APPENDIX

Cauchy's Formula for C^∞ Functions

Let D be an open disc in the complex numbers, and let D^c be the closed disc, so the boundary of D^c is a circle. Cauchy's formula gives us the value as an integral over the circle C:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}$$

if f is holomorphic on D^c, that is on some open set containing the closed disc. But what happens if f is not holomorphic but merely smooth, say its real and imaginary parts are infinitely differentiable in the real sense? There is also a formula, which unfortunately is not usually taught in basic courses, although it gives a beautiful application of several notions which arise in both real and complex analysis, and advanced calculus. We shall give this theorem here, together with an application, which occurs in the theory of partial differential equations.

Let us write $z = x + iy$. We introduce two new derivatives. Let

$$f(z) = f_1(x, y) + if_2(x, y),$$

where $f_1 = \operatorname{Re} f$ and $f_2 = \operatorname{Im} f$ are the real and imaginary parts of f respectively. We say that f is C^∞ if f_1, f_2 are infinitely differentiable in the naive sense of functions of two real variables x and y. In other words, all partial derivatives of all orders exist and are continuous. We write $f \in C^\infty(D^c)$ to mean that f is C^∞ on some open set containing D^c.

For such f we define

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Symbolically, we put

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The **Cauchy–Riemann equations** can be formulated neatly by saying that f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

See Chapter VIII, §1.

We shall need the Stokes–Green formula for a simple type of region. In advanced calculus, one integrates expressions

$$\int_C P\,dx + Q\,dy,$$

where P, Q are C^∞ functions, and C is some curve. The Stokes–Green theorem relates such integrals over a boundary to a double integral

$$\iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

taken over a region B which is bounded by the curve C. The precise statement is this.

Stokes–Green Formula. *Let B be a region of the plane, bounded by a finite number of curves, oriented so that the region lies to the left of each curve. Let γ be the boundary, so oriented. Let P, Q have continuous first partial derivatives on B and its boundary. Then*

$$\int_\gamma P\,dx + Q\,dy = \iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

It is useful to express the Stokes–Green formula in terms of the derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$. Writing

$$dz = dx + i\,dy \quad \text{and} \quad d\bar{z} = dx - i\,dy,$$

we can solve for dx and dy in terms of dz and $d\bar{z}$, to give

$$dx = \tfrac{1}{2}(dz + d\bar{z}) \quad \text{and} \quad dy = \frac{1}{2i}(dz - d\bar{z}).$$

Then

$$P\,dx + Q\,dy = g\,dz + h\,d\bar{z},$$

where g, h are suitable functions. Let us write symbolically

$$dz \wedge d\bar{z} = -2i\,dx\,dy.$$

Then by substitution, we find the following version of the **Stokes–Green Formula**:

$$\boxed{\int_\gamma g\,dz + h\,d\bar{z} = \iint_B \left(\frac{\partial h}{\partial z} - \frac{\partial g}{\partial \bar{z}}\right) dz \wedge d\bar{z}.}$$

Remark. Directly from the definition of $\partial/\partial z$ and $\partial/\partial \bar{z}$ one verifies that the usual expression for df is given by

$$\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Consider the special case where $B = B(a)$ is the region obtained from the disc D^c by deleting a small disc of radius a centered at a point z_0, as shown on the figure.

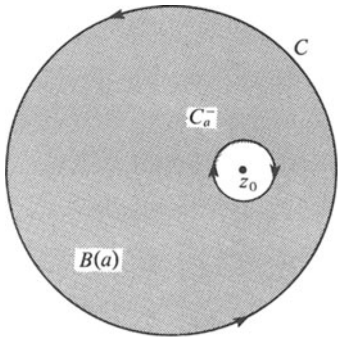

Figure 1

Then the boundary consists of two circles C and C_a^-, oriented as shown so that the region lies to the left of each curve. We have written C_a^- to indicate the circle with clockwise orientation, so that the region $B(a)$ lies to the left of C_a^-. As before, C is the circle around D, oriented counterclockwise. Then the boundary of $B(a)$ can be written

$$\gamma = C + C_a^-.$$

We shall deal with integrals

$$\iint_D \varphi(z) \frac{dz \wedge d\bar{z}}{z - z_0},$$

where $\varphi(z)$ is a smooth function, and where z_0 is some point in the interior of the disc. Such an integral is an improper integral, and is supposed to be interpreted as a limit

$$\lim_{a \to 0} \iint_{B(a)} \varphi(z) \frac{dz \wedge d\bar{z}}{z - z_0},$$

where $B(a)$ is the complement of a disc of radius a centered at z_0. The limit exists, as one sees by using polar coordinates. Letting $z = z_0 + re^{i\theta}$ with polar coordinates around the fixed point z_0, we have

$$dx\, dy = r\, dr\, d\theta$$

and $z - z_0 = re^{i\theta}$, so r cancels and we see that the limit exists, since the integral becomes simply

$$\iint_D \varphi(z)\, dr\, d\theta.$$

The region $B(a)$ is precisely of the type where we apply the Stokes–Green formula.

Cauchy's Theorem for C^∞ functions. *Let $f \in C^\infty(D^c)$ and let z_0 be a point in the interior D. Let C be the circle around D. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{z - z_0} + \frac{1}{2\pi i} \iint_D \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0}.$$

Proof. Let a be a small positive number, and let $B(a)$ be the region obtained by deleting from D a disc of radius a centered at z_0. Then $\partial f/\partial \bar{z}$ is C^∞ on $B(a)$ and we can apply the Stokes–Green formula to

$$\frac{f(z)\, dz}{z - z_0} = g(z)\, dz$$

over this region. Note that this expression has no term with $d\bar{z}$. Furthermore

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{z - z_0} \right) = 0 \qquad \text{and} \qquad \frac{\partial g}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}} \frac{1}{z - z_0}$$

because $1/(z - z_0)$ is holomorphic in this region. Then by Stokes–Green we find

$$\int_{C_a^-} \frac{f(z)\,dz}{z - z_0} + \int_C \frac{f(z)\,dz}{z - z_0} = -\iint_{B(a)} \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0}.$$

The limit of the integral on the right-hand side as a approaches 0 is the double integral (with a minus sign) which occurs in Cauchy's formula. We now determine the limit of the curve integral over C_a^- on the left-hand side. We parametrize C_a (counterclockwise orientation) by

$$z = z_0 + ae^{i\theta}, \qquad 0 \leq \theta \leq 2\pi.$$

Then $dz = aie^{i\theta}\,dz$, so

$$\int_{C_a^-} \frac{f(z)\,dz}{z - z_0} = -\int_{C_a} \frac{f(z)\,dz}{z - z_0} = -\int_0^{2\pi} f(z_0 + ae^{i\theta})i\,d\theta.$$

Since f is continuous at z_0, we can write

$$f(z_0 + ae^{i\theta}) = f(z_0) + h(a, \theta)$$

where $h(a, \theta)$ is a function such that

$$\lim_{a \to 0} h(a, \theta) = 0$$

uniformly in θ. Therefore

$$\begin{aligned} \lim_{a \to 0} \int_{C_a^-} \frac{f(z)\,dz}{z - z_0} &= -2\pi i f(z_0) - \lim_{a \to 0} \int_0^{2\pi} h(a, \theta)i\,d\theta \\ &= -2\pi i f(z_0). \end{aligned}$$

Cauchy's formula now follows at once.

Remark 1. Suppose that f is holomorphic on D. Then

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

and so the double integral disappears from the general formula to give the Cauchy formula as we encountered it previously.

Remark 2. Consider the special case when the function f is 0 on the boundary of the disc. Then the integral over the circle C is equal to 0, and we obtain the formula

$$f(z_0) = \frac{1}{2\pi i} \iint_D \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0}.$$

This allows us to recover the values of the function from its derivative $\partial f/\partial \bar{z}$. Conversely, one has the following result.

Theorem. *Let $g \in C^\infty(D^c)$ be a C^∞ function on the closed disc. Then the function*

$$f(w) = \frac{1}{2\pi i} \iint_D \frac{g(z)}{z - w} dz \wedge d\bar{z}$$

is defined and C^∞ on D, and satisfies

$$\frac{\partial f}{\partial \bar{w}} = g(w) \qquad \text{for} \quad w \in D.$$

The proof is essentially a corollary of Cauchy's theorem if one has the appropriate technique for differentiating under the integral sign. However, we have now reached the boundary of this course, and we omit the proof.

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