

Appendix

Theorem A1. Let $K(z) = k_0 + k_1z + k_2z^2 + \dots$ be a p.g.f. with $k_0 > 0$ and $0 < K'(1) = \alpha$. Then the equation

$$\xi = zK(\xi) \tag{1}$$

has a unique continuous root $\xi \equiv \xi(z)$ in $(0, 1)$ such that $\xi(0+) = 0$. Also, as $z \rightarrow 1-$, $\xi(z)$ converges to the least positive root ζ of the equation $\zeta = K(\zeta)$, and $0 < \zeta < 1$ iff $\alpha > 1$.

PROOF. Consider the function

$$f(x) = \frac{K(x)}{x} = \frac{k_0}{x} + k_1 + k_2x + k_3x^2 + \dots \quad (0 < x < 1).$$

We have $f(0+) = \infty, f(1-) = 1$. Moreover,

$$f''(x) = \frac{2k_0}{x^3} + 2k_3 + 6k_4x + \dots > 0.$$

It follows that $f(x)$ is monotone decreasing and > 1 for $x < \zeta$ where ζ is the least positive root of the equation $f(x) = 1$, and $\zeta < 1$ or $= 1$ depending on whether $f'(1) = \alpha - 1 > 0$ or ≤ 0 . Therefore for a given z in $(0, 1)$ there is a unique x such that $f(x) = z^{-1}$ in the range $(0 < x < \zeta)$ and from (1) it is clear that $x = \xi(z)$. Clearly ξ is a continuous function of z and $\rightarrow 0$ as $z \rightarrow 0+$. Also $z \rightarrow 1-, \xi(z) \rightarrow \zeta$.

Theorem A2. Let $\psi(\theta)$ be the Laplace transform of a distribution ($\theta > 0$) and $\rho = -\lambda\psi'(0)/k$ where $0 < \lambda < \infty$ and k is a positive integer. Then for $0 < z < 1$ the equation

$$\gamma^k = z\psi(\theta + \lambda - \lambda\gamma) \tag{2}$$

has exactly k distinct roots $\gamma_r \equiv \gamma_r(z, \theta)$ with $|\gamma_r| < 1$ ($r = 1, 2, \dots, k$). As $z \rightarrow 1-$ and $\theta \rightarrow 0+$, $\gamma_r(z, \theta) \rightarrow \zeta_r$ where ζ_r are the roots of the equation

$$\zeta^k = z\psi(\lambda - \lambda\zeta) \tag{3}$$

with $|\zeta_r| < 1$ ($r = 1, 2, \dots, k$) if $\rho > 1$, while $|\zeta_r| < 1$ ($r = 1, 2, \dots, k - 1$) and $\zeta_k = 1$ if $\rho \leq 1$.

PROOF. Let $|x| = 1 - \varepsilon$ for ε sufficiently small and positive.

Then $|z\psi(\theta + \lambda - \lambda x)| < (1 - \varepsilon)^k$

and by Rouché's theorem the equation $x^k = z\psi(\theta + \lambda - \lambda x)$ has exactly k roots with $|x| < 1 - \varepsilon$. The remaining results follow from continuity arguments.

Theorem A3. Let $\phi(\theta) = \int_0^\infty (1 - e^{-\theta x})v(dx)$ ($\theta > 0$), where v is a Lévy measure, and $\rho = \int_0^\infty xv(dx)$. Then for $s > 0$ the equation

$$\eta = s + \phi(\eta) \tag{4}$$

has a unique continuous solution $\eta \equiv \eta(s)$ with $\eta(\infty) = \infty$. Furthermore:

- (i) as $s \rightarrow 0+$, $\eta(s) \rightarrow \eta_0$ is the largest positive root of the equation $\eta_0 = \phi(\eta_0)$, and $\eta_0 > 0$ iff $\rho > 1$;
- (ii) $\eta'(0+) = (1 - \rho)^{-1}$ if $\rho < 1$, and $= \infty$ if $\rho = 1$.

PROOF. Consider the function $f(x) = x - \phi(x)$ ($x > 0$). We have $f(0+) = 0$, $f(\infty) = \infty$ and

$$f''(x) = \int_0^\infty y^2 e^{-xy}v(dy) > 0.$$

It follows that $f(x)$ is positive and monotone increasing for $x > \eta_0$, where η_0 is the largest positive root of $f(x) = 0$, and $\eta_0 = 0$ or > 0 depending on whether $f'(0) = 1 - \rho \geq 0$ or < 0 . Thus for a given $s > 0$, there is a unique x such that $f(x) = s$ in the range $x > \eta_0$ and it is clear from (4) that $x = \eta(s)$. It is clear that $\eta(s)$ is a continuous function of s and $\rightarrow \infty$ as $s \rightarrow \infty$. Also, as $s \rightarrow 0+$, $\eta(s) \rightarrow \eta_0$, which proves (i). The remaining result follows easily.

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