Appendix A
Construction of Real Numbers

The purpose of this appendix is to give a construction of the field $\mathbb{R}$ of *real* numbers from the field $\mathbb{Q}$ of *rational* numbers which, we assume, is known to the reader. Let us point out that we did not give the axiomatic construction of the set $\mathbb{N}$ of *natural numbers* from which one can first construct the set $\mathbb{Z}$ of *integers* and, subsequently, the set $\mathbb{Q}$ of *rationals*. These constructions may be found in most textbooks on abstract algebra, e.g., *A Survey of Modern Algebra*, by Birkhoff and MacLane [BM77].

Most authors use the so-called Dedekind Cuts to construct the set of real numbers from that of rational numbers. Since, however, the reader is now familiar with sequences and series, it is more natural to use Georg Cantor's method of construction, which is based on Cauchy sequences of rational numbers, and can be extended to more abstract situations. This abstraction, which is referred to as the *completion* of a *metric space*, was discussed in Chap. 5. We begin our discussion by introducing some notation and definitions.

Notation. We recall that the set of *rational* numbers is denoted by $\mathbb{Q}$ and the set of *positive rationals* by $\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$. Also, the set of all sequences of rational numbers, i.e., the set of all functions from $\mathbb{N}$ to $\mathbb{Q}$, is denoted by $\mathbb{Q}^{\mathbb{N}}$.

Next, we define the *Cauchy* sequences of rational numbers. Although the definition of Cauchy sequences was given earlier, since we have not yet constructed the set of real numbers, we must insist that the $\varepsilon > 0$ in our definition take on *rational values only*.

Definition A.1 (Cauchy Sequences in $\mathbb{Q}$). A sequence $x \in \mathbb{Q}^{\mathbb{N}}$ is called a *Cauchy* sequence if the following holds:

$$(\forall \varepsilon \in \mathbb{Q}^+)(\exists N \in \mathbb{N})(m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon).$$

The set of all Cauchy sequences in $\mathbb{Q}$ will be denoted by $\mathcal{C}$.

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Next, we define *null* sequences in $\mathbb{Q}$. Again, this was defined earlier, but we must be careful to use *only rational* $\varepsilon > 0$.

Definition A.2 (Null Sequences in $\mathbb{Q}$). A sequence $x \in \mathbb{Q}^{\mathbb{N}}$ is called a *null* sequence if the following holds:

$$(\forall \varepsilon \in \mathbb{Q}^{+})(\exists N \in \mathbb{N})(n \geq N \Rightarrow |x_n| < \varepsilon).$$

The set of all null sequences in $\mathbb{Q}$ will be denoted by $\mathcal{N}$.

Remark. Note that $\mathcal{N}$ is precisely the set of all rational sequences that converge to *zero* and that we obviously have $\mathcal{N} \subset \mathcal{C}$. (Why?)

As we have seen, the set $\mathbb{Q}$ of rationals is *dense* in the set $\mathbb{R}$ of real numbers, which we introduced axiomatically. It follows that each real number ξ is the limit of a (*not unique*) sequence (x_n) of rational numbers. It is tempting, therefore, to take such a sequence (x_n) as the *definition* of the real number ξ. The nonuniqueness of (x_n) poses a problem, however, for *two* such sequences in fact represent the *same* ξ. This motivates the following definition.

Definition A.3 (Equivalent Cauchy Sequences). We say that two Cauchy sequences $x, y \in \mathcal{C}$ are *equivalent* and write $x \sim y$, if and only if $x - y \in \mathcal{N}$.

Exercise A.1. Show that the relation $\sim$ is indeed an *equivalence relation* on the set $\mathcal{C}$.

Notation. For each sequence $x \in \mathcal{C}$, its *equivalence class* is denoted by $[x]$ and, we recall, is defined by $[x] = \{y \in \mathcal{C} : y \sim x\}$. The set of all equivalence classes of elements of $\mathcal{C}$ is denoted by $\mathcal{C}/\mathcal{N}$.

Definition A.4 (Real Number). The set $\mathbb{R}$ of *real numbers* is defined to be $\mathbb{R} := \mathcal{C}/\mathcal{N}$. Thus ξ is a real number if $\xi = [x]$ for some $x \in \mathcal{C}$. The sequence $x \in \mathcal{C}$ is then called a *representative* of ξ. Clearly, if x and y both represent ξ, then $x - y \in \mathcal{N}$.

Exercise A.2.

1. Show that a Cauchy sequence in $\mathbb{Q}$ is bounded.
2. Show that $\mathcal{C}$ is *closed* under addition and multiplication; i.e., $\forall x,\ y \in \mathcal{C}$, we have $x + y,\ xy \in \mathcal{C}$.
3. Show that $\mathcal{N}$ is an *ideal* in $\mathcal{C}$; i.e., it is closed under addition and satisfies the stronger condition that $\forall x \in \mathcal{N}$ and $\forall y \in \mathcal{C}$ we have $xy \in \mathcal{N}$. *Hints:* For the addition, use an $\varepsilon/2$-argument. For the multiplication, use the inequalities $|x_m y_m - x_n y_n| \leq |y_m||x_m - x_n| + |x_n||y_m - y_n| \leq B|x_m - x_n| + A|y_m - y_n|$, for some constants $A,\ B \in \mathbb{Q}^{+}$, where the second inequality follows from part (1).

Definition A.5 (Addition, Subtraction, Multiplication). Let $\xi = [x]$ and $\eta = [y]$ be any real numbers. We define $\xi + \eta,\ -\eta,\ \xi - \eta$, and $\xi\eta$ (or $\xi \cdot \eta$) as follows:

1. $\xi + \eta := [x + y]$,
2. $-\eta := [-y]$,
3. $\xi - \eta := \xi + (-\eta) = [x - y]$, and
4. $\xi\eta := [xy]$.

Exercise A.3. Show that the definitions of $\xi + \eta$ and $\xi\eta$ are *independent* of the representatives x and y of ξ and η, respectively. In other words, show that, if $x \sim x'$ and $y \sim y'$, then we have $x + y \sim x' + y'$ and $xy \sim x'y'$. *Hint:* You will need arguments similar to those needed in Exercise A.2.

Proposition A.1 (Ring Properties of $\mathbb{R}$). *The set $\mathbb{R}$ of real numbers is a commutative ring with identity. In other words, for all real numbers ξ, η, and ζ, we have*

1. $\xi + \eta = \eta + \xi$,
2. $(\xi + \eta) + \zeta = \xi + (\eta + \zeta)$,
3. $\exists\ 0 \in \mathbb{R}$ *with* $0 + \xi = \xi$,
4. $\exists\ -\xi \in \mathbb{R}$ *with* $\xi + (-\xi) = 0$,
5. $\xi\eta = \eta\xi$,
6. $(\xi\eta)\zeta = \xi(\eta\zeta)$,
7. $\exists\ 1 \in \mathbb{R}$, $1 \neq 0$, *with* $1 \cdot \xi = \xi$, *and*
8. $\xi(\eta + \zeta) = \xi\eta + \xi\zeta$.

Proof. The proofs of these properties are straightforward. For example, to prove (2), note that if $\xi = [x]$, $\eta = [y]$, and $\zeta = [z]$, then $(\xi + \eta) + \zeta = [(x + y) + z]$, while $\xi + (\eta + \zeta) = [x + (y + z)]$. Since we obviously have $(x + y) + z = x + (y + z)$ in $\mathcal{C}$, (2) follows. Note that the *additive identity* ("0" in (3)) is in fact $0 = [(0, 0, 0, \dots)] \in \mathcal{N}$ and that the *multiplicative identity* ("1" in (7)) is $1 = [(1, 1, 1, \dots)]$. Also, $1 \neq 0$ is obvious, because the sequences $(0, 0, 0, \dots)$ and $(1, 1, 1, \dots)$ are *not* equivalent. □

Proposition A.2. *Let $\phi : \mathbb{Q} \to \mathbb{R}$ be defined by $\phi(r) = [(r, r, r, \dots)]$. Then ϕ is an injective "ring homomorphism." In other words, ϕ is a one-to-one map satisfying $\phi(r + s) = \phi(r) + \phi(s)$, $\phi(rs) = \phi(r)\phi(s)$, $\phi(0) = 0$, and $\phi(1) = 1$, $\forall\ r, s \in \mathbb{Q}$.*

Exercise A.4. Prove Proposition A.2.

Remark. By Proposition A.2, the map ϕ is a *field isomorphism* of $\mathbb{Q}$ onto its image $\phi(\mathbb{Q}) \subset \mathbb{R}$; i.e., a one-to-one correspondence between $\mathbb{Q}$ and $\phi(\mathbb{Q})$ that preserves all the algebraic properties of $\mathbb{Q}$. Therefore, we henceforth *identify* the two sets and, by abuse of notation, will write $\mathbb{Q} = \phi(\mathbb{Q}) \subset \mathbb{R}$. Based on this identification, the *field* $\mathbb{Q}$ of rational numbers becomes a *subfield* of the field $\mathbb{R}$ of real numbers. Here, by a *field* we mean a set **F** together with two *operations* "+" of *addition* and "·" of *multiplication*, i.e., two maps $+ : (x, y) \mapsto x + y$ and $\cdot : (x, y) \mapsto x \cdot y$, from $\mathbf{F} \times \mathbf{F}$ to **F**, satisfying the nine (algebraic) axioms $(A_1 - A_4,\ M_1 - M_4,\ D)$ stated for real numbers in Sect. 2.1 of Chapter 2.

Proposition A.1 only shows that $\mathbb{R}$ is a *commutative ring with identity*. To prove that $\mathbb{R}$ is actually a *field*, the only property we need to check is the existence of *reciprocals* for *nonzero* real numbers (cf. Axiom (M_4) at the beginning of Chap. 2). To this end, we shall need the following.

Proposition A.3. *Let ξ be a nonzero element of $\mathbb{R}$. Then, there exists a rational number $r \in \mathbb{Q}^+$ and a representative $x \in \mathcal{C}$ of ξ such that either $x_n \geq r \quad \forall n \in \mathbb{N}$ or $x_n \leq -r \quad \forall n \in \mathbb{N}$.*

Proof. Let $y \in \mathcal{C}$ be a representative of ξ. Since $\xi \neq 0$, the sequence (y_n) is *not* equivalent to $(0, 0, 0, \ldots)$ and we have

$$(\exists \varepsilon \in \mathbb{Q}^+)(\forall N \in \mathbb{N})(\exists n \geq N)(|y_n - 0| \geq \varepsilon). \tag{$*$}$$

On the other hand, $(y_n) \in \mathcal{C}$ implies that

$$(\exists K \in \mathbb{N})(m,\ n \geq K \ \Rightarrow\ |y_m - y_n| < \varepsilon/2). \tag{$**$}$$

Now, by ($*$), we can find $k \geq K$ such that $|y_k| \geq \varepsilon$. Changing ξ to $-\xi$, if necessary, we may assume that $y_k \geq \varepsilon$. Therefore, using ($**$),

$$m \geq K \ \Rightarrow\ |y_m - y_k| \ < \varepsilon/2 \ \Rightarrow\ y_m \geq y_k - |y_m - y_k| \geq \varepsilon - \varepsilon/2 = \varepsilon/2.$$

Let $x_n := \varepsilon/2$ for $n < K$, and $x_n = y_n$ for $n \geq K$. It is then clear that $\xi = [(x_n)]$ and that, with $r := \varepsilon/2$, we have $x_n \geq r$ for all $n \in \mathbb{N}$. □

Definition A.6 (Positive and Negative Cauchy Sequences). We say that a sequence $x \in \mathcal{C}$ is *positive* (resp., *negative*) if it satisfies the first (resp., second) alternative in Proposition A.3. The set of all positive (resp., negative) sequences in $\mathcal{C}$ is denoted by $\mathcal{C}^+$ (resp., $\mathcal{C}^-$).

Remark. It is obvious that the two alternatives in Proposition A.3 are mutually exclusive, i.e., that $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$. Moreover, the condition in the first (and hence also second) alternative needs only be satisfied *ultimately*; i.e., it can be replaced by

$$(\exists r \in \mathbb{Q}^+)(\exists N \in \mathbb{N})(n \geq N \Rightarrow x_n \geq r).$$

Indeed, one can always replace x by the equivalent sequence x' defined by $x'_k := r \quad \forall k < N$ and $x'_k := x_k \quad \forall k \geq N$.

Proposition A.4. *We have $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{N} \cup \mathcal{C}^-$, where the union is disjoint. In other words, $\{\mathcal{C}^+, \mathcal{N}, \mathcal{C}^-\}$ is a partition of $\mathcal{C}$.*

Proof. This is an obvious consequence of Proposition A.3. □

We are now going to prove that $\mathbb{R}$ is indeed a field.

Theorem A.1. *The set $\mathbb{R}$ of real numbers is a field. In other words, in addition to the ring properties (1)–(8) of Proposition A.1, we also have the following:*

$$(\forall \xi \in \mathbb{R} \setminus \{0\})(\exists\ 1/\xi \in \mathbb{R} \setminus \{0\})(\xi \cdot (1/\xi) = 1).$$

Proof. Suppose that $\xi \in \mathbb{R} \setminus \{0\}$. By Proposition A.3, we can then find $r \in \mathbb{Q}^+$ and a representative (x_n) of ξ such that $|x_n| \geq r \ \forall n \in \mathbb{N}$. If we can show that $(1/x_n) \in \mathcal{C}$, then, setting $1/\xi := [(1/x_n)]$, we clearly get $\xi \cdot (1/\xi) = 1$. However, $(x_n) \in \mathcal{C}$ implies

$$(\forall \varepsilon \in \mathbb{Q}^+)(\exists N \in \mathbb{N})(m,\ n \geq N \Rightarrow |x_m - x_n| < \varepsilon r^2).$$

Therefore,

$$m,\ n \geq N \ \Rightarrow \ |1/x_m - 1/x_n| = \frac{|x_m - x_n|}{|x_m||x_n|} < \frac{\varepsilon r^2}{r^2} = \varepsilon,$$

which proves indeed that $(1/x_n) \in \mathcal{C}$ and completes the proof. □

Having established the field properties of $\mathbb{R}$, we now turn our attention to its *order* properties. Recall that this was treated axiomatically (cf. Axioms $(O)_1 - (O)_3$ at the beginning of Chap. 2) by means of a subset $P \subset \mathbb{R}$ called the subset of *positive* real numbers. In what follows we will define this subset and will denote it by $\mathbb{R}^+$, rather than P.

Definition A.7 (Positive and Negative Real Numbers). We define a real number $\xi \in \mathbb{R}$ to be *positive* (resp., *negative*) and write $\xi > 0$ (resp., $\xi < 0$), if $\xi = [x]$ for some $x \in \mathcal{C}^+$ (resp., $x \in \mathcal{C}^-$). The set of all positive (resp., negative) real numbers will be denoted by $\mathbb{R}^+$ (resp., $\mathbb{R}^-$).

Proposition A.5. *We have* $\mathbb{R}^- = -\mathbb{R}^+ := \{\xi \in \mathbb{R} : -\xi \in \mathbb{R}^+\}$, *and the set* $\mathbb{R}^+$ *of positive real numbers satisfies the following properties:*

1. $\mathbb{R}^+ + \mathbb{R}^+ \subset \mathbb{R}^+$,
2. $\mathbb{R}^+ \cdot \mathbb{R}^+ \subset \mathbb{R}^+$, *and*
3. $\mathbb{R} = \mathbb{R}^+ \cup \{0\} \cup \mathbb{R}^-$, *where the union is disjoint* *(Trichotomy).*

Exercise A.5. Prove Proposition A.5.

Now that the existence of the set $\mathbb{R}^+$ of positive real numbers has been established and that, in view of Proposition A.5, the order axioms (O_1), (O_2), and (O_3) are satisfied, all the order properties of the set $\mathbb{R}$ of real numbers can be proved as before. For instance, given $\xi,\ \eta \in \mathbb{R}$, we write $\xi \leq \eta$ to mean $\eta - \xi \in \mathbb{R}^+ \cup \{0\}$ and the set $\mathbb{R}$ is then *totally ordered* by the ordering $\leq$.

Remark.

1. We have defined the notion of *Cauchy sequence* once for (axiomatically defined) *real* numbers in Chap. 2 and again, in this appendix, for *rational* numbers (which are real numbers), using exclusively *rational* $\varepsilon > 0$. To show that, for rational sequences, the two definitions are identical, we need only show the following:

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists \varepsilon' \in \mathbb{Q})(0 < \varepsilon' \leq \varepsilon).$$

This, however, follows at once from Proposition A.3.

2. Since the set $\mathbb{R}$ we have constructed satisfies all the *algebraic* and *order* properties treated axiomatically in Chap. 2, the notion of *convergent sequence* can be defined as before. In other words, a sequence $(\xi_n) \in \mathbb{R}^{\mathbb{N}}$ of real numbers *converges* to the *limit* $\lambda \in \mathbb{R}$ (in symbols $\lim(\xi_n) = \lambda$), if the following holds:

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists N \in \mathbb{N})(n \geq N \Rightarrow |\xi_n - \lambda| < \varepsilon).$$

Our construction of real numbers was motivated by the intuitive idea that a real number should be the limit of a convergent sequence of rationals. The following proposition shows that this is indeed the case.

Proposition A.6. *Let ξ be a real number. For a sequence $x \in \mathcal{C}$ to be a representative of ξ, it is necessary and sufficient that* $\lim(x_n) = \xi$.

Proof. Suppose that $\xi = [x]$, and let $\varepsilon \in \mathbb{R}^+$ be given. Then, we can find $\varepsilon' \in \mathbb{Q}^+$ with $\varepsilon' \leq \varepsilon$. We can also find $N \in \mathbb{N}$ such that

$$m, \ n \geq N \ \Rightarrow \ -\varepsilon' < x_m - x_n < \varepsilon'. \tag{$*$}$$

Given $m \geq N$, the real number $x_m - \xi$ is the class of the sequence $(x_m - x_1, x_m - x_2, \ldots)$ which, using $(*)$, can be replaced by an equivalent one, $(y_n) \in \mathcal{C}$, such that $x_m - \xi = [(y_n)]$ and $-\varepsilon' < y_n < \varepsilon' \ \forall n \in \mathbb{N}$. Therefore, $-\varepsilon' < x_m - \xi < \varepsilon'$, and hence $|x_m - \xi| < \varepsilon$. This shows that we have $\lim(x_n) = \xi$. Conversely, suppose that $\lim(x_n) = \xi$ and that $\xi = [(y_n)]$ for a sequence $(y_n) \in \mathcal{C}$. Then, as we just proved, $\lim(y_n) = \xi$. It then follows that $(x_n) \sim (y_n)$ (why?), and we have $\xi = [(x_n)]$. □

All the algebraic and order properties we have proved for the set $\mathbb{R} := \mathcal{C}/\mathcal{N}$ are also shared by its subfield $\mathbb{Q}$ of rational numbers. We are finally ready to prove the *completeness* of $\mathbb{R}$ which, in the axiomatic treatment, was called the *Supremum Property* or *Completeness Axiom*. This property is *not* satisfied by the subfield $\mathbb{Q}$. Since the Supremum Property is *equivalent* to *Cauchy's Criterion* [as was pointed out in Remark 2.2.47 (2)], all we need is to prove this criterion for our set $\mathbb{R} := \mathcal{C}/\mathcal{N}$.

Theorem A.2 (Cauchy's Criterion). *A sequence $(\xi_n) \in \mathbb{R}^{\mathbb{N}}$ is convergent if and only if it is a Cauchy sequence.*

Proof. The necessity of the condition is obvious, as we saw in the proof of Theorem 2.2.46. To prove the sufficiency, note that, by Proposition A.6, for each $n \in \mathbb{N}$, we can find a *rational* number $x_n \in \mathbb{Q}$ (recall that $x_n = [(x_n, x_n, \ldots)]$) such that $|\xi_n - x_n| < 1/n$. Now

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists N \in \mathbb{N})(m, \ n \geq N \ \Rightarrow \ |\xi_m - \xi_n| < \ \varepsilon/3).$$

Thus, if $m,\ n \geq \max\{N, 3/\varepsilon\}$, then

$$\begin{aligned}|x_m - x_n| &\leq |x_m - \xi_m| + |\xi_m - \xi_n| + |\xi_n - x_n| \\ &< \frac{1}{m} + \frac{\varepsilon}{3} + \frac{1}{n} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

and hence $(x_n) \in \mathcal{C}$. Let $\xi = [(x_n)]$. We then have $\lim(x_n) = \xi$ and, since $\lim(\xi_n - x_n) = 0$, we get $\lim(\xi_n) = \xi$. □

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