbers consistent with the ordering on the rational numbers.

Artin then established Hilbert's classic result that the division ring is commutative if and only if Pappus's theorem is true, and so, by Wedderburn's theorem, in a finite Desarguian plane Pappus's theorem is true – although, most intriguingly, no synthetic proof of that result was known.

Finally, by consideration of orderings that I have omitted, Artin showed that for an ordered geometry to come from a field which is isomorphic to a subfield of the field of real numbers with its natural ordering, it is necessary and sufficient that the Archimedean postulate holds. It follows that in an Archimedean field the theorem of Pappus holds and the field is necessarily commutative.

It seems that it is the introduction of non-Archimedean fields that provoked an attempt to eliminate continuity considerations from abstract projective geometry, say by the use of segment arithmetic (as done by Hilbert and again by Hölder). Hilbert showed that in plane projective geometry with congruence and parallels but not continuity or an Archimedean axiom Pappus's theorem can be proved. Also, Pappus's theorem cannot be proved in simple projective geometry without continuity or congruence.

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