

Vector Identities

The following table summarizes some general formulas that are useful for calculations with vector fields in \mathbb{R}^3 . Some expressions in this table require explanation. In identity **7**, $\mathbf{V} = (\mathbf{F} \cdot \nabla)\mathbf{G}$ has components $V_i = \mathbf{F} \cdot (\nabla \mathbf{G}_i)$, for $i = 1, 2, 3$, where $\mathbf{G} = (G_1, G_2, G_3)$. In identity **13**, the vector field $\nabla^2 \mathbf{F}$ has components $\nabla^2 F_i$, where $\mathbf{F} = (F_1, F_2, F_3)$. In identity **20**, $(\mathbf{F} \times \nabla) \times \mathbf{G}$ means ∇ is to operate only on \mathbf{G} in the following way: To calculate $(\mathbf{F} \times \nabla) \times \mathbf{G}$, we define $(\mathbf{F} \times \nabla) \times \mathbf{G} = \mathbf{U} \times \mathbf{G}$ where we define $\mathbf{U} = \mathbf{F} \times \nabla$ by:

$$\mathbf{U} = \mathbf{F} \times \nabla = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_1 & F_2 & F_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}.$$

- 1** $\nabla(f + g) = \nabla f + \nabla g$
- 2** $\nabla(cf) = c\nabla f$, for a constant c
- 3** $\nabla(fg) = f\nabla g + g\nabla f$
- 4** $\nabla\left(\frac{f}{g}\right) = \frac{(g\nabla f - f\nabla g)}{g^2}$
- 5** $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$
- 6** $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$
- 7** $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \operatorname{curl} \mathbf{G} + \mathbf{G} \times \operatorname{curl} \mathbf{F}$
- 8** $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla f$
- 9** $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$
- 10** $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$
- 11** $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + \nabla f \times \mathbf{F}$
- 12** $\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$
- 13** $\operatorname{curl} \operatorname{curl} \mathbf{F} = \operatorname{grad} \operatorname{div} \mathbf{F} - \nabla^2 \mathbf{F}$
- 14** $\operatorname{curl} \nabla f = 0$
- 15** $\nabla(\mathbf{F} \cdot \mathbf{F}) = 2(\mathbf{F} \cdot \nabla)\mathbf{F} + 2\mathbf{F} \times (\operatorname{curl} \mathbf{F})$
- 16** $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
- 17** $\operatorname{div}(\nabla f \times \nabla g) = 0$
- 18** $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
- 19** $\mathbf{H} \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\mathbf{H} \times \mathbf{F}) = \mathbf{F} \cdot (\mathbf{G} \times \mathbf{H})$
- 20** $\mathbf{H} \cdot ((\mathbf{F} \times \nabla) \times \mathbf{G}) = ((\mathbf{H} \cdot \nabla)\mathbf{G}) \cdot \mathbf{F} - (\mathbf{H} \cdot \mathbf{F})(\nabla \cdot \mathbf{G})$
- 21** $\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) = (\mathbf{F} \cdot \mathbf{H})\mathbf{G} - \mathbf{H}(\mathbf{F} \cdot \mathbf{G})$

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