A

Measure and Integral over Unbounded Sets

As presented in Chaps. 2 and 3, Lebesgue's theory of measure and integral is limited to functions defined over bounded sets. There are several ways of introducing the theory for arbitrary domains (and even for functions with values in the set of extended reals). However, instead of developing a general theory from scratch, in this Appendix we lay out an approach utilizing properties of measure and integral that were established in the main text for functions over bounded domains.

A.1 The Measure of an Arbitrary Set

In this section we extend the concept of measurability from bounded to arbitrary sets of real numbers.

Definition A.1. *A set E of real numbers is said to be* measurable *if all the sets*

$$E \cap [-n, n], \quad n \in \mathbb{N}$$

are measurable in the sense of Definition 2.5.

It is clear that a bounded set is measurable in the sense of Definition 2.5 if and only if it is measurable in the sense of Definition A.1.

Another possible way of introducing the concept of measurability is to define a set E to be measurable if the intersection $E \cap A$ of this set with an arbitrary bounded measurable set A is measurable. However, this approach produces the same class of measurable sets as defined above.

Theorem A.1. *A set E is measurable in the sense of Definition A.1 if and only if, for any bounded measurable set A, the set $E \cap A$ is measurable in the sense of Definition 2.5.*

S. Ovchinnikov, *Measure, Integral, Derivative: A Course on Lebesgue's Theory*, Universitext, DOI 10.1007/978-1-4614-7196-7,
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Proof. Because the sufficiency part is trivial, we proceed with the necessity part. Let us assume that the set E is measurable in the sense of Definition A.1 and let A be a bounded measurable set of real numbers. Inasmuch as A is bounded, there is $n \in \mathbb{N}$ such that $A \subseteq [-n, n]$. Then

$$E \cap A = (E \cap [-n, n]) \setminus ([-n, n] \setminus A).$$

The sets $E \cap [-n, n]$, $[-n, n]$, and A are bounded and measurable. It follows that the set $E \cap A$ is bounded and measurable. □

Definition A.2. *Let E be a measurable set of real numbers. The* measure *$m(E)$ is defined by*

$$m(E) = \sup\{m(E \cap [-n, n]) : n \in \mathbb{N}\}.$$

Because

$$E \cap [-1, 1] \subseteq E \cap [-2, 2] \subseteq \cdots \subseteq E \cap [-n, n] \subseteq \cdots,$$

the sequence of numbers $(m(E \cap [-n, n]))$ is increasing. If it is bounded, then it is convergent (cf. Theorem 1.3) and

$$m(E) = \lim m(E \cap [-n, n]).$$

Otherwise, we set $m(E) = \infty$ (see conventions in Sect. 1.2). Thus $m(E)$ is defined for every measurable set and assumes its values in the set of extended real numbers (recall that ∞ stands for $+\infty$). By convention, $a + \infty = \infty$ and $a \leq \infty$ for all extended real numbers a. We also make the convention that $\lim a_n = \infty$ if (a_n) is an increasing sequence of extended real numbers with $a_k = \infty$ for some $k \in \mathbb{N}$.

Example A.1.
1. Let I be an unbounded interval. By choosing a sufficiently large n, we can make the length of the interval $I \cap [-n, n]$ larger than any given number. Therefore, $m(I) = \infty$. In particular, $m(\mathbb{R}) = \infty$.
2. Let E be a finite or countable set of points. Then

$$m(E \cap [-n, n]) = 0, \quad \text{for every } n \in \mathbb{N}.$$

It follows that $m(E) = 0$.

3. Let

$$E = \bigcup_{k=2}^{\infty} \left[k - \frac{1}{k}, k + \frac{1}{k}\right].$$

Then

$$m(E) = \lim m(E \cap [-n, n]) = \lim \left(\sum_{k=2}^{n-1} \frac{2}{k} + \frac{1}{n} \right) = \infty.$$

4. Let $E = \cup_{k=1}^{\infty}[k - 2^{-k}, k + 2^{-k}]$. It is easy to verify that $m(E) = \sum_{k=1}^{\infty} 2^{-k+1} = 2$.

In what follows we present generalizations of the following properties of bounded measurable sets:

1. Open and closed sets are measurable (Theorem 2.16).
2. Countable additivity (Theorem 2.18).
3. The union and intersection of a countable family of measurable sets is measurable (Theorems 2.22 and 2.23).
4. The "continuity" properties of Lebesgue's measure (Theorems 2.24 and 2.25).

Some other properties are found in Exercises A.2–A.5.

Theorem A.2. *Any open or closed set of real numbers is measurable.*

Proof. Let E be an open subset of $\mathbb{R}$. Because E is a union of an at most countable family of pairwise disjoint open intervals (cf. Theorem 1.7), its intersection with any interval $[-n, n]$, $n \in \mathbb{N}$, is the union of an at most countable family of pairwise disjoint intervals. The latter set is measurable by Theorem 2.18.

Now let E be a closed subset of $\mathbb{R}$. Then, for every $n \in \mathbb{N}$, the intersection $E \cap [-n, n]$ is a bounded closed set and therefore is measurable. □

We establish the additivity property of measure separately for finite and countable families of sets.

Theorem A.3. *Let $\{E_i\}_{i\in J}$ be a finite family of pairwise disjoint measurable sets. Then*

$$m\Big(\bigcup_{i\in J} E_i\Big) = \sum_{i\in J} m(E_i).$$

Proof. By Definition A.2 and Theorem 2.18, we have

$$\begin{aligned} m\Big(\bigcup_{i\in J} E_i\Big) &= \lim m\Big(\Big(\bigcup_{i\in J} E_i\Big) \cap [-n, n]\Big) \\ &= \lim m\Big(\bigcup_{i\in J} \big(E_i \cap [-n, n]\big)\Big) \\ &= \lim \sum_{i\in J} m(E_i \cap [-n, n]) \\ &= \sum_{i\in J} \lim m(E_i \cap [-n, n]) \\ &= \sum_{i\in J} m(E_i), \end{aligned}$$

because J is a finite set (cf. Exercise A.1). □

Theorem A.4. *Let $\{E_i\}_{i\in J}$ be a countable family of pairwise disjoint measurable sets. Then*

$$m\Big(\bigcup_{i\in J} E_i\Big) = \sum_{i\in J} m(E_i).$$

Proof. First, we assume that $\{m(E_i)\}_{i\in J}$ is a summable family. Then, given $\varepsilon > 0$, there is a finite set $J_0 \subseteq J$ such that

$$\sum_{i\in J_0} m(E_i) > \sum_{i\in J} m(E_i) - \varepsilon.$$

Inasmuch as

$$\bigcup_{i\in J} E_i = \Big(\bigcup_{i\in J_0} E_i\Big) \cup \Big(\bigcup_{i\in J\setminus J_0} E_i\Big)$$

we have, by Theorem A.3,

$$m\Big(\bigcup_{i\in J} E_i\Big) = \sum_{i\in J_0} m(E_i) + m\Big(\bigcup_{i\in J\setminus J_0} E_i\Big).$$

Hence,

$$m\Big(\bigcup_{i\in J} E_i\Big) > \sum_{i\in J} m(E_i) - \varepsilon + m\Big(\bigcup_{i\in J\setminus J_0} E_i\Big).$$

Because ε is an arbitrary positive number, we conclude that

$$m\Big(\bigcup_{i\in J} E_i\Big) \geq \sum_{i\in J} m(E_i).$$

To prove the reverse inequality, we observe that, by the definition of m,

$$m(E_i \cap [-n, n]) \leq m(E_i),$$

for every $i, n \in \mathbb{N}$. Because $\{m(E_i)\}_{i\in J}$ is a summable family, we have, by Theorem 1.15,

$$\sum_{i\in J} m(E_i \cap [-n, n]) \leq \sum_{i\in J} m(E_i).$$

By Theorem 2.18,

$$m\Big(\Big(\bigcup_{i\in J} E_i\Big) \cap [-n, n]\Big) = \sum_{i\in J} m(E_i \cap [-n, n]).$$

It follows that

$$m\Big(\Big(\bigcup_{i\in J} E_i\Big) \cap [-n, n]\Big) \leq \sum_{i\in J} m(E_i).$$

By taking the limit as $n \to \infty$, we obtain the desired inequality:

$$m\Big(\bigcup_{i\in J} E_i\Big) \leq \sum_{i\in J} m(E_i).$$

It remains to consider the case when $\sum_{i \in J} m(E_i) = \infty$. Then, for any given real number M, there is a finite subset $J_0 \subseteq J$ such that

$$\sum_{i \in J_0} m(E_i) \geq M.$$

By Theorem A.3,

$$m\Big(\bigcup_{i \in J} E_i\Big) = \sum_{i \in J_0} m(E_i) + m\Big(\bigcup_{i \in J \setminus J_0} E_i\Big) \geq M.$$

It follows that $m\Big(\bigcup_{i \in J} E_i\Big) = \infty = \sum_{i \in J} m(E_i)$. □

The proof of the following theorem is straightforward and left as an exercise (cf. Exercise A.6).

Theorem A.5. *Let $\{E_i\}_{i \in J}$ be a finite or countable family of measurable sets. Then*

(i) *The union $\bigcup_{i \in J} E_i$ is measurable.*
(ii) *The intersection $\bigcap_{i \in J} E_i$ is measurable.*

Finally, we establish two "continuity" properties of the extended measure.

Theorem A.6. *Let (E_n) be a sequence of measurable sets such that*

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots$$

and let $E = \bigcup_{i=1}^{\infty} E_i$. Then

$$m(E) = \lim m(E_n).$$

Proof. By Theorem A.5, the set E is measurable. We consider two possible cases.

1. There is k such that $m(E_k) = \infty$. Because $E_k \subseteq E$, we have $m(E) = \infty$ (cf. Exercise A.3) and the result follows.
2. $m(E_k) < \infty$ for all $k \in \mathbb{N}$. Since

$$E_{k+1} = E_k \cup (E_{k+1} \setminus E_k), \qquad k \in \mathbb{N},$$

we have, by Theorem A.3,

$$m(E_{k+1} \setminus E_k) = m(E_{k+1}) - m(E_k).$$

Furthermore,

$$E = E_1 \cup (E_2 \setminus E_1) \cup \cdots \cup (E_{k+1} \setminus E_k) \cup \cdots$$

with pairwise disjoint sets on the right side. By Theorem A.4,

$$m(E) = m(E_1) + \sum_{i=1}^{\infty} [m(E_{k+1}) - m(E_k)].$$

The nth partial sum of the series on the right side is

$$m(E_1) + \sum_{k=1}^{n-1} [m(E_{k+1}) - m(E_k)] = m(E_n).$$

Therefore, $m(E) = \lim m(E_n)$. □

Theorem A.7. *Let (E_n) be a sequence of measurable sets such that*

$$E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots .$$

If $m(E_1) < \infty$, then

$$m\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim m(E_n).$$

Proof. Let $E = \bigcap_{k=1}^{\infty} E_k$. We have

$$E_n = E \cup \bigcup_{k=n}^{\infty} (E_k \setminus E_{k+1}).$$

By Theorem A.4,

$$m(E_n) = m(E) + \sum_{k=n}^{\infty} m(E_k \setminus E_{k+1}). \tag{A.1}$$

In particular,

$$m(E_1) = m(E) + \sum_{k=1}^{\infty} m(E_k \setminus E_{k+1}).$$

Because $m(E_1) < \infty$, the series on the right side converges, that is,

$$\lim \sum_{k=n}^{\infty} m(E_k \setminus E_{k+1}) = 0.$$

The result follows from (A.1). □

A.2 Measurable Functions over Arbitrary Sets

It is not difficult to verify that the statements of Theorem 2.29 hold for arbitrary measurable sets which were introduced in Sect. A.1. Therefore we use the same definition of measurable functions as in Definition 2.6. The only difference is that the measurable sets under consideration are allowed to be unbounded. Almost all statements in Sects. 2.7 and 2.8 are immediately seen to hold for this extended class of measurable functions (cf. Exercise A.8). The only exception is Egorov's Theorem 2.34 as evidenced by the following counterexample.

Let (f_n) be a sequence of functions on $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 0, & \text{if } x \leq n, \\ 1, & \text{if } x > n, \end{cases} \qquad \text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

The terms of this sequence are measurable functions and the sequence converges pointwise to the zero function on $\mathbb{R}$. Suppose that the statement of Theorem 2.34 holds for this sequence and let $\delta = 1$ in the theorem. According to this theorem, there is a measurable set E_1 such that $m(E_1) < 1$ and (f_n) converges uniformly to the zero function on the set $E = \mathbb{R} \setminus E_1$. Then, for $\varepsilon = 1$, there is N such that

$$|f_n(x) - 0| < 1, \qquad \text{for all } x \in E \text{ and } n \geq N.$$

In particular, $f_N(x) = 0$ on the set E. However, this is impossible because the set E is not bounded above (cf. Exercise A.9).

We conclude this section by proving a theorem which is an analog of Egorov's Theorem 2.34 for arbitrary measurable subsets of $\mathbb{R}$.

Theorem A.8. *Let E be an arbitrary measurable set of real numbers, and let (f_n) be a sequence of measurable functions on E converging pointwise to a function f. Then E can be written as the union*

$$E = A \cup \bigcup_{k=1}^{\infty} B_k,$$

where the sets A, $B_1, B_2 \ldots$ are measurable, $m(A) = 0$, and (f_n) converges uniformly on each of the sets $B_1, B_2 \ldots$.

Proof. First, we consider the case of a bounded set E. By Theorem 2.34, for every $n \in \mathbb{N}$ there is a measurable set E_n such that $m(E_n) < 1/n$ and (f_n) converges uniformly on $B_n = E \setminus E_n$. We have

$$m\Big(E \setminus \bigcup_{k=1}^{n} B_k\Big) = m\Big(\bigcap_{k=1}^{n} E_k\Big) \leq m(E_n) < \frac{1}{n}.$$

Let

$$A = E \setminus \bigcup_{k=1}^{\infty} B_k,$$

so

$$E = A \cup \bigcup_{k=1}^{\infty} B_k.$$

Because

$$A = E \setminus \bigcup_{k=1}^{\infty} B_k \subseteq E \setminus \bigcup_{k=1}^{n} B_k,$$

for every $n \in \mathbb{N}$, and

$$m\Big(E \setminus \bigcup_{k=1}^{n} B_k\Big) < \frac{1}{n},$$

we have $m(A) = 0$, and the result follows.

Now suppose that E is unbounded. We can represent E as the union of pairwise disjoint bounded sets

$$E = \{0\} \cup \bigcup_{k=1}^{\infty} \Big(E \cap ([-k, -k+1) \cup (k-1, k])\Big).$$

By applying the result of the theorem for each bounded set in the above countable union, we obtain the desired representation. □

A.3 Integration over Arbitrary Sets

As in Chap. 3, we begin by defining the integral of a nonnegative measurable function on a measurable set E (bounded or not).

Definition A.3. *Let f be a nonnegative measurable function on an arbitrary measurable set E. We define*

$$\int_E f = \sup\Big\{ \int_{E\cap[-n,n]} f : n \in \mathbb{N}\Big\}.$$

Each of the integrals on the right side is well defined though it may be equal to ∞ (cf. Definition 3.2). According to our conventions, we set $\int_E f = \infty$ if one of the integrals on the right side is infinite. If $\int_E f < \infty$, we say that the function f is integrable *over E.*

Inasmuch as

$$\int_{E\cap[-n,n]} f \le \int_{E\cap[-m,m]} f, \qquad \text{for } n \le m,$$

for a nonnegative function f, we can write

$$\int_E f = \lim \left(\int_{E \cap [-n,n]} f \right).$$

The following theorem provides for a partial justification for the above definition (cf. the definition of a measurable set in Sect. A.1 and the subsequent theorem there).

Theorem A.9. *For a nonnegative measurable function f on E we have*

$$\sup \left\{ \int_{E \cap A} f : A \textit{ is a bounded measurable set} \right\} = \int_E f.$$

Proof. We denote

$$S = \sup \left\{ \int_{E \cap A} f : A \text{ is a bounded measurable set} \right\}.$$

If $\int_{E \cap A} f = \infty$ for some bounded set A, then $S = \infty$. Then $\int_E f = \infty$ because there is n such that $A \subseteq [-n, n]$.

Suppose that $\int_{E \cap A} f < \infty$ for all bounded sets A. Because for any $n \in \mathbb{N}$, the interval $[-n, n]$ is a bounded set, we have $S \geq \int_E f$. On the other hand, for any bounded set A there is n such that $A \subseteq [-n, n]$. Hence, $S \leq \int_E f$, and the result follows. □

For a function f of arbitrary sign, we use its positive and negative parts, f^+ and f^-, to define the integral $\int_E f$ (cf. Sect. 3.5).

Definition A.4. *Let f be a measurable function on an arbitrary measurable set E. We define*

$$\int_E f = \int_E f^+ - \int_E f^-,$$

provided that at least one of the integrals on the right side is finite. Otherwise, the integral $\int_E f$ is undefined. If $\int_E f$ exists and is finite, then the function f is said to be integrable *over E.*

Thus a function f is integrable on an arbitrary measurable set if and only if both nonnegative functions f^+ and f^- are integrable.

Theorem A.10. *If f is an integrable function over a set E, then*

$$\lim \int_{E \cap [-n,n]} f = \int_E f.$$

Proof. Inasmuch as f is integrable, the integrals $\int_E f^+$ and $\int_E f^-$ exist and are finite. Hence we have

$$\lim \int_{E\cap[-n,n]} f^+ = \int_E f^+ \quad \text{and} \quad \lim \int_{E\cap[-n,n]} f^- = \int_E f^-.$$

Therefore,

$$\begin{aligned}\lim \int_{E\cap[-n,n]} f &= \lim \left(\int_{E\cap[-n,n]} f^+ - \int_{E\cap[-n,n]} f^- \right) \\ &= \lim \int_{E\cap[-n,n]} f^+ - \lim \int_{E\cap[-n,n]} f^- \\ &= \int_E f^+ - \int_E f^- = \int_E f,\end{aligned}$$

and the result follows. □

However, the existence of the limit $\lim \int_{E\cap[-n,n]} f$ does not imply Lebesgue's integrability of the function f as the following two examples demonstrate.

Example A.2. Let $f(x) = x$ on $\mathbb{R}$. Then

$$\int_{\mathbb{R}} f^+ = \int_{\mathbb{R}} \max\{x, 0\} = \infty \quad \text{and} \quad \int_{\mathbb{R}} f^- = \int_{\mathbb{R}} \max\{-x, 0\} = \infty.$$

Hence, the function x is not Lebesgue integrable over $\mathbb{R}$. However, $\int_{-n}^n f = 0$ for all $n \in \mathbb{N}$, so the limit $\lim \int_{-n}^n f$ exists and equals zero.

The limit $\lim \int_{-n}^n f$ may exist even for a bounded function over $\mathbb{R}$ which is not Lebesgue integrable as the next example shows.

Example A.3. We define

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{for } x = 0, \end{cases} \qquad x \in \mathbb{R}.$$

It can be shown that the limit of the sequence $(\int_{-n}^n f)$ exists (and equals π), but both Lebesgue integrals $\int_{\mathbb{R}} f^+$ and $\int_{\mathbb{R}} f^-$ are infinite (cf. Exercise A.11), so f is not Lebesgue integrable.

Many properties of the integral over an arbitrary measurable domain can be easily established as consequences of properties already established in the case of bounded domains (cf. Exercises A.10–A.15). However, some arguments are subtle. As an example, we give a proof of Fatou's Lemma (cf. Theorem 3.16).

Theorem A.11. (Fatou's Lemma) *Let* $(f_1, \ldots, f_k, \ldots)$ *be a sequence of nonnegative measurable functions converging pointwise to a function* f *a.e. on* E*. Then*

$$\int_E f \le \liminf \int_E f_k.$$

Proof. As in the proof of Theorem 3.16, we may assume that convergence takes place over the entire set E.

For any given $m, n \in \mathbb{N}$ we have

$$\inf \left\{ \int_{E\cap[-n,n]} f_k : k \ge m \right\} \le \int_{E\cap[-n,n]} f_p, \qquad \text{for all } p \ge m.$$

It follows that

$$\lim_{n\to\infty} \inf \left\{ \int_{E\cap[-n,n]} f_k : k \ge m \right\} \le \int_E f_p, \quad \text{for all } p \ge m. \tag{A.2}$$

By Theorem 3.16,

$$\int_{E\cap[-n,n]} f \le \lim_{m\to\infty} \left(\inf \left\{ \int_{E\cap[-n,n]} f_k : k \ge m \right\} \right).$$

By taking the limits as $n \to \infty$ on both sides, we obtain

$$\begin{aligned} \int_E f &\le \lim_{n\to\infty} \lim_{m\to\infty} \left(\inf \left\{ \int_{E\cap[-n,n]} f_k : k \ge m \right\} \right) \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \left(\inf \left\{ \int_{E\cap[-n,n]} f_k : k \ge m \right\} \right) \\ &\le \liminf \int_E f_k. \end{aligned}$$

Here, the two repeated limits are equal by Exercise A.16, and the last inequality follows from (A.2). □

The next two theorems can be proven by repeating verbatim the arguments used in the proofs of Theorems 3.17 and 3.25. The proofs are left to the reader as exercises (cf. Exercise A.17).

Theorem A.12 (The Monotone Convergence Theorem). *Let* (f_n) *be an increasing sequence of nonnegative measurable functions on* E*. If* (f_n) *converges pointwise to* f *a.e. on* E*, then*

$$\lim \int_E f_n = \int_E f.$$

Theorem A.13. (The Dominated Convergence Theorem) *Let (f_n) be a sequence of measurable functions on E. Suppose that there is an integrable function g on E that dominates (f_n) on E in the sense that*

$$|f_n| \leq g \quad \text{on } E \text{ for all } n.$$

If (f_n) converges pointwise to f a.e. on E, then

$$\lim \int_E f_n = \int_E f.$$

Notes

The classes of measurable sets and integrable functions introduced in the Appendix are the same as obtained by more conventional methods. Of course, we cannot prove it here.

Theorem A.8 is known as Lusin's version of Egorov's Theorem (Theorem 2.34) for unbounded domains. Egorov's Theorem also holds in the following form for unbounded sets:

Theorem A.14. (Egorov's Theorem) *Let (f_n) be a sequence of measurable functions on a set E of finite measure that converges pointwise a.e. on E to a function f. Then for each $\delta > 0$, there is a measurable set $E_\delta \subseteq E$ such that $m(E_\delta) < \delta$ and (f_n) converges uniformly to f on $E \setminus E_\delta$.*

The following theorem is an important result which is also due to Lusin.

Theorem A.15. (Lusin's Theorem) *Let f be a measurable function on a set E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set $F \subseteq E$ for which*

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

The function f in Example A.3 has an improper Riemann integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

On the other hand, this function is not Lebesgue integrable over the set of reals $\mathbb{R}$ [cf. Apo74, Exercise 10.9]. In this connection, see Exercise A.12 below.

As we observed in Chap. 3, Riemann integrable functions over a bounded interval form a proper subset of the set of Lebesgue integrable functions over the same interval. The situation in the case of unbounded intervals is different—the corresponding classes are incomparable in this case. In other words, there are functions which are Riemann (improper) integrable, say, over $\mathbb{R}$, but not Lebesgue integrable over that set (cf. f from the previous paragraph). On the other hand, the Dirichlet function,

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}, \end{cases} \qquad x \in \mathbb{R},$$

is not Riemann integrable over any bounded interval, whereas it is Lebesgue integrable with $\int_{\mathbb{R}} f = 0$.

Note that we exchanged order of limits in the proof of Theorem A.11 (cf. Exercise A.16). For a general result see Sect. 8.20 in [Apo74].

Exercises

A.1. (i) Let (a_n) and (b_n) be two increasing sequences of real numbers. Prove that

$$\lim(a_n + b_n) = \lim a_n + \lim b_n.$$

(ii) Extend the result of part (i) to finite sums of increasing sequences.

A.2. Let E be a measurable set of real numbers. Prove that the set $\complement E = \mathbb{R}\backslash E$ is also measurable.

A.3. Prove that if E and F are measurable sets and $E\subseteq F$, then $m(E)\leq m(F)$.

A.4. Let E be a measurable set. Show that

(a) If $m(E) < \infty$ and $\varepsilon > 0$, then there exist an open set G and a bounded closed set F, such that $F \subseteq E \subseteq G$ and

$$m(G) - m(E) < \varepsilon, \quad m(E) - m(F) < \varepsilon.$$

(b) If $m(E) = \infty$, then for any $M > 0$, there exists a bounded closed set $F \subseteq E$ such that $m(F) > M$.

A.5. Prove that the image of a measurable set E under the translation $x \mapsto x + a$ is measurable with $m(E + a) = m(E)$.

A.6. Prove Theorem A.5.

A.7. Let E_1 and E_2 be measurable sets. Show that

(a) The set $E_1 \setminus E_2$ is measurable.
(b) The set $E_1 \triangle E_2$ is measurable.

A.8. Show that the statements of Theorems 2.30, 2.31, Corollary 2.1, and Theorems 2.32, 2.33 hold for measurable functions over arbitrary sets.

A.9. Let A be a subset of $\mathbb{R}$ such that $m(A) < \infty$. Show that

$$\sup(\mathbb{R} \setminus A) = \infty.$$

A.10. Show that a measurable function f is integrable over E if and only if the function $|f|$ is integrable over E and that

$$\left|\int_E f\right| \leq \int_E |f|$$

in this case (cf. Theorem 3.20).

A.11. Let f be the function from Example A.3. Show that

(a) $\lim \int_{-n}^{n} f$ exists.
(b) $\int_{\mathbb{R}} f^+ = \int_{\mathbb{R}} f^- = \infty$.

(cf. [Apo74, Exercise 10.9]).

A.12. Let $f : [0, \infty) \to \mathbb{R}$ be Riemann integrable on every bounded subinterval of $[0, \infty)$. Prove that f is Lebesgue integrable over $[0, \infty)$ if and only if the limit (the improper Riemann integral)

$$\lim \int_0^n |f| = \int_0^\infty |f|$$

exists. Show also that in this case

$$(L) \int_0^\infty f = (R) \int_0^\infty f.$$

A.13. Establish the linearity and monotonicity properties of the integral over arbitrary measurable sets (cf. Theorems 3.23 and 3.24).

A.14. Let f be an integrable function over the finite union $E = \bigcup_{k=1}^n E_k$ of pairwise disjoint measurable sets. Show that

$$\int_E f = \sum_{k=1}^n \int_{E_k} f.$$

A.15. Let E be a set of measure zero and f be a function on E. Show that f is measurable with $\int_E f = 0$.

A.16. Let (a_{mn}) be a double sequence of nonnegative real numbers such that $a_{mn} \geq a_{pq}$ for all $m \geq p$, $n \geq q$. Show that

$$\lim_{m,n\to\infty} a_{mn} = \lim_{n\to\infty} \lim_{m\to\infty} a_{mn} = \lim_{m\to\infty} \lim_{n\to\infty} a_{mn}.$$

A.17. Prove Theorems A.12 and A.13.

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S. Ovchinnikov, *Measure, Integral, Derivative: A Course on Lebesgue's Theory*, Universitext, DOI 10.1007/978-1-4614-7196-7,

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