Appendix

Theorem A1. *Let* $K(z) = k_0 + k_1 z + k_2 z^2 + \cdots$ *be a p.g.f. with* $k_0 > 0$ *and* $0 < K'(1) = \alpha$. *Then the equation*

$$\xi = zK(\xi) \tag{1}$$

has a unique continuous root $\xi \equiv \xi(z)$ *in* $(0, 1)$ *such that* $\xi(0+) = 0$. *Also, as* $z \to 1-$, $\xi(z)$ *converges to the least positive root* ζ *of the equation* $\zeta = K(\zeta)$, *and* $0 < \zeta < 1$ *iff* $\alpha > 1$.

Proof. Consider the function

$$f(x) = \frac{K(x)}{x} = \frac{k_0}{x} + k_1 + k_2 x + k_3 x^2 + \cdots \qquad (0 < x < 1).$$

We have $f(0+) = \infty, f(1-) = 1$. Moreover,

$$f''(x) = \frac{2k_0}{x^3} + 2k_3 + 6k_4 x + \cdots \qquad >0.$$

It follows that $f(x)$ is monotone decreasing and >1 for $x < \zeta$ where ζ is the least positive root of the equation $f(x) = 1$, and $\zeta < 1$ or $= 1$ depending on whether $f'(1) = \alpha - 1 > 0$ or ≤ 0. Therefore for a given z in $(0, 1)$ there is a unique x such that $f(x) = z^{-1}$ in the range $(0 < x < \zeta)$ and from (1) it is clear that $x = \xi(z)$. Clearly ξ is a continuous function of z and $\to 0$ as $z \to 0+$. Also $z \to 1-$, $\xi(z) \to \zeta$.

Theorem A2. *Let* $\psi(\theta)$ *be the Laplace transform of a distribution* $(\theta > 0)$ *and* $\rho = -\lambda\psi'(0)/k$ *where* $0 < \lambda < \infty$ *and* k *is a positive integer. Then for* $0 < z < 1$ *the equation*

$$\gamma^k = z\psi(\theta + \lambda - \lambda\gamma) \tag{2}$$

has exactly k distinct roots $\gamma_r \equiv \gamma_r(z, \theta)$ *with* $|\gamma_r| < 1$ $(r = 1, 2, \ldots k)$. *As* $z \to 1-$ *and* $\theta \to 0+$, $\gamma_r(z, \theta) \to \zeta_r$ *where* ζ_r *are the roots of the equation*

$$\zeta^k = z\psi(\lambda - \lambda\zeta) \tag{3}$$

with $|\zeta_r| < 1$ $(r = 1, 2, \ldots, k)$ *if* $\rho > 1$, *while* $|\zeta_r| < 1$ $(\mathrm{r} = 1, 2, \ldots, k-1)$ *and* $\zeta_k = 1$ *if* $\rho \leq 1$.

PROOF. Let $|x| = 1 - \varepsilon$ for ε sufficiently small and positive.

Then
$$|z\psi(\theta + \lambda - \lambda x)| < (1 - \varepsilon)^k$$

and by Rouche's theorem the equation $x^k = z\psi(\theta + \lambda - \lambda x)$ has exactly k roots with $|x| < 1 - \varepsilon$. The remaining results follow from continuity arguments.

Theorem A3. *Let* $\phi(\theta) = \int_0^\infty (1 - e^{-\theta x})\nu(dx)$ $(\theta > 0)$, *where* ν *is a Lévy measure, and* $\rho = \int_0^\infty x\nu(dx)$. *Then for* $s > 0$ *the equation*

$$\eta = s + \phi(\eta) \tag{4}$$

has a unique continuous solution $\eta \equiv \eta(s)$ *with* $\eta(\infty) = \infty$. *Furthermore:*

(*i*) *as* $s \to 0+$, $\eta(s) \to \eta_0$ *is the largest positive root of the equation* $\eta_0 = \phi(\eta_0)$, *and* $\eta_0 > 0$ *iff* $\rho > 1$;
(*ii*) $\eta'(0+) = (1 - \rho)^{-1}$ *if* $\rho < 1$, *and* $= \infty$ *if* $\rho = 1$.

PROOF. Consider the function $f(x) = x - \phi(x)$ $(x > 0)$. We have $f(0+) = 0$, $f(\infty) = \infty$ and

$$f''(x) = \int_0^\infty y^2 e^{-xy}\nu(dy) > 0.$$

It follows that $f(x)$ is positive and monotone increasing for $x > \eta_0$, where η_0 is the largest positive root of $f(x) = 0$, and $\eta_0 = 0$ or > 0 depending on whether $f'(0) = 1 - \rho \geq 0$ or < 0. Thus for a given $s > 0$, there is a unique x such that $f(x) = s$ in the range $x > \eta_0$ and it is clear from (4) that $x = \eta(s)$. It is clear that $\eta(s)$ is a continuous function of s and $\to \infty$ as $s \to \infty$. Also, as $s \to 0+$, $\eta(s) \to \eta_0$, which proves (i). The remaining result follows easily.

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