Appendix

A.1. Ergodic decompositions.

In this section we suppose that M is a Borel subset of a Polish space and $P(x,\cdot)$ is a family of transition probabilities of a Markov chain X_n on M i.e. $P(x,\cdot)$ is a Borel probability measure for each fixed $x \in M$ and $P(x,\Gamma)$ is a Borel function of $x \in M$ for any given Γ from the Borel σ- field $\mathcal{B}(M)$. Next we define the transition operator P and its adjoint P^* in the same way as in (I.2.8) and (I.2.9). Again, a measure $\eta \in \mathcal{P}(M)$ is called P^*-invariant if $P^*\eta = \eta$. Furthermore we shall say that a Borel subset $A \subset M$ is P-invariant if

$$P\chi_A \geq \chi_A \tag{1.1}$$

i.e. $P(x,A) = 1$ provided $x \in A$. To connect this definition with the notion of (P,ρ)-invariant sets introduced in Section 1.2 we shall prove

Lemma 1.1.

(i) *If A is P-invariant then A is (P,ρ)-invariant for any P^*-invariant measure $\rho \in \mathcal{P}(M)$;*

(ii) *Suppose that $\rho \in \mathcal{P}(M)$ is P^*-invariant and B is a (P,ρ)-invariant subset of M. Then there exists a P- invariant subset $\tilde{B} \subset M$ such that $\rho(B \,\Delta\, \tilde{B}) = 0$ where Δ denotes the symmetric difference.*

Proof. If ρ is P-invariant then $\int P\chi_A \, d\rho = \int \chi_A d\rho = \rho(A)$ which together with (1.1) implies (i). Next, let $B \subset M$ be (P,ρ)-

invariant i.e.

$$P\chi_B = \chi_B \quad \rho\text{-}a.s. \tag{1.2}$$

Then there is a Borel set $C \supset B$ such that $\rho(C \setminus B) = 0$. Since ρ is P^*-invariant then by (1.2),

$$\rho(B) = \rho(C) = \int_M P(x,C)d\rho(x) \geq \int_M P(x,B)d\rho(x) \geq$$

$$\geq \int_B P(x,B)d\rho(x) = \rho(B).$$

Hence

$$P(x,C) = P(x,B) \qquad \rho\text{-}a.s. \tag{1.3}$$

and so

$$P\chi_C = \chi_C \qquad \rho\text{-}a.s. \tag{1.4}$$

Now define inductively $C_0 = C$ and $C_{i+1} = \{x \in C_i : P(x,C_i) = 1\}$, $i = 1,2, \cdots$. Since $P(x,C_i)$ are Borel functions of x then all C_i are Borel sets. Besides, $C_0 \supset C_1 \supset C_2 \supset \cdots$ and by (1.4), $\rho(C_0 \setminus C_1) = 0$. But then

$$\rho(C_1) = \int_M P(x,C_1)d\rho(x) = \int_{C_1} P(x,C_1)d\rho(x).$$

This together with (1.4) give

$$P\chi_{C_1} = \chi_{C_1} \quad \rho\text{-}a.s.$$

Repeating this argument we obtain that

$$P\chi_{C_i} = \chi_{C_i} \quad \rho\text{-}a.s \quad and \ \rho(C_i \setminus C_{i+1}) = 0 \quad \text{for } all \quad i = 0,1,2, \cdot \quad (1.5)$$

Finally, $\widetilde{B} \equiv \bigcap_{i\geq 0} C_i$ satisfies the conditions of (ii). Indeed, if $x \in \widetilde{B}$ then $P(x,C_i) = 1$ for all $i = 0,1,2,\cdots$. Since $P(x,\cdot)$ is a measure then also $P(x,\widetilde{B}) = 1$ and so $P\chi_{\widetilde{B}} \geq \chi_{\widetilde{B}}$. Besides, (1.5) implies $\rho(C_0 \setminus \widetilde{B}) = 0$ which concludes the proof. ▪

Remark 1.1. The collection $\mathcal{A}$ of all P-invariant sets does not form a σ-field since not every $A \in \mathcal{A}$ has the complement belonging to $\mathcal{A}$. Still, by Lemma 1.1 given any P^*-invariant measure η the completion $\mathcal{A}_\eta$ of $\mathcal{A}$ coincides with the family of (P,η)-invariant set and so it forms a σ-field. On the other hand if we add to $\mathcal{A}$ complements of all sets then the new collection $\widetilde{\mathcal{A}}$ will be already a σ-field.

Next, we shall call a P^*-invariant measure ρ ergodic if $\rho(A) = 0$ or $= 1$ for any $A \in \mathcal{A}$. By Lemma 1.1 it is easy to see that this definition coincides with the definition given in Section 1. 2. Let $\mathcal{M}$ be the space of all P^*-invariant probability measures. We shall introduce a measurable structure on $\mathcal{M}$ by saying that any function $G(\eta) = \int g d\eta$ on $\mathcal{M}$ is measurable provided g is a function on M measurable with respect to the completions of the Borel σ-field for any P^*-invariant probability measure. The main result of this section is the following (cf. Rohlin [40] for the deterministic dynamical systems).

Theorem 1.1. *The set $\mathcal{M}_e$ of all ergodic measures is a measurable subset of $\mathcal{M}$ and each measure η from $\mathcal{M}$ can be uniquely represented as an integral*

$$\eta = \int_{\mathcal{M}_e} \rho d\nu_\eta \tag{1.6}$$

i.e.

$$\eta(\Gamma) = \int_{\mathcal{M}_e} \rho(\Gamma) d\nu_\eta(\rho) \qquad (1.7)$$

for any Borel $\Gamma \subset M$ *where* ν_η *is a probability measure on* $\mathcal{M}$ *concentrated on* $\mathcal{M}_e$.

The proof of this theorem proceeds in the same way as in Kifer and Pirogov [24] by constracting certain conditional probabilities. We shall start with

Lemma 1.2. (Dynkin [14]) *There exists a function q on M such that the family of functions* $W_q = \{1, q, q^2, \cdots\}$ *has the following properties:*

(i) Suppose that a set of functions $\mathcal{H}$ *contains* W_q *and satisfies the conditions:*

a) if $g_1, g_2 \in \mathcal{H}$ *then for any numbers* c_1 *and* c_2, $c_1 g_1 + c_2 g_2 \in \mathcal{H}$;

b) if a sequence $f_n \in \mathcal{H}$ *is uniformly bounded and pointwise converges to g then* $g \in \mathcal{H}$;

Then $\mathcal{H}$ *contains all bounded Borel functions.*

(ii) W_q *separates probability measures on M i.e. for any two different measures* $\eta_1, \eta_2 \in \mathcal{P}(M)$ *there exists an integer k such that* $\eta_1(q^k) \neq \eta_2(q^k)$.

(ii) If for $\eta_n \in \mathcal{P}(M)$ *the sequence* $\eta_n(g)$ *converges for each* $g \in W_q$ *then there exists a probability measure* η *such that* $\eta_n(g) \to \eta(g)$ *when* $g \in W_q$.

Proof. According to §36–37 of Kuratowski [31] any Borel subset M of a Polish space is Borel measurably isomorphic to a closed subset of the interval $\mathbb{I} = \{x : 0 \leq x \leq 1\}$ considered with its Borel σ-field $\mathcal{B}(\mathbb{I})$. This isomorphism is given by a function

$q : M \to \mathbb{I}$. We shall introduce convergences of points by $x_n \overset{q}{\to} x$ if $q(x_n) \to q(x)$, and measures by $\mu_n \overset{q}{\to} \mu$ if $\int q^k d\mu_n \to \int q^k d\mu$ for all $k = 0,1,2, \cdots$.

With respect to this q-topology the spaces M and $\mathcal{P}(M)$ are compact since if we shall identify x with $q(x)$ then M is transformed into a compact subset of $\mathbb{I}$ and $\mathcal{P}(M)$ is transformed into a compact subset of $\mathcal{P}(\mathbb{I})$ considered with the topology of weak convergence.

To prove that $W_q = \{1, q, q^2, \cdots\}$ satisfies (i) notice that if $\mathcal{H}$ satisfies a) then $\mathcal{H}$ must contain all polinoms and so by b) $\mathcal{H}$ contains all bounded Borel functions.

If $\eta_1, \eta_2 \in \mathcal{P}(M)$ and $\int q^k d\eta_1 = \int q^k d\eta_2$ for all $k = 0,1,2, \cdots$ then the set $\mathcal{H}$ of bounded Borel functions such that $\int g d\mu_1 = \int g d\mu_2$ satisfies a) and b) and so by (i) it contains all bounded Borel functions, proving (ii).

To prove (iii) remark that any measure η which is a limit point of a sequence η_n in q-topology satisfies

$$\int q^k d\eta = \lim_{n \to \infty} \int q^k d\eta$$

for all $k = 0,1,2, \cdots$ and so by (ii) it follows that this sequence has a unique limit point i.e. it converges, proving (iii). ▪

Proof of Theorem 1.1. By a partial case of the Chacon-Ornstein theorem due to E. Hopf (see Neveu [37], Proposition V.6.3 or Rosenblatt [41], Corollary 2 of Section 2 in Ch. IV) if $\int |g| d\eta < \infty$ then η-almost surely the limit

$$\hat{g}(x) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (P^k g)(x) = E_\eta(g \mid \mathcal{A}_\eta) \tag{1.8}$$

exists where $\eta \in \mathcal{P}(M)$ is P^*-invariant, $\mathcal{A}_\eta$ was defined in Remark 1.1, $E_\eta(g \mid \mathcal{A}_\eta)$ denotes the conditional expectation on the probability space $(M, \mathcal{B}_\eta, \eta)$ with respect to the σ-field $\mathcal{A}_\eta$ and $\mathcal{B}_\eta$ is the completion of the Borel σ-field with respect to η. This means that $E_\eta(g \mid \mathcal{A}_\eta)$ is an $\mathcal{A}_\eta$-measurable function satisfying

$$\int_A E_\eta(g \mid \mathcal{A}_\eta) d\eta = \int_A g \; d\eta \tag{1.9}$$

for any $A \in \mathcal{A}_\eta$.

Let $\tilde{M}$ be the set of those x for which the limit (1.8) exists for all functions from W_q constructed in Lemma 1.2. Then it follows that $\tilde{M} \in \mathcal{A}_\eta$ and $\eta(\tilde{M}) = 1$ for any P^*-invariant $\eta \in \mathcal{P}(M)$. If δ_x denotes the unit mass at x then we can write

$$\hat{g}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int g d(P^*)^k \delta_x$$

provided $g \in W_q$ and $x \in \tilde{M}$. Hence by (ii) and (iii) of Lemma 1.2 there exists a unique measure $\rho^x \in \mathcal{P}(M)$ such that if $x \in \tilde{M}$ then

$$\hat{g}(x) = \int g d\rho^x \quad \text{for } any \quad g \in W_q. \tag{1.10}$$

Therefore for any $\mathcal{A}_\eta$-measurable function h (i.e. all sets $\{x : h(x) < a\}$ belong to $\mathcal{A}_\eta$) one has

$$\int gh \; d\eta = \int (\int g \; d\rho^x) h(x) d\eta(x) \tag{1.11}$$

provided $\eta \in \mathcal{P}(M)$ is P^*-invariant. From (i) of Lemma 1.2 it follows that (1.11) holds for any bounded Borel function g. Since

$\hat{g}$ is $\mathcal{A}_\eta$-measurable then $\int g d\rho^x$ as a function of x is $\mathcal{A}_\eta$-measurable for each $g \in W_q$ and so by (i) of Lemma 1.2 it is $\mathcal{A}_\eta$-measurable for any bounded Borel function g.

This together with (1.11) give

$$\int g \; d\rho^x = E_\eta(g \mid \mathcal{A}_\eta)(x) \qquad \eta\text{-}a.s. \tag{1.12}$$

It follows from (1.8) and (1.10) that for any $g \in W_q$ and $x \in \widetilde{M}$,

$$\int g \; d\rho^x = \int Pg d\rho^x = \int g dP^* \rho^x$$

and so by (ii) of Lemma 1.2,

$$P^* \rho^x = \rho^x \quad \text{for} \quad all \quad x \in \widetilde{M}. \tag{1.13}$$

Next, we shall proceed in the same way as in Dynkin [14], Theorem 2.1. Let $\eta \in \mathcal{M}_e$ then for any bounded Borel function g,

$$E_\eta(g \mid \mathcal{A}_\eta)(x) = \int g \; d\eta \quad \eta\text{-}a.s.$$

and so by (1.12),

$$\int g \; d\rho^x = \int g \; d\eta \quad \eta\text{-}a.s. \tag{1.14}$$

This is true, in particular, for all $g \in W_q$ which together with (ii) of Lemma 1.2 imply that

$$\eta\{x : \rho^x = \eta\} = 1 \tag{1.15}$$

i.e. in this case η coincides with one of the measures ρ^x. In other words

$$\mathcal{M}_e \subset \{\rho^x,\ x \in \widetilde{M}\}. \tag{1.16}$$

On the other hand if (1.15) is true then by Lemma 1.1 for any P-invariant set A

$$\eta(A) = \rho^x(A) = E_\eta(\chi_A \mid \mathcal{A}_\eta)(x) = \chi_A(x) \quad \eta\text{-}a.s. \tag{1.17}$$

and so η is ergodic. Thus (1.15) is equivalent to the ergodicity of η. But $\{x : \rho^x = \eta\} = \{x : \int g d\rho^x = \int g d\eta$ for all $g \in W_q\}$. Hence we can say that η is ergodic if and only if

$$\int g d\rho^x = \int g\ d\rho \quad \text{for} \quad all \quad g \in W_q \quad \eta\text{-}a.s. \tag{1.18}$$

This is equivalent to

$$\Phi_g(\eta) = \int_{\widetilde{M}} \left(\int g d\rho^x - \int g d\eta\right)^2 d\eta(x) = 0 \tag{1.19}$$

for all $g \in W_q$ which implies that $\mathcal{M}_e$ is a measurable subset of $\mathcal{M}$.

Next we are going to show that the measures ρ^x are ergodic η-a.s. for any P^*-invariant measure η. Indeed, by (1.12),

$$0 \le \Phi_g(\rho^x) = \int \left(\int g d\rho^y\right)^2 d\rho^x(y) - \left(\int g d\rho^x\right)^2. \tag{1.20}$$

Taking an integral in (1.20) with respect to a P^*-invariant measure η we shall obtain in view of (1.12) that

$$\int \Phi_g(\rho^x) d\eta(x) = 0.$$

Since $\Phi_g(\rho^x) \ge 0$ it follows that

$$\Phi_g(\rho^x) = 0 \quad \eta\text{-}a.s. \tag{1.21}$$

which implies, as we have seen it above, the ergodicity of those ρ^x which satisfy (1.21).

Now by (1.12),

$$\eta = \int \rho^x d\eta(x) \tag{1.22}$$

and putting $\nu_\eta(G) = \eta\{x : \rho^x \in G\}$ one obtains the desired representation (1.6). To get the uniqueness notice that for any measurable subset G of $\mathcal{M}_e$,

$$\eta\{x : \rho^x \in G\} = \int_{\mathcal{M}_e} \rho\{x : \rho^x \in G\} d\nu_\eta(\rho)$$

$$= \int_{\mathcal{M}_e} \chi_G(\rho) d\nu_\eta(\rho) = \nu_\eta(G)$$

since $\rho\{x : \rho^x \in G\} = \chi_G(\rho)$ provided $\rho \in \mathcal{M}_e$. This completes the proof of Theorem 1.1. ▪

Remark 1.1. The map $\varphi : M \to \mathcal{M}_e$ acting by $\varphi(x) = \rho^x$ determines also a measurable partition (see Rohlin [40]) of M into pre-images $\varphi^{-1}(\rho)$ which are called ergodic components.

Remark 1.2. In the circumstances of Section 1.2 we may need an ergodic decomposition of $\eta \times \mathbf{p}$. But if η has an ergodic decomposition $\eta = \int \rho d\nu(\rho)$ then $\eta \times \mathbf{p}$ has the ergodic decomposition $\eta \times \mathbf{p} = \int \rho \times \mathbf{p} d\nu(\rho)$.

A.2. Subadditive ergodic theorem.

We shall prove in this section Kingman's subadditive ergodic theorem [29] under the circumstances when an ergodic decomposition exists which is the case of the main interest in this book. A proof for the general case the reader can find in the original Kingman's paper [29]. A shorter proof was given by Derriennic [13] (see also Appendix A of Ruelle [43]).

We shall start with the ergodic case where we shall follow Ledrappier [33]. Suppose that f is a measure-preserving transformation of a probability space $(M,\mathcal{E},\mu)$ i.e. $\mu(f^{-1}E) = \mu(E)$ for any $E \in \mathcal{E}$, where $\mathcal{E}$ is a σ-field of measurable subsets of M and $\mu(M) = 1$. An f-invariant measure μ is called ergodic if any f-invariant set $A \in \mathcal{E}$, i.e. $f^{-1}A = A$, satisfies $\mu(A) = 0$ or 1.

Theorem 2.1. *Let a sequence of functions $g_1, g_2, \cdots$ satisfies $g_1^+ \in \mathbb{L}^1(M,\mu)$ and $g_{n+m} \le g_n + g_m \circ f^n$. If μ is ergodic then μ-a.s. there exists a limit*

$$\lim_{n\to\infty} \frac{1}{n} g_n = c \equiv \inf_n \frac{1}{n} \int g_n \, d\mu. \tag{2.1}$$

Proof. Notice, first that the sequence $\int g_n d\mu$ is subadditive and so, by the well known argument which we have demonstrated already in the proof of Theorem II.1.1,

$$\lim_{n\to\infty} \frac{1}{n} \int g_n d\mu = c = \inf_n \frac{1}{n} \int g_n \, d\mu. \tag{2.2}$$

Introduce

$$\bar{g}(x) = \limsup_{n\to\infty} \frac{1}{n} g_n(x) \quad \text{and} \quad \underline{g}(x) = \liminf_{n\to\infty} \frac{1}{n} g_n(x).$$

Then $\bar{g}$ and $\underline{g}$ are f-invariant. Indeed, in view of the subadditivity

$$\bar{g} \circ f = \limsup_n \frac{1}{n}(g_n \circ f)$$

$$\geq \limsup_n \frac{1}{n}(g_{n+1} - g_1) = \bar{g}$$

and similarly, $\underline{g} \circ f \geq \underline{g}$ μ-a.s. Since

$$\int \bar{g} \circ f d\mu = \int \bar{g} d\mu \quad and \quad \int \underline{g} \circ f d\mu = \int \underline{g}\, d\mu$$

the f-invariance of $\bar{g}$ and $\underline{g}$ follows. But μ - is ergodic and so $\bar{g}$ and $\underline{g}$ are μ-a.s. constants.

Next, we shall show that

$$\underline{g} \geq c. \tag{2.3}$$

Indeed, assume that $\underline{g} \geq -\infty$ and take an arbitrary $\varepsilon > 0$. One can choose a measurable function $n(x)$ such that μ-a.s.

$$g_{n(x)} \leq n(x)(\underline{g} + \varepsilon). \tag{2.4}$$

For $N > 1$ put $A_N = \{n(x) \geq N\} \cap \{x :$ (2.4) is not true$\}$ and define

$$\tilde{g}(x) = \begin{cases} \underline{g}(x) & \text{if } x \in M \setminus A_N \\ \max(\underline{g}, g_1(x)) & \text{if } x \in A_N \end{cases}$$

and

$$\tilde{n}(x) = \begin{cases} n(x) & \text{if } x \in M \setminus A_N \\ 1 & \text{if } x \in A_N \end{cases}.$$

Then by the subadditivity condition,

$$g_{\tilde{n}(x)}(x) \le \sum_{i=0}^{\tilde{n}(x)-1} (\tilde{g}+\varepsilon)(f^i x) \quad \mu\text{-a.s.} \tag{2.5}$$

Define by induction

$$n_j(x) = n_{j-1}(x) + \tilde{n}(f^{n_{j-1}(x)}x)$$

where $n_0(x) = 0$ and $n_1(x) = \tilde{n}(x)$. For any integer $P > N$ put

$$\mathcal{J}_P(x) = \inf\{j : n_j(x) \ge P-N\}.$$

By the subadditivity,

$$g_P(x) \le \sum_{0 \le j \le n_{\mathcal{J}_P(x)-1}} g_{\tilde{n}(f^{n_j(x)}x)}(f^{n_j(x)}x) \tag{2.6}$$

$$+ \sum_{n_{\mathcal{J}_P(x)}(x) \le j \le P-1} g_1(f^j x).$$

Summing (2.5) taken at each $f^{n_j(x)}x$ we have

$$g_P(x) \le \sum_{0 \le j \le n_{\mathcal{J}_P(x)}(x)-1} (\tilde{g}+\varepsilon)(f^i x) \tag{2.7}$$

$$= \sum_{n_{\mathcal{J}_P(x)}(x) \le j \le P-1} g_1(f^j(x)).$$

Since $n_{\mathcal{J}_P(x)}(x) \ge P-N$ then (2.7) implies

$$g_P(x) \leq \sum_{0 \leq j \leq P-N-1} (\tilde{g}+\varepsilon)(f^i x) \tag{2.8}$$

$$\sum_{P-N \leq j \leq P-1} (\tilde{g}^+ + g_1^+ + \varepsilon)(f^i x) \quad \mu\text{-}a.s.$$

Therefore

$$c \leq \frac{1}{P} \int g_P \, d\mu \leq \frac{P-N}{P} \int (\tilde{g}+\varepsilon) d\mu \tag{2.9}$$

$$+ \frac{N}{P} \int (g^+ + g_1^+ + \varepsilon) d\mu.$$

Letting $P \to \infty$, then $N \to \infty$ and, finally, $\varepsilon \to 0$ we derive (2.3) provided $\underline{g} > -\infty$. If $\underline{g} = -\infty$ then for each K one can find a measurable integer-valued $k(x) > 0$ such that

$$g_{k(x)} \leq k(x) \, K. \tag{2.10}$$

The same proof as above with (2.10) in place of (2.4) enables us to show that $c \leq K$. This is true for any K and so $c = -\infty \leq \underline{g}$ proving (2.3) completely.

We shall need also the following inequality

$$\bar{g} \leq \int g_1 d\mu. \tag{2.11}$$

The proof of (2.11) is similar to the arguments above. Suppose that $\bar{g} < \infty$ and take an arbitrary $\varepsilon > 0$. One can find a measurable function $n(x)$ such that μ-a.s.

$$n(x)\bar{g} \leq g_{n(x)} + \varepsilon n(x). \tag{2.12}$$

By the subadditivity,

$$n(x)\bar{g} \leq \sum_{i=0}^{n(x)-1} (g_1+\varepsilon)(f^i x). \tag{2.13}$$

Put, again $A_N = \{n(x) \geq N\} \cap \{x : \text{ (2.13) is not true}\}$ and

$$\tilde{g}(x) = \begin{cases} g_1(x) & \text{if } x \in M \setminus A_N \\ \max(g_1(x), \bar{g}) & \text{if } x \in A_N \end{cases}$$

and

$$\tilde{n}(x) = \begin{cases} n(x) & \text{if } x \in M \setminus A_N \\ 1 & \text{if } x \in A_N \end{cases}$$

Then

$$\tilde{n}(x)\bar{g} \leq \sum_{0 \leq i \leq n(x)-1} (\tilde{g} + \varepsilon)(f^i x). \tag{2.14}$$

In the same way as in the proof of (2.8) we can derive from (2.14) for any integer $P > N$ that

$$P\bar{g} \leq \sum_{0 \leq j \leq P-N-1} (\tilde{g} + \varepsilon)(f^j x) \tag{2.15}$$

$$+ \sum_{P-N \leq j \leq P-1} (\bar{g}^+ + \tilde{g}^+ + \varepsilon)(f^j x).$$

Letting $P \to \infty$ one obtains $\bar{g} \leq \int \tilde{g} d\mu + \varepsilon$. When $N \to \infty$, $\int \tilde{g} d\mu$ tends to $\int g_1 d\mu$ which gives (2.11) after taking $\varepsilon \to 0$ for the case $\bar{g} < \infty$. When $\bar{g} = \infty$ then

$$k(x)K \leq g_{k(x)}(x) + \varepsilon k(x) \tag{2.16}$$

for some $K > \int g_1 d\mu$ and a measurable function $k(x)$. The same proof as above gives $K \leq \int g_1 d\mu$ for any K and so $\int g_1 d\mu = \infty$ which is impossible.

Now we can assert that for any integer $j > 0$,

$$\bar{g} \le \frac{1}{j} \int g_j \, d\mu \tag{2.17}$$

which together with (2.3) proves (2.1). Indeed, if (2.17) is true then

$$\bar{g} \le \lim_{j \to \infty} \frac{1}{j} \int g_j \, d\mu = c. \tag{2.18}$$

Since $\bar{g} \ge \underline{g}$ then by (2.3) and (2.18),

$$\bar{g} = \underline{g} = c \tag{2.19}$$

proving (2.1).

It remains to establish (2.17). Put $\bar{g}_j = \mathit{limsup}_n \frac{1}{n} g_{jn}$. It is easy to see in the same way as at the beginning of the proof that $\bar{g}_j$ is a constant μ-a.s. Moreover

$$\bar{g}_j = j \, \bar{g}. \tag{2.20}$$

Indeed, $\bar{g}_j \le j \, \bar{g}$ since in the definition of $\bar{g}_j$ the *limsup* is taken along a subsequence. On the other hand, by the subadditivity

$$g_n \le g_{kj} + \sum_{0 \le i \le j-1} g_1 \circ f^{kj+i} \tag{2.21}$$

where k is the integral part of n.

Notice that

$$\limsup_{\ell} \frac{1}{\ell} g_1 \circ f^{\ell} \leq 0 \quad \mu\text{-}a.s. \tag{2.22}$$

Indeed,

$$\sum_{\ell > 0} \mu\{x : \frac{1}{\ell} g_1 \circ f^{\ell} \geq \delta\} \tag{2.23}$$

$$= \sum_{\ell > 0} \mu\{x : g_1 \geq \ell\delta\} \leq \frac{1}{\delta} \int g_1^{+} d\mu$$

since μ is f-invariant. Thus by the Borel-Cantelli lemma (see, for instance, Neveu [37]) the left hand side of (2.20) is less or equal to δ μ-a.s. Since δ is arbitrary we obtain (2.22).

Now (2.22) applied to (2.21) gives $\bar{g} \leq \frac{1}{j} \bar{g}_j$ which together with the inequality in the opposite direction proved earlier give (2.20). Next, we can use (2.11) with $\bar{g}_j$ and f^j in place of $\bar{g}$ and f, respectively, to obtain $\bar{g}_j \leq \int g_j d\mu$ which together with (2.20) gives (2.17). As we have explained it above this completes the proof of Theorem 2.1. ▪

Next, we shall consider a non-ergodic case.

Corollary 2.1. *Let in the conditions of Theorem 2.1 a measure μ is not necessarily ergodic but it can be represented as an integral*

$$\mu = \int \rho d\nu(\rho) \tag{2.24}$$

over the space of f-invariant ergodic measures. Then

$$\hat{g} = \lim_{n \to \infty} \frac{1}{n} g_n \quad \mu\text{-}a.s. \tag{2.25}$$

exist and $\hat{g} \circ f = \hat{g}$, μ-*a.s.*

Proof. Let $\widetilde{M} = \{x :$ the limit (2.25) does not exist$\}$. Then by Theorem 2.1 $\rho(\widetilde{M}) = 0$ for any ergodic ρ and so by (2.24), $\mu(\widetilde{M}) = 0$. This means that the limit $\hat{g}$ exists μ-a.s. Consider $h = \hat{g} - \hat{g} \circ f$. As we have seen it at the beginning of the proof of Theorem 2.1 $\hat{g}$ is f-invariant ρ-a.s. with respect to any P^*-invariant ergodic ρ and so $h = 0$ ρ-a.s. Let $\approx{M} = \{x : h \neq 0\}$ then by (2.24), $\mu(\approx{M}) = \int \rho(\approx{M}) d\nu(\rho) = 0$ and so $h = 0$ μ-a.s. Hence $\hat{g} \circ f = \hat{g}$ μ-a.s. concluding the proof. ▪

Remark 2.1. To be sure that an ergodic decomposition exists we can employ Theorem 1.1 of Appendix for the case when $\mathbf{P}(x, \cdot) = \delta_{fx}$ where δ_y is the Dirac measure at y. The conditions of Theorem 1.1 will be satisfied if the space under consideration is a Borel subset of a Polish space.

Remark 2.2. According to Remark 1.2, in order to apply Corollary 2.1 to the transformation τ from Section 1.2 we only have to be sure that a certain P^*-invariant measure η on $\mathbf{M}$ has an ergodic decomposition. If $\mathbf{M}$ is a Borel subset of a Polish space then by Theorem 1.1 this will be the case .

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