A

The Fourier and Laplace Transforms of Distributions

Below we introduce the Fourier and Laplace transformations for test functions and distributions and list (without proof) some of their most important properties.

A.1 Definition. A complex-valued function φ defined on $\mathbb{R}^m$ is called a test function (of rapid descent) if $\varphi \in C^\infty(\mathbb{R}^m)$ and for any $k \in \mathbb{Z}_+$,

$$\langle\varphi\rangle_k = \sup_{x\in\mathbb{R}^m} (1+|x|)^k \sum_{|\alpha|\le k} |\partial^\alpha \varphi(x)| < \infty,$$

where $|x| = (x_1^2 + \cdots + x_m^2)^{1/2}$. The space of all such functions is denoted by $\mathcal{S}(\mathbb{R}^m)$.

A.2 Definition. The Fourier transform $\tilde{\varphi}$ of a test function (of rapid descent) $\varphi \in \mathcal{S}(\mathbb{R}^m)$ is defined by

$$\tilde{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = \int\limits_{\mathbb{R}^m} e^{i(x,\xi)}\varphi(x)\,dx. \tag{A.1}$$

A.3 Theorem. *The operator $\mathcal{F}$ is a homeomorphism from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{R}^2)$, and its inverse $\mathcal{F}^{-1}$ acts according to the formula*

$$\varphi(x) = (\mathcal{F}^{-1}\tilde{\varphi})(x) = (2\pi)^{-m} \int\limits_{\mathbb{R}^m} e^{-i(x,\xi)}\tilde{\varphi}(\xi)\,d\xi, \quad \tilde{\varphi} \in \mathcal{S}(\mathbb{R}^m).$$

A.4 Remark. The continuity of $\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^m)$ to $\mathcal{S}(\mathbb{R}^m)$ means that for any $k \in \mathbb{Z}_+$, there is $l(k)$ such that

$$\langle\mathcal{F}\varphi\rangle_k \le c_k \langle\varphi\rangle_{l(k)} \quad \forall\varphi \in \mathcal{S}(\mathbb{R}^m).$$

A.5 Theorem. (i) *For any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^m)$, there holds Parseval's equality*

$$\int_{\mathbb{R}^m} \tilde{\varphi}(\xi)\overline{\tilde{\psi}(\xi)}\, d\xi = (2\pi)^m \int_{\mathbb{R}^m} \varphi(x)\overline{\psi(x)}\, dx. \tag{A.2}$$

(ii) $\mathcal{F}(\varphi * \psi)(\xi) = \tilde{\varphi}(\xi)\tilde{\psi}(\xi)$ *for any φ, $\psi \in \mathcal{S}(\mathbb{R}^m)$.*
(iii) *For any $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and any multiindex α,*

$$\mathcal{F}(\partial^\alpha \varphi)(\xi) = (-i)^{|\alpha|}\xi^\alpha \tilde{\varphi}(\xi),$$
$$\mathcal{F}(x^\alpha \varphi)(\xi) = (-i)^{|\alpha|}\partial^\alpha \tilde{\varphi}(\xi).$$

A.6 Example. If $m = 1$ and $\varphi(x) = e^{-a^2x^2}$, $a > 0$, then

$$\begin{aligned}(\mathcal{F}e^{-a^2x^2})(\xi) &= \int_{\mathbb{R}} e^{ix\xi - a^2x^2}\, dx \\ &= \frac{1}{a}\, e^{-\xi^2/(4a^2)} \int_{\mathbb{R}} e^{-(\sigma + i\xi/(2a))^2}\, d\sigma \\ &= \frac{1}{a}\, e^{-\xi^2/(4a^2)} \int_{\operatorname{Im}\zeta = \zeta/(2a)} e^{-\zeta^2}\, d\zeta.\end{aligned}$$

Using Cauchy's theorem, we can easily check that

$$\int_{\operatorname{Im}\zeta = \zeta/(2a)} e^{-\zeta^2}\, d\zeta = \int_{\mathbb{R}} e^{-\zeta^2}\, d\zeta = \sqrt{\pi};$$

therefore,

$$(\mathcal{F}e^{-a^2x^2})(\xi) = \frac{\sqrt{\pi}}{a}\, e^{-\xi^2/(4a^2)}.$$

A.7 Remark. Since $\mathcal{S}(\mathbb{R}^m)$ is dense in $L^1(\mathbb{R}^m)$ and $L^2(\mathbb{R}^m)$, the Fourier transformation can be extended by continuity to the latter spaces. Thus, if $f \in L^1(\mathbb{R}^m)$, then $\mathcal{F}f$ is simply defined by (A.1) and is continuous and bounded in $\mathbb{R}^m$.

If $f \in L^2(\mathbb{R}^m)$, then we approximate f by a sequence $\{f\}_{n=1}^\infty$ in $L^2(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$, for example, the sequence of truncations

$$f_n(x) = \begin{cases} f(x), & |x| < n, \\ 0, & |x| > n. \end{cases}$$

It is not difficult to show that $f_n \to f$ and $\tilde{f}_n \to \tilde{f} \in L^2(\mathbb{R}^m)$. Then we again define the Fourier transform of f by (A.1), understanding that equality in the above sense.

A.8 Theorem. *If f, $g \in L^2(\mathbb{R}^m)$, then*

(i) *the inverse transformation $\mathcal{F}^{-1}$ acts according to the formula*

$$f(x) = (\mathcal{F}^{-1}\tilde{f})(x) = (2\pi)^{-m} \int\limits_{\mathbb{R}^m} e^{-i(x,\xi)} \tilde{f}(\xi)\, d\xi.$$

(ii) *Parseval's equality* (A.2) *holds.*

A.9 Definition. The (generalized) Fourier transform

$$\tilde{f} = \mathcal{F}f$$

of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^m)$ (an element of the dual of $\mathcal{S}(\mathbb{R}^m)$) is defined by

$$(\tilde{f}, \tilde{\varphi}) = (2\pi)^m (f, \varphi) \quad \forall \tilde{\varphi} \in \mathcal{S}(\mathbb{R}^m),$$

where

$$\varphi = \mathcal{F}^{-1}\tilde{\varphi}$$

and $(\cdot\,, \cdot)$ is the duality generated by the inner product in $L^2(\mathbb{R}^m)$.

A.10 Theorem. (i) *The distributional operator $\mathcal{F}$ is a homeomorphism from $\mathcal{S}'(\mathbb{R}^m)$ to $\mathcal{S}'(\mathbb{R}^m)$.*

(ii) *For any $f \in \mathcal{S}'(\mathbb{R}^m)$ and any $\varphi \in \mathcal{S}(\mathbb{R}^m)$,*

$$\mathcal{F}(f * \varphi) = \tilde{\varphi}\tilde{f}.$$

(iii) *For any $f \in \mathcal{S}'(\mathbb{R}^m)$ and any multiindex α,*

$$\mathcal{F}(\partial^\alpha f) = (-i)^{|\alpha|}\xi^\alpha \tilde{f},$$
$$\mathcal{F}(x^\alpha f) = (-i)^{|\alpha|}\partial^\alpha \tilde{f}.$$

A.11 Examples. (i) If δ is the Dirac delta, then, by Theorem A.3(iii),

$$\mathcal{F}(\partial^\alpha \delta) = (-i)^{|\alpha|}\xi^\alpha \tilde{\delta}.$$

Since for any $\tilde{\varphi} \in \mathcal{S}(\mathbb{R}^m)$,

$$(\tilde{\delta}, \tilde{\varphi}) = (2\pi)^m (\delta, \varphi) = (2\pi)^m \overline{\varphi(0)} = \int\limits_{\mathbb{R}^m} \overline{\tilde{\varphi}(\xi)}\, d\xi = (1, \tilde{\varphi}),$$

it follows that

$$\mathcal{F}(\partial^\alpha \delta) = (-i)^{|\alpha|}\xi^\alpha.$$

(ii) For the one-dimensional distribution $\widetilde{x^{-1}}$ generated by the function x^{-1} (in the sense of principal value) we have

$$(\mathcal{F}\widetilde{x^{-1}}, \tilde{\varphi}) = 2\pi(x^{-1}, \varphi) = 2\pi \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int\limits_{\varepsilon<|x|<R} \frac{1}{x}\,\overline{\varphi(x)}\,dx$$

$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int\limits_{\mathbb{R}} \Bigg(\int\limits_{\varepsilon<|x|<R} \frac{1}{x}\,e^{ix\xi}\,dx\Bigg)\overline{\tilde{\varphi}(\xi)}\,d\xi$$

$$= 2i \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int\limits_{\mathbb{R}} \Bigg(\int\limits_{\varepsilon}^{R} \frac{1}{x}\,\sin(x\xi)\,dx\Bigg)\overline{\tilde{\varphi}(\xi)}\,d\xi.$$

Since

$$\int\limits_0^\infty \frac{1}{x}\,\sin(x\xi)\,dx = \tfrac{1}{2}\,\pi\,\operatorname{sgn}\xi,$$

we finally obtain

$$\mathcal{F}\widetilde{x^{-1}} = i\pi\,\operatorname{sgn}\xi.$$

We now turn our attention to the Laplace transformation for one-dimensional distributions. Let $g \in L^2(\mathbb{R})$, $\operatorname{supp} g \in [0;\infty)$, and suppose that there is $a \in \mathbb{R}$ such that $e^{-\xi t}g(t)$ belongs to $L^2(0;\infty)$ for $\xi > a$ and $e^{-\xi t}g(t)$ does not belong to $L^1(0;\infty)$ for $\xi < a$. If $a < a' < \xi$, then the equality

$$e^{-\xi t}g(t) = e^{-(\xi-a')t}e^{-a't}g(t)$$

implies that $e^{-\xi t}g(t)$ belongs to $L^1(0;\infty)$ for $\xi > a$. In this case, the Laplace transform $(\mathcal{L}g)(p)$, $p = \xi + i\eta$, $\xi > a$, exists and is defined by

$$G(p) = (\mathcal{L}g)(p) = \int\limits_0^\infty e^{-pt}g(t)\,dt.$$

The number a is called the Dedekind abscissa of absolute convergence of the Laplace transform.

A.12 Theorem. *The function $G(p) = G(\xi + i\eta)$ is holomorphic in the half-space $\operatorname{Re} p = \xi > a$.*

A.13 Remark. We may write the Laplace transform G of g in the form

$$G(p) = G(\xi + i\eta) = \int\limits_0^\infty e^{-it\eta}e^{-\xi t}g(t)\,dt = \big(F(e^{-\xi t}g)\big)(-\eta). \tag{A.3}$$

By Parseval's equality,

$$2\pi \int_0^\infty e^{-2\xi t} |g(t)|^2 \, dt = \int_{\mathbb{R}} |G(\xi + i\eta)|^2 \, d\eta;$$

hence,

$$\int_0^\infty e^{-2at} |g(t)|^2 \, dt = \sup_{\xi > a} \int_0^\infty e^{-2\xi t} |g(t)|^2 \, dt = \frac{1}{2\pi} \sup_{\xi > a} \int_{\mathbb{R}} |G(\xi + i\eta)|^2 \, d\eta.$$

From (A.3) it now follows that if $\xi > a$, then

$$e^{-\xi t} g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\eta t} G(\xi - i\eta) \, d\eta = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta t} G(\xi + i\eta) \, d\eta,$$

so the inverse Laplace transformation is defined by the formula

$$g(t) = (\mathcal{L}^{-1} G)(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} p = \xi} e^{pt} G(p) \, dp.$$

Clearly, the convergence of these integrals is understood in the L^2-sense.

Let g and h be functions of the above class with abscissas of absolute convergence a and b, respectively. The convolution $u = g * h$ of these functions is defined by

$$u(t) = (g * h)(t) = \int_0^t g(t - \tau) h(\tau) \, d\tau.$$

A.14 Theorem. *The Laplace transform $U(p) = (\mathcal{L}u)(p)$ exists, is holomorphic for $\xi > \max\{a, b\}$, and*

$$U(p) = G(p) H(p),$$

where G and H are the Laplace transforms of f and h, respectively.

Next, we define the Laplace transform of a distribution. We restrict the space of test functions (of rapid descent) for $m = 1$ to the subspace $\mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$ of all infinitely differentiable functions with compact support in $\mathbb{R}$, and denote by $\mathcal{D}'_+(\mathbb{R})$ the set of distributions in $\mathcal{D}'(\mathbb{R})$ (the dual of $\mathcal{D}(\mathbb{R})$) with support in $[0; \infty)$.

Let $g \in \mathcal{D}'_+(\mathbb{R})$, and let $a \in \mathbb{R}$ be such that $e^{-\xi t} g(t)$ belongs to $\mathcal{S}'(\mathbb{R})$ for $\xi > a$ and $e^{-\xi t} g(t)$ does not belong to $\mathcal{S}'(\mathbb{R})$ for $\xi < a$. The number a in this case is called the abscissa of convergence of the Laplace transform of the distribution g. It is easy to show that if $\xi > a$, then $e^{-\xi t} g(t)$ belongs to $\mathcal{S}'(\mathbb{R})$.

To define the distributional (generalized) Laplace transformation, we first introduce the distributional equivalent of the change of variable for functions, which takes us from $f(x)$ to $f(-x)$. Specifically, for any $f \in \mathcal{S}'(\mathbb{R})$, we define $f_{-x} \in \mathcal{S}'(\mathbb{R})$ by

$$\big(f_{-x}, \varphi(x)\big) = \big(f, \varphi(-x)\big) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

A.15 Definition. If $g \in \mathcal{D}'_+(\mathbb{R})$ is a distribution with abscissa of convergence $a \in \mathbb{R}$, then its (generalized) Laplace transform $G(p) = (\mathcal{L}g)(p)$ is defined for $\xi = \operatorname{Re} p > a$ by

$$G(p) = (\mathcal{L}g)(p) = \big(F(e^{-\xi t} g)\big)_{-\eta}(p), \quad p = \xi + i\eta.$$

A.16 Theorem. *If g and h are elements of $\mathcal{D}'_+(\mathbb{R})$ with abscissas of convergence a and b and Laplace transforms G and H, respectively, then*

(i) *$G(p)$, $p = \xi + i\eta$, is holomorphic for $\xi > a$;*

(ii) *$(\mathcal{L}(\partial_t^l g))(p) = p^l G(p)$;*

(iii) *$u = g * h$ exists and*

$$U(p) = (\mathcal{L}u)(p) = G(p)H(p) \quad \textit{for } \xi = \operatorname{Re} p > \max\{a, b\}.$$

A.17 Theorem. *Let $g \in \mathcal{D}'_+(\mathbb{R})$ be a distribution with abscissa of convergence a and Laplace transform G.*

(i) *For any $\varepsilon > 0$, there are positive numbers c and l such that*

$$|G(p)| \le c(1 + |p|)^l \quad \textit{for } \operatorname{Re} p = \xi \ge a + \varepsilon; \tag{A.4}$$

that is, $G(p)$ grows at infinity no faster than a polynomial.

(ii) *If $G(p)$ is holomorphic for $\xi > a$ and satisfies* (A.4)*, then there is a unique distribution $g \in \mathcal{D}'_+(\mathbb{R})$ such that*

$$(\mathcal{L}g)(p) = G(p) \quad \textit{for } \xi > a.$$

A.18 Example. It is easy to verify that the Laplace transform of the one-dimensional Dirac delta is $(\mathcal{L}\delta)(p) = 1$.

References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions. Dover (1964)
2. Agranovich, M.S., Vishik, M.I.: Elliptic problems with a parameter and parabolic problems of a general form. Russian Math. Surveys, 19:3, 53–161 (1964)
3. Akhiezer, N.I., Glazman, I.M.: Theory of Linear Operators in Hilbert Space, vol. 1. F. Ungar (1961–63)
4. Chudinovich, I., Constanda, C.: Non-stationary integral equations for elastic plates. C.R. Acad. Sci. Paris Sér. I, 329, 1115–1120 (1999)
5. Chudinovich, I., Constanda, C.: Solvability of initial-boundary value problems in bending of plates. J. Appl. Math. Phys., 51, 449–466 (2000)
6. Chudinovich, I., Constanda, C.: The Cauchy problem in the theory of plates with transverse shear deformation. Math. Models Methods Appl. Sci., 10, 463–477 (2000)
7. Chudinovich, I., Constanda, C.: Variational and Potential Methods in the Theory of Bending of Plates with Transverse Shear Deformation. Chapman & Hall/CRC (2000)
8. Chudinovich, I., Constanda, C.: Dynamic transmission problems for plates. J. Appl. Math. Phys. (ZAMP), 53, 1060–1074 (2002)
9. Constanda, C.: A Mathematical Analysis of Bending of Plates with Transverse Shear Deformation. Longman/Wiley (1990)
10. Constanda, C.: Direct and Indirect Boundary Integral Equation Methods. Chapman & Hall/CRC (1999)
11. Doetsch, G.: Anleitung zum praktischen Gebrauch der Laplace-Transformation und der Z-Transformation. R. Oldenbourg (1967)
12. Eskin, G.I.: Boundary Value Problems for Elliptic Pseudodifferential Equations. Amer. Math. Soc. (1981)
13. Lions, J.-L., Magenes, E.: Nonhomogeneous Boundary Value Problems and Applications, vol. 1. Springer-Verlag (1972)
14. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity. Cambridge University Press (1959)
15. Lubich, Ch.: On the multistep time discretization of linear initial-boundary value problems and their boundary integral equations. Numer. Math., 67, 364–389 (1994)
16. Mizohata, S.: The Theory of Partial Differential Equations. Cambridge University Press (1973)
17. Sokolnikoff, I.S.: Mathematical Theory of Elasticity. McGraw-Hill (1956)
18. Yosida, K.: Functional Analysis. Springer-Verlag (1966)

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