

A

The Fourier and Laplace Transforms of Distributions

Below we introduce the Fourier and Laplace transformations for test functions and distributions and list (without proof) some of their most important properties.

A.1 Definition. A complex-valued function φ defined on \mathbb{R}^m is called a test function (of rapid descent) if $\varphi \in C^\infty(\mathbb{R}^m)$ and for any $k \in \mathbb{Z}_+$,

$$\langle \varphi \rangle_k = \sup_{x \in \mathbb{R}^m} (1 + |x|)^k \sum_{|\alpha| \leq k} |\partial^\alpha \varphi(x)| < \infty,$$

where $|x| = (x_1^2 + \cdots + x_m^2)^{1/2}$. The space of all such functions is denoted by $\mathcal{S}(\mathbb{R}^m)$.

A.2 Definition. The Fourier transform $\tilde{\varphi}$ of a test function (of rapid descent) $\varphi \in \mathcal{S}(\mathbb{R}^m)$ is defined by

$$\tilde{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}^m} e^{i(x,\xi)} \varphi(x) dx. \quad (\text{A.1})$$

A.3 Theorem. *The operator \mathcal{F} is a homeomorphism from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{R}^2)$, and its inverse \mathcal{F}^{-1} acts according to the formula*

$$\varphi(x) = (\mathcal{F}^{-1}\tilde{\varphi})(x) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{-i(x,\xi)} \tilde{\varphi}(\xi) d\xi, \quad \tilde{\varphi} \in \mathcal{S}(\mathbb{R}^m).$$

A.4 Remark. The continuity of \mathcal{F} from $\mathcal{S}(\mathbb{R}^m)$ to $\mathcal{S}(\mathbb{R}^m)$ means that for any $k \in \mathbb{Z}_+$, there is $l(k)$ such that

$$\langle \mathcal{F}\varphi \rangle_k \leq c_k \langle \varphi \rangle_{l(k)} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^m).$$

A.5 Theorem. (i) For any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^m)$, there holds Parseval's equality

$$\int_{\mathbb{R}^m} \tilde{\varphi}(\xi) \overline{\tilde{\psi}(\xi)} d\xi = (2\pi)^m \int_{\mathbb{R}^m} \varphi(x) \overline{\psi(x)} dx. \quad (\text{A.2})$$

(ii) $\mathcal{F}(\varphi * \psi)(\xi) = \tilde{\varphi}(\xi) \tilde{\psi}(\xi)$ for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^m)$.

(iii) For any $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and any multiindex α ,

$$\mathcal{F}(\partial^\alpha \varphi)(\xi) = (-i)^{|\alpha|} \xi^\alpha \tilde{\varphi}(\xi),$$

$$\mathcal{F}(x^\alpha \varphi)(\xi) = (-i)^{|\alpha|} \partial^\alpha \tilde{\varphi}(\xi).$$

A.6 Example. If $m = 1$ and $\varphi(x) = e^{-a^2 x^2}$, $a > 0$, then

$$\begin{aligned} (\mathcal{F}e^{-a^2 x^2})(\xi) &= \int_{\mathbb{R}} e^{ix\xi - a^2 x^2} dx \\ &= \frac{1}{a} e^{-\xi^2/(4a^2)} \int_{\mathbb{R}} e^{-(\sigma + i\xi/(2a))^2} d\sigma \\ &= \frac{1}{a} e^{-\xi^2/(4a^2)} \int_{\text{Im } \zeta = \zeta/(2a)} e^{-\zeta^2} d\zeta. \end{aligned}$$

Using Cauchy's theorem, we can easily check that

$$\int_{\text{Im } \zeta = \zeta/(2a)} e^{-\zeta^2} d\zeta = \int_{\mathbb{R}} e^{-\zeta^2} d\zeta = \sqrt{\pi};$$

therefore,

$$(\mathcal{F}e^{-a^2 x^2})(\xi) = \frac{\sqrt{\pi}}{a} e^{-\xi^2/(4a^2)}.$$

A.7 Remark. Since $\mathcal{S}(\mathbb{R}^m)$ is dense in $L^1(\mathbb{R}^m)$ and $L^2(\mathbb{R}^m)$, the Fourier transformation can be extended by continuity to the latter spaces. Thus, if $f \in L^1(\mathbb{R}^m)$, then $\mathcal{F}f$ is simply defined by (A.1) and is continuous and bounded in \mathbb{R}^m .

If $f \in L^2(\mathbb{R}^m)$, then we approximate f by a sequence $\{f\}_{n=1}^\infty$ in $L^2(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$, for example, the sequence of truncations

$$f_n(x) = \begin{cases} f(x), & |x| < n, \\ 0, & |x| > n. \end{cases}$$

It is not difficult to show that $f_n \rightarrow f$ and $\tilde{f}_n \rightarrow \tilde{f} \in L^2(\mathbb{R}^m)$. Then we again define the Fourier transform of f by (A.1), understanding that equality in the above sense.

A.8 Theorem. *If $f, g \in L^2(\mathbb{R}^m)$, then*

(i) *the inverse transformation \mathcal{F}^{-1} acts according to the formula*

$$f(x) = (\mathcal{F}^{-1}\tilde{f})(x) = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{-i(x,\xi)} \tilde{f}(\xi) d\xi.$$

(ii) *Parseval's equality (A.2) holds.*

A.9 Definition. The (generalized) Fourier transform

$$\tilde{f} = \mathcal{F}f$$

of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^m)$ (an element of the dual of $\mathcal{S}(\mathbb{R}^m)$) is defined by

$$(\tilde{f}, \tilde{\varphi}) = (2\pi)^m (f, \varphi) \quad \forall \tilde{\varphi} \in \mathcal{S}(\mathbb{R}^m),$$

where

$$\varphi = \mathcal{F}^{-1}\tilde{\varphi}$$

and (\cdot, \cdot) is the duality generated by the inner product in $L^2(\mathbb{R}^m)$.

A.10 Theorem. (i) *The distributional operator \mathcal{F} is a homeomorphism from $\mathcal{S}'(\mathbb{R}^m)$ to $\mathcal{S}'(\mathbb{R}^m)$.*

(ii) *For any $f \in \mathcal{S}'(\mathbb{R}^m)$ and any $\varphi \in \mathcal{S}(\mathbb{R}^m)$,*

$$\mathcal{F}(f * \varphi) = \tilde{\varphi}\tilde{f}.$$

(iii) *For any $f \in \mathcal{S}'(\mathbb{R}^m)$ and any multiindex α ,*

$$\mathcal{F}(\partial^\alpha f) = (-i)^{|\alpha|} \xi^\alpha \tilde{f},$$

$$\mathcal{F}(x^\alpha f) = (-i)^{|\alpha|} \partial^\alpha \tilde{f}.$$

A.11 Examples. (i) If δ is the Dirac delta, then, by Theorem A.3(iii),

$$\mathcal{F}(\partial^\alpha \delta) = (-i)^{|\alpha|} \xi^\alpha \tilde{\delta}.$$

Since for any $\tilde{\varphi} \in \mathcal{S}(\mathbb{R}^m)$,

$$(\tilde{\delta}, \tilde{\varphi}) = (2\pi)^m (\delta, \varphi) = (2\pi)^m \overline{\varphi(0)} = \int_{\mathbb{R}^m} \overline{\tilde{\varphi}(\xi)} d\xi = (1, \tilde{\varphi}),$$

it follows that

$$\mathcal{F}(\partial^\alpha \delta) = (-i)^{|\alpha|} \xi^\alpha.$$

(ii) For the one-dimensional distribution $\widetilde{x^{-1}}$ generated by the function x^{-1} (in the sense of principal value) we have

$$\begin{aligned} (\mathcal{F}\widetilde{x^{-1}}, \tilde{\varphi}) &= 2\pi(x^{-1}, \varphi) = 2\pi \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\varepsilon < |x| < R} \frac{1}{x} \overline{\varphi(x)} dx \\ &= \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\mathbb{R}} \left(\int_{\varepsilon < |x| < R} \frac{1}{x} e^{ix\xi} dx \right) \overline{\tilde{\varphi}(\xi)} d\xi \\ &= 2i \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\mathbb{R}} \left(\int_{\varepsilon}^R \frac{1}{x} \sin(x\xi) dx \right) \overline{\tilde{\varphi}(\xi)} d\xi. \end{aligned}$$

Since

$$\int_0^{\infty} \frac{1}{x} \sin(x\xi) dx = \frac{1}{2} \pi \operatorname{sgn} \xi,$$

we finally obtain

$$\mathcal{F}\widetilde{x^{-1}} = i\pi \operatorname{sgn} \xi.$$

We now turn our attention to the Laplace transformation for one-dimensional distributions. Let $g \in L^2(\mathbb{R})$, $\operatorname{supp} g \in [0; \infty)$, and suppose that there is $a \in \mathbb{R}$ such that $e^{-\xi t}g(t)$ belongs to $L^2(0; \infty)$ for $\xi > a$ and $e^{-\xi t}g(t)$ does not belong to $L^1(0; \infty)$ for $\xi < a$. If $a < a' < \xi$, then the equality

$$e^{-\xi t}g(t) = e^{-(\xi-a')t}e^{-a't}g(t)$$

implies that $e^{-\xi t}g(t)$ belongs to $L^1(0; \infty)$ for $\xi > a$. In this case, the Laplace transform $(\mathcal{L}g)(p)$, $p = \xi + i\eta$, $\xi > a$, exists and is defined by

$$G(p) = (\mathcal{L}g)(p) = \int_0^{\infty} e^{-pt}g(t) dt.$$

The number a is called the Dedekind abscissa of absolute convergence of the Laplace transform.

A.12 Theorem. *The function $G(p) = G(\xi + i\eta)$ is holomorphic in the half-space $\operatorname{Re} p = \xi > a$.*

A.13 Remark. We may write the Laplace transform G of g in the form

$$G(p) = G(\xi + i\eta) = \int_0^{\infty} e^{-it\eta} e^{-\xi t} g(t) dt = (F(e^{-\xi t}g))(-\eta). \tag{A.3}$$

By Parseval's equality,

$$2\pi \int_0^\infty e^{-2\xi t} |g(t)|^2 dt = \int_{\mathbb{R}} |G(\xi + i\eta)|^2 d\eta;$$

hence,

$$\int_0^\infty e^{-2at} |g(t)|^2 dt = \sup_{\xi > a} \int_0^\infty e^{-2\xi t} |g(t)|^2 dt = \frac{1}{2\pi} \sup_{\xi > a} \int_{\mathbb{R}} |G(\xi + i\eta)|^2 d\eta.$$

From (A.3) it now follows that if $\xi > a$, then

$$e^{-\xi t} g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\eta t} G(\xi - i\eta) d\eta = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta t} G(\xi + i\eta) d\eta,$$

so the inverse Laplace transformation is defined by the formula

$$g(t) = (\mathcal{L}^{-1}G)(t) = \frac{1}{2\pi i} \int_{\text{Re } p = \xi} e^{pt} G(p) dp.$$

Clearly, the convergence of these integrals is understood in the L^2 -sense.

Let g and h be functions of the above class with abscissas of absolute convergence a and b , respectively. The convolution $u = g * h$ of these functions is defined by

$$u(t) = (g * h)(t) = \int_0^t g(t - \tau)h(\tau) d\tau.$$

A.14 Theorem. *The Laplace transform $U(p) = (\mathcal{L}u)(p)$ exists, is holomorphic for $\xi > \max\{a, b\}$, and*

$$U(p) = G(p)H(p),$$

where G and H are the Laplace transforms of f and h , respectively.

Next, we define the Laplace transform of a distribution. We restrict the space of test functions (of rapid descent) for $m = 1$ to the subspace $\mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$ of all infinitely differentiable functions with compact support in \mathbb{R} , and denote by $\mathcal{D}'_+(\mathbb{R})$ the set of distributions in $\mathcal{D}'(\mathbb{R})$ (the dual of $\mathcal{D}(\mathbb{R})$) with support in $[0; \infty)$.

Let $g \in \mathcal{D}'_+(\mathbb{R})$, and let $a \in \mathbb{R}$ be such that $e^{-\xi t}g(t)$ belongs to $\mathcal{S}'(\mathbb{R})$ for $\xi > a$ and $e^{-\xi t}g(t)$ does not belong to $\mathcal{S}'(\mathbb{R})$ for $\xi < a$. The number a in

this case is called the abscissa of convergence of the Laplace transform of the distribution g . It is easy to show that if $\xi > a$, then $e^{-\xi t}g(t)$ belongs to $\mathcal{S}'(\mathbb{R})$.

To define the distributional (generalized) Laplace transformation, we first introduce the distributional equivalent of the change of variable for functions, which takes us from $f(x)$ to $f(-x)$. Specifically, for any $f \in \mathcal{S}'(\mathbb{R})$, we define $f_{-x} \in \mathcal{S}'(\mathbb{R})$ by

$$(f_{-x}, \varphi(x)) = (f, \varphi(-x)) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

A.15 Definition. If $g \in \mathcal{D}'_+(\mathbb{R})$ is a distribution with abscissa of convergence $a \in \mathbb{R}$, then its (generalized) Laplace transform $G(p) = (\mathcal{L}g)(p)$ is defined for $\xi = \operatorname{Re} p > a$ by

$$G(p) = (\mathcal{L}g)(p) = (F(e^{-\xi t}g))_{-\eta}(p), \quad p = \xi + i\eta.$$

A.16 Theorem. *If g and h are elements of $\mathcal{D}'_+(\mathbb{R})$ with abscissas of convergence a and b and Laplace transforms G and H , respectively, then*

- (i) $G(p)$, $p = \xi + i\eta$, is holomorphic for $\xi > a$;
- (ii) $(\mathcal{L}(\partial_t^l g))(p) = p^l G(p)$;
- (iii) $u = g * h$ exists and

$$U(p) = (\mathcal{L}u)(p) = G(p)H(p) \quad \text{for } \xi = \operatorname{Re} p > \max\{a, b\}.$$

A.17 Theorem. *Let $g \in \mathcal{D}'_+(\mathbb{R})$ be a distribution with abscissa of convergence a and Laplace transform G .*

- (i) *For any $\varepsilon > 0$, there are positive numbers c and l such that*

$$|G(p)| \leq c(1 + |p|)^l \quad \text{for } \operatorname{Re} p = \xi \geq a + \varepsilon; \tag{A.4}$$

that is, $G(p)$ grows at infinity no faster than a polynomial.

- (ii) *If $G(p)$ is holomorphic for $\xi > a$ and satisfies (A.4), then there is a unique distribution $g \in \mathcal{D}'_+(\mathbb{R})$ such that*

$$(\mathcal{L}g)(p) = G(p) \quad \text{for } \xi > a.$$

A.18 Example. It is easy to verify that the Laplace transform of the one-dimensional Dirac delta is $(\mathcal{L}\delta)(p) = 1$.

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Index

- Area potential, time-dependent 129
- Bessel function, modified
 - of order one 12
 - of order zero 12
- Boundary conditions
 - combined 6, 108
 - Dirichlet 5, 19, 43, 119
 - elastic 114
 - mixed 5, 99
 - Neumann 5, 37, 43, 120
 - transmission (contact) 6
- integral equations,
 - nonstationary 1, 16, 51
- operators 73, 126
 - algebra of 69
- Cauchy
 - problem 131, 135
 - theorem 140
- Crack 6
- Dirac distribution 8
- Direct method 53
- Displacement vector 5
- Distribution, tempered 141
- Dual 19, 141
- Equations, nonhomogeneous 129
- Forces, transverse shear 4
- Form
 - energy bilinear 82
 - sesquilinear 22
- Fourier
 - transform 9, 139
 - distributional 19
 - inverse 11
 - transformation 8
- Fredholm Alternative 21, 24, 68
- Functional, bounded antilinear
 - (conjugate linear) 23, 68
- Homeomorphism 23, 68, 95, 139
- Initial conditions, nonhomogeneous 129
- Internal energy density 22
- Jump formulas 14, 15
- Lamé constants 2
- Laplace
 - transform 2, 142
 - of a distribution 143
 - transformation 2
 - distributional (generalized) 144
- Lemma
 - Lax–Milgram 26
 - Rellich’s 23
- Mapping
 - absolutely continuous 35
 - holomorphic 20

Matrix

- boundary operator 4
- of fundamental solutions 8, 77, 121

Moments, bending and twisting 4

Operators

- averaging 3
- boundary 73, 126
 - algebra of 69
- extension 2, 65
- Poincaré–Steklov 37, 39, 61, 70, 102
- restriction 2
- trace 70, 109

Parseval's equality 8, 21, 31, 140

Plate

- constitutive relations 3
- elastic 1
- equations of motion 3
 - nonhomogeneous 131
- finite, with a crack 93
- infinite, with a finite inclusion 57
- middle plane of 1
- multiply connected 57
- on an elastic foundation 7, 119
- piecewise homogeneous finite 75

- potentials, time-dependent 13, 15
- thickness 1
- weakened by a crack 82
- with transverse shear deformation 3

Potentials 8

- boundary operators generated by 16
- density 14
- initial 133
- in terms of Laplace transforms 13
- time-dependent 43, 73, 87, 126

Sobolev spaces with a

- parameter 37, 69, 85

Solution, weak 22, 34, 74, 97, 135

Somigliana representation formula 53

Test function (of rapid descent) 139

Theorem

- Dunford's 41
- Lebesgue's dominated convergence 136
- Paley–Wiener 21
- trace 87, 96, 103, 110

Variational

- equation 23
- problem 94