Appendix A
Review of Basic Material

A.1 Tensor Products of Vector Spaces

Given two vector spaces V_1 and V_2 over $\mathbb{C}$, the tensor product is a new vector space $V_1 \otimes V_2$, together with a bilinear "product" map $\otimes : V_1 \times V_2 \to V_1 \otimes V_2$. If V_1 and V_2 are finite dimensional with bases $\{u_j\}$ and $\{v_k\}$, then $V_1 \otimes V_2$ is finite dimensional with $\{u_j \otimes v_k\}$ forming a basis for $V_1 \otimes V_2$. In the finite-dimensional case, we could simply *define* the tensor product by this basis property, but then we would have to worry about whether the construction is basis independent. Instead, we define $V_1 \otimes V_2$ by a "universal property."

Definition A.1 *Suppose V_1 and V_2 are vector spaces over a field $\mathbb{F}$. Then a* ***tensor product*** *of V_1 and V_2 is a vector space W over $\mathbb{F}$ together with a bilinear map $T : V_1 \times V_2 \to W$ having the following "universal property": If U is any vector space over $\mathbb{F}$ and $\Phi : V_1 \times V_2 \to U$ is a bilinear map, then there exists a unique linear map $\tilde{\Phi} : W \to U$ such that the following diagram commutes:*

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{T} & W \\ \Phi \downarrow & \swarrow \tilde{\Phi} & \\ U & & \end{array} .$$

Proposition A.2 *For any two vector spaces V_1 and V_2, a tensor product of V_1 and V_2 exists and is unique up to "canonical isomorphism." That is, for two tensor products (W_1, T_1) and (W_2, T_2), there is a unique invertible linear map $\Psi : W_1 \to W_2$ such that $T_2 = \Psi \circ T_1$.*

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In light of the uniqueness result, we may speak of “the” tensor product of V_1 and V_2. We choose any one tensor product and we denote it by $V_1 \otimes V_2$. We also denote the linear map $T : V_1 \times V_2 \to V_1 \otimes V_2$ as $(u, v) \mapsto u \otimes v$. In this notation, the universal property reads as follows: Given any *bilinear* map Φ of $V_1 \times V_2$ into a vector space U, there exists a unique *linear* map $\tilde{\Phi} : V_1 \otimes V_2 \to U$ such that

$$\tilde{\Phi}(u \otimes v) = \Phi(u, v).$$

Proposition A.3 *If V_1 and V_2 are finite-dimensional vector spaces with bases $\{u_j\}_{j=1}^{n_1}$ and $\{v_k\}_{k=1}^{n_2}$, then $V_1 \otimes V_2$ is finite dimensional and the set of elements of the form $u_j \otimes v_k$, $1 \leq j \leq n_1$, $1 \leq k \leq n_2$, forms a basis for $V_1 \otimes V_2$. In particular,*

$$\dim(V_1 \otimes V_2) = (\dim V_1)(\dim V_2).$$

It should be emphasized that, in general, not every element of $V_1 \otimes V_2$ is of the form $u \otimes v$ with $u \in V_1$ and $v \in V_2$. All we can say is that each element of $V_1 \otimes V_2$ can be decomposed as a *linear combination* of elements of the form $u \otimes v$. This decomposition, furthermore, is far from canonical; even in the finite-dimensional case, it depends on a choice of bases for V_1 and V_2. Nevertheless, the universal property of the tensor product tells us that we can define linear maps from $V_1 \otimes V_2$ to any vector space U, simply by defining them on elements of the form $u \otimes v$. Provided that $\Phi(u, v)$ is bilinear in u and v, the universal property tells us that there is a unique linear map $\tilde{\Phi}$ on $V_1 \otimes V_2$ such that on element of the form $u \otimes v$, $\tilde{\Phi}$ is equal to $\Phi(u, v)$. A representative application of the universal property is in the following result.

Proposition A.4 *If $A \in \mathrm{End}(V_1)$ and $B \in \mathrm{End}(V_2)$, there exists a unique linear map $A \otimes B : V_1 \otimes V_2 \to V_1 \otimes V_2$ such that*

$$(A \otimes B)(u \otimes v) = (Au) \otimes (Bv).$$

For $A_1, A_2 \in \mathrm{End}(V_1)$ and $B_1, B_2 \in \mathrm{End}(V_2)$, we have

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2).$$

To construct $A \otimes B$, we apply the universal property with $U = V_1 \otimes V_2$ and $\Phi(u, v) = (Au) \otimes (Bv)$. Since A and B are linear and $\otimes$ is bilinear, Φ is bilinear. The linear map $\tilde{\Phi} : V_1 \otimes V_2 \to V_1 \otimes V_2$ is then the map that we denote $A \otimes B$.

The tensor product, as we have defined it in this section, applies to all vector spaces, whether finite dimensional or infinite dimensional. The construction, however, is purely algebraic; if there is a topology on V_1 and V_2, the tensor product takes no account of that topology. In the Hilbert space setting, then, we will have to refine the notion of the tensor product so that the tensor product of two Hilbert spaces will again be a Hilbert space. See Sect. A.4.5.

A.2 Measure Theory

It is assumed that the reader is familiar with the basic notions of measure theory, including the concepts of σ-algebras, measures, measurable functions, and the Lebesgue integral. A triple (X, Ω, μ), consisting of a set X, a σ-algebra Ω of subsets of X, and a (non-negative) measure μ on Ω is called a *measure space*. A measurable function $\psi : X \to \mathbb{C}$ is said to be *integrable* if $\int_X |\psi| \; d\mu < \infty$. The σ-algebra *generated by* any collection of subsets of a set X is the smallest σ-algebra of subsets of X containing that collection.

We assume those parts of measure theory that are entirely standard: the monotone convergence and dominated convergence theorems, L^p spaces, and Fubini's theorem. We briefly review a few other topics that might not be as familiar.

A measure μ on a measurable space (X, μ) is said to be *σ-finite* if X can be written as a countable union of measurable sets of finite measure.

Definition A.5 *Suppose μ and ν are two σ-finite measures on a measure space (X, Ω). Then we say that μ is **absolutely continuous** with respect to ν if for all $E \in \Omega$, if $\nu(E) = 0$ then $\mu(E) = 0$. We say that μ and ν are **equivalent** if each measure is absolutely continuous with respect to the other.*

Theorem A.6 (Radon–Nikodym) *Suppose μ and ν are two σ-finite measures on a measure space (X, Ω) and that μ is absolutely continuous with respect to ν. Then there exists a non-negative, measurable function ρ on X such that*

$$\mu(E) = \int_E \rho \; d\nu,$$

*for all $E \in \Omega$. The function ρ is called the **density** of μ with respect to ν.*

Definition A.7 *A collection $\mathcal{M}$ of subsets of a set X is called a **monotone class** if $\mathcal{M}$ is closed under countable* increasing *unions and countable* decreasing *intersections.*

A countable increasing union means the union of a sequence E_j of sets where E_j is contained in E_{j+1} for each j, with a similar definition for countable decreasing intersections.

Theorem A.8 (Monotone Class Lemma) *Suppose $\mathcal{M}$ is a monotone class of subsets of a set X and suppose $\mathcal{M}$ contains an algebra $\mathcal{A}$ of subsets of X. Then $\mathcal{M}$ contains the σ-algebra generated by $\mathcal{A}$.*

Corollary A.9 *Suppose μ and ν are two finite measures on a measure space (X, Ω). Suppose μ and ν agree on an algebra $\mathcal{A} \subset \Omega$. Then μ and ν agree on the σ-algebra generated by $\mathcal{A}$.*

Note that in general, the collection of sets on which two measures agree is *not* a σ-algebra, nor even an algebra.

Theorem A.10 *Suppose μ is a measure on the Borel σ-algebra in a locally compact, separable metric space X. Suppose also that $\mu(K) < \infty$ for each compact subset K of X. Then the space of continuous functions of compact support on X is dense in $L^p(X, \mu)$, for all p with $1 \leq p < \infty$.*

A word of clarification is in order here. If ψ is a continuous function on X with compact support, then $\int_X |\psi|^p \; d\mu$ is finite, since ψ is bounded and μ is finite on compact sets. Thus, we can define a map from $C_c(X)$ into $L^p(X, \mu)$ by mapping a continuous function ψ of compact support to the equivalence class $[\psi]$. The theorem is asserting, more precisely, that the *image* of $C_c(X)$ under this map is dense in $L^p(X, \mu)$. It should be noted, however, that the map $\psi \mapsto [\psi]$ need not be injective. After all, if there is a nonempty open set U inside X with $\mu(U) = 0$, then for any ψ with support contained in U, the equivalence class $[\psi]$ will be the zero element of $L^p(X, \mu)$. Nevertheless, we will allow ourselves a small abuse of terminology and say that $C_c(X)$ is dense in $L^p(X, \mu)$.

A.3 Elementary Functional Analysis

In this section, we briefly review some of the results from elementary functional analysis that we make use of the text. Most of these results can be found in the book of Rudin [32].

A.3.1 The Stone–Weierstrass Theorem

The Weierstrass theorem states that every continuous, real-valued function on an interval can be uniformly approximated by polynomials. A substantial generalization of this was obtained by Stone. If X is a compact metric space, let $\mathcal{C}(X;\mathbb{R})$ and $\mathcal{C}(X;\mathbb{C})$ denote the space of continuous real- and complex-valued continuous functions, respectively. A subset $\mathcal{A}$ of $\mathcal{C}(X;\mathbb{F})$ is called an *algebra* if it is closed under pointwise addition, pointwise multiplication, and multiplication by elements of $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An algebra $\mathcal{A}$ is said to *separate points* if for any two distinct points x and y in X, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. We use on $\mathcal{C}(X;\mathbb{F})$ the *supremum norm*, given by

$$\|f\|_{\sup} := \sup_{x \in X} |f(x)|,$$

and $\mathcal{C}(X, \mathbb{F})$ is complete with respect to the associated distance function, $d(f, g) = \|f - g\|_{\sup}$.

Theorem A.11 (Stone–Weierstrass, Real Version) *Let X be a compact metric space and let $\mathcal{A}$ be an algebra in $\mathcal{C}(X;\mathbb{R})$. If $\mathcal{A}$ contains the constant functions and separates points, then $\mathcal{A}$ is dense in $\mathcal{C}(X;\mathbb{R})$ with respect to the supremum norm.*

Theorem A.12 (Stone–Weierstrass, Complex Version) *Let X be a compact metric space and let $\mathcal{A}$ be an algebra in $\mathcal{C}(X;\mathbb{C})$. If $\mathcal{A}$ contains the constant functions, separates points, and is closed under complex conjugation, then $\mathcal{A}$ is dense in $\mathcal{C}(X;\mathbb{C})$ with respect to the supremum norm.*

A consequence of the complex version of the Stone–Weierstrass theorem is the following: If K is a compact subset of $\mathbb{C}$, then every continuous, complex-valued function on K can be uniformly approximated by polynomials in z and $\bar{z}$.

A.3.2 The Fourier Transform

We now describe the Fourier transform on $\mathbb{R}^n$, in various forms.

Definition A.13 *For any $\psi \in L^1(\mathbb{R}^n)$, define the **Fourier transform** of ψ to be the function $\hat{\psi}$ on $\mathbb{R}^n$ given by*

$$\hat{\psi}(\mathbf{k}) = (2\pi)^{-n/2} \int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}} \psi(\mathbf{x})\ d\mathbf{x}.$$

Proposition A.14 *For any $\psi \in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{\psi}$ of ψ has the following properties: (1) $\left|\hat{\psi}(\mathbf{k})\right| \leq (2\pi)^{-n/2} \left\|\psi\right\|_{L^1}$, (2) $\hat{\psi}$ is continuous, and (3) $\hat{\psi}(\mathbf{k})$ tends to zero as $|\mathbf{k}|$ tends to ∞.*

The bound on $\hat{\psi}$ is obvious and the continuity of $\hat{\psi}$ follows from dominated convergence. To show that $\hat{\psi}$ tends to zero at infinity, we first establish this on a dense subspace of $L^1(\mathbb{R}^n)$ (e.g., the Schwartz space; see below) and then take uniform limits.

Definition A.15 *The **Schwartz space** $\mathcal{S}(\mathbb{R}^n)$ is the space of all C^∞ functions ψ on $\mathbb{R}^n$ such that*

$$\lim_{x\to\pm\infty} \left|\mathbf{x}^{\mathbf{j}} \partial^{\mathbf{k}} \psi(\mathbf{x})\right| = 0$$

for all n-tuples of non-negative integers $\mathbf{j}$ and $\mathbf{k}$. Here if $\mathbf{j} = (j_1, \ldots, j_n)$ then $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} \cdots x_n^{j_n}$ and

$$\partial^{\mathbf{j}} = \left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{j_n}.$$

*An element of the Schwartz space is called a **Schwartz function.***

Proposition A.16 *If ψ belongs to $\mathcal{S}(\mathbb{R}^n)$, then $\hat{\psi}$ also belongs to $\mathcal{S}(\mathbb{R}^n)$.*

The proof of this result hinges on the behavior of the Fourier transform under differentiation and under multiplication by x, results which are of interest in their on right.

Proposition A.17 *If ψ is a Schwartz function, the following properties hold*

1. *We have*
$$\widehat{\frac{\partial\psi}{\partial x_j}}(\mathbf{k}) = ik_j\hat{\psi}(\mathbf{k}). \tag{A.1}$$

2. *The function $\hat{\psi}$ is differentiable at every point and the Fourier transform of the function $x_j\psi(x)$ is given by*
$$\widehat{x_j\psi}(\mathbf{k}) = i\frac{\partial}{\partial k_j}\hat{\psi}(\mathbf{k}). \tag{A.2}$$

The first point is proved by integration by parts and the second by differentiation under the integral in the definition of $\hat{\psi}$.

Theorem A.18 (Fourier Inversion and Plancherel Formula, I) *The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ has the following properties.*

1. *The Fourier transform maps the Schwartz space* onto *the Schwartz space.*

2. *For all $\psi \in \mathcal{S}(\mathbb{R}^n)$, the function ψ can be recovered from its Fourier transform by the* ***Fourier inversion formula****:*
$$\psi(\mathbf{x}) = (2\pi)^{-n/2}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\psi}(k)\ d\mathbf{k}.$$

3. *For all $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have the* ***Plancherel theorem****:*
$$\int_{\mathbb{R}^n} |\psi(\mathbf{x})|^2\ d\mathbf{x} = \int_{\mathbb{R}^n} |\hat{\psi}(\mathbf{k})|^2\ d\mathbf{k}.$$

Since the Schwartz space is dense in $L^2(\mathbb{R}^n)$, the BLT theorem and Theorem A.18 imply that the Fourier transform extends uniquely to an isometric map of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$.

Theorem A.19 (Fourier Inversion and Plancherel Theorem, II) *The Fourier transform extends to an isometric map $\mathcal{F}$ of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. This map may be computed as*
$$\mathcal{F}(\psi)(\mathbf{k}) = (2\pi)^{-n/2}\lim_{A\to\infty}\int_{|\mathbf{x}|\le A} e^{-i\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{x})\ d\mathbf{x}, \tag{A.3}$$
where the limit is in the norm topology of $L^2(\mathbb{R}^n)$. The inverse map $\mathcal{F}^{-1}$ may be computed as
$$\left(\mathcal{F}^{-1}f\right)(\mathbf{x}) = (2\pi)^{-n/2}\lim_{A\to\infty}\int_{|\mathbf{x}|\le A} e^{i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{k})\ d\mathbf{k}.$$

If ψ belongs to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then by dominated convergence, the limit in coincides with the L^1 Fourier transform in Definition A.13.

Definition A.20 *For two measurable functions ϕ and ψ, define the* ***convolution*** *$\phi * \psi$ of ϕ and ψ by the formula*

$$(\phi * \psi)(\mathbf{x}) = \int_{\mathbb{R}^n} \phi(\mathbf{x} - \mathbf{y})\psi(\mathbf{y})\ d\mathbf{y},$$

provided that the integral is absolutely convergent for all $\mathbf{x}$.

Proposition A.21 *Suppose that ϕ and ψ belong to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\phi * \psi$ is defined and belongs to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and we have*

$$(2\pi)^{-n/2}\mathcal{F}(\phi * \psi) = \mathcal{F}(\phi)\mathcal{F}(\psi).$$

This result is proved by plugging $\phi * \psi$ into the definition of the Fourier transform, writing $e^{-i\mathbf{k}\cdot\mathbf{x}}$ as $e^{-i\mathbf{k}\cdot\mathbf{y}}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$, and using Fubini's theorem.

We will have occasion to use the following Gaussian integral.

Proposition A.22 *For all $a > 0$ and $b \in \mathbb{C}$, we have*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/(2a)} e^{bx}\ dx = \sqrt{a} e^{ab^2/2}.$$

Taking $b = ik$ in the last part of the proposition gives us the Fourier transform of the Gaussian function $e^{-x^2/(2a)}$. Taking $b = 0$ allows us to determine the proper normalization of the Gaussian probability density.

A.3.3 Distributions

In this section we give a brief account of the theory of distributions—what physicists call "generalized functions"—including the notion of "derivative in the distribution sense."

The idea is that we study functions by studying their integral against some class of very nice "test functions." Consider, for example, a locally integrable function f and consider integrals of the form

$$\int_{\mathbb{R}^n} \chi(\mathbf{x}) f(\mathbf{x})\ d\mathbf{x}, \tag{A.4}$$

where χ belongs to $C_c^\infty(\mathbb{R}^n)$, the space of smooth, compactly supported functions. We might think, for example, that χ is positive, has integral equal to 1, and is supported near some point $\mathbf{a} \in \mathbb{R}^n$. In that case, the integral (A.4) is an approximation to the value of f at $\mathbf{a}$, what physicists describe as a "smeared out" version of $f(\mathbf{a})$.

Proposition A.23 *Suppose f_1 and f_2 are locally integrable functions on $\mathbb{R}^n$. If*

$$\int_{\mathbb{R}^n} \chi(\mathbf{x}) f_1(\mathbf{x})\ d\mathbf{x} = \int_{\mathbb{R}^n} \chi(\mathbf{x}) f_2(\mathbf{x})\ d\mathbf{x}$$

for all $\chi \in C_c^\infty(\mathbb{R}^n)$, then $f_1(\mathbf{x}) = f_2(\mathbf{x})$ for almost every $\mathbf{x}$.

The idea now is that we allow objects that do not have values at points, but for which something like (A.4) makes sense. Mathematically, we think of (A.4) as a linear functional on $C_c^\infty(\mathbb{R}^n)$.

Definition A.24 *A sequence $\chi_m \in C_c^\infty(\mathbb{R}^n)$ is said to **converge** to $\chi \in C_c^\infty(\mathbb{R}^n)$ if (1) there exists a single compact set K containing the support of all the χ_n's, (2) χ_m converges uniformly to χ, and (3) each derivative of χ_m converges uniformly to the corresponding derivative of χ.*

Definition A.25 *A **distribution** on $\mathbb{R}^n$ is a linear map $T : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ having the following continuity property: If χ_m converges to χ in the sense of Definition A.24, $T(\chi_m)$ converges to $T(\chi)$.*

The continuity condition on T should be regarded as a technicality, in that any functional that is well defined and linear on all of $C_c^\infty(\mathbb{R}^n)$ and is obtained in a reasonably constructive fashion will satisfy this property.

Example A.26 *The Dirac δ-"function" is the distribution δ defined by*

$$\delta(\chi) = \chi(0).$$

Definition A.27 *If T is a distribution and f is a locally integrable function, the expression "T is equal to f" or "T is given by f" means that*

$$T(\chi) = \int_{\mathbb{R}^n} \chi(\mathbf{x}) f(\mathbf{x})\ d\mathbf{x}$$

for all $\chi \in C_c^\infty(\mathbb{R}^n)$.

Definition A.28 *If T is a distribution, define the distribution $\partial T/\partial x_j$ by the formula*

$$\frac{\partial T}{\partial x_j}(\chi) = -T\left(\frac{\partial \chi}{\partial x_j}\right).$$

It is easy to verify that if T has the continuity property in Definition A.25, then so does $\partial T/\partial x_j$. Furthermore, if T is given by a continuously differentiable function, then the derivative of T is in the distribution sense coincides with the derivative of T in the classical sense, as can easily be shown using integration by parts. If T is a distribution, we may define ΔT by repeated applications of Definition A.28, with the result that

$$(\Delta T)(\chi) = T(\Delta \chi).$$

Proposition A.29 *If ϕ and ψ are L^2 functions, the equation $\partial\psi/\partial x_j = \phi$ holds in the distribution sense if and only if*

$$-\left\langle \frac{\partial\chi}{\partial x_j}, \psi \right\rangle = \langle \chi, \phi \rangle$$

for all $\chi \in C_c^\infty(\mathbb{R}^n)$. Similarly, the equation $\Delta\psi = \phi$ holds in the distribution sense if and only if

$$\langle \Delta\chi, \psi \rangle = \langle \chi, \phi \rangle$$

for all $\chi \in C_c^\infty(\mathbb{R}^n)$.

Proposition A.30 *If T is a distribution on $\mathbb{R}$ and dT/dx is the zero distribution, then T is a constant, meaning that there is some constant c such that*

$$T(\chi) = \int_{-\infty}^{\infty} \chi(x)c \; dx. \tag{A.5}$$

Suppose, in particular, that if T is given by a locally integrable function f, and the derivative of T is zero. Then Proposition A.30 tells us that for some constant c, we have $\int_{-\infty}^{\infty} \chi(x)(f(x) - c) \; dx = 0$ for all $\chi \in C_c^\infty(\mathbb{R})$. Then Proposition A.23 tells us that $f(x) = c$ almost everywhere. This means that if the derivative of f is zero, even in the weak (or distributional) sense, then f must be constant.

A.3.4 Banach Spaces

In this section, we define Banach spaces and describe some of their elementary properties.

Definition A.31 *A* ***norm*** *on a vector space V over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a map from V into $\mathbb{R}$, denoted $\psi \mapsto \|\psi\|$, with the following properties.*

1. *For all $\psi \in V$, $\|\psi\| \geq 0$, with equality if and only if $\psi = 0$.*
2. *For all $\psi \in V$ and $c \in \mathbb{F}$, we have $\|c\psi\| = |c| \, \|\psi\|$.*
3. *For all $\phi, \psi \in V$, we have $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$.*

If $\|\cdot\|$ is a norm on V, then we can define a distance function d on V by setting $d(\phi, \psi) = \|\psi - \phi\|$.

Definition A.32 *A normed vector space is said to be a* ***Banach space*** *if it is complete with respect to the associated distance function. A Banach space is said to be* ***separable*** *if contains a countable dense subset.*

One important class of examples of Banach spaces are the L^p spaces.

Definition A.33 *An infinite series,* $\sum_{n=1}^{\infty}\psi_n$*, with values in normed space* V*, is said to* ***converge*** *if there exists some* $L \in V$ *such that*

$$\lim_{N\to\infty}\|S_n - L\| = 0,$$

where $S_N = \sum_{n=1}^{N}\psi_n$*.*

Proposition A.34 *If* V *is a Banach space, then absolute convergence implies convergence in* V*. That is, if*

$$\sum_{n=1}^{\infty}\|\psi_n\| < \infty,$$

then $\sum_{n=1}^{\infty}\psi_n$ *converges in* V*.*

Definition A.35 *If* V_1 *and* V_2 *are normed spaces, a linear map* $T : V_1 \to V_2$ *is* ***bounded*** *if*

$$\sup_{\psi\in V_1\setminus\{0\}}\frac{\|T\psi\|}{\|\psi\|} < \infty. \tag{A.6}$$

If T *is bounded, then the supremum in (A.6) is called the* ***operator norm*** *of* T*, denoted* $\|T\|$*.*

Theorem A.36 (Bounded Linear Transformation Theorem) *Let* V_1 *be a normed space and* V_2 *a Banach space. Suppose* W *is a dense subspace of* V_1 *and* $T : W \to V_2$ *is a bounded linear map. Then there exists a unique bounded linear map* $\tilde{T} : V_1 \to V_2$ *such that* $\tilde{T}|_W = T$*. Furthermore, the norm of* $\tilde{T}$ *equals the norm of* T*.*

Definition A.37 *If* V *is a normed space over* $\mathbb{F}$ *(*$\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*), then a* ***bounded linear functional*** *on* V *is a bounded linear map of* V *into* $\mathbb{F}$*, where on* $\mathbb{F}$ *we use the norm given by the absolute value. The collection of all bounded linear functionals, with the norm given by (A.6), is called the* ***dual space*** *to* V*, denoted* V^**.*

Theorem A.38 *If* V *is a normed vector space, then the following results hold.*

1. *The dual space* V^* *is a Banach space.*
2. *For all* $\psi \in V$*, there exists a nonzero* $\xi \in V^*$ *such that*

   $$|\xi(\psi)| = \|\xi\|\,\|\psi\|\,.$$

   In particular, if $\xi(\psi) = 0$ *for all* $\xi \in V^*$*, then* $\psi = 0$*.*

Theorem A.39 (Closed Graph Theorem) *Suppose that* V_1 *is a Banach space and* V_2 *a normed vector space. For any linear map* $T : V_1 \to V_2$*, let* $Graph(T)$ *denote the set of pairs* $(\psi, T\psi)$ *in* $V_1 \times V_2$ *such that* $\psi \in V_1$*. If the graph of* T *is a closed subset of* $V_1 \times V_2$*, then* T *is bounded.*

Here is a simple example of how the closed graph theorem can be applied. Suppose V_1 and V_2 are Banach spaces and $T : V_1 \to V_2$ is a linear map that is one-to-one, onto, and bounded. Then the inverse map $T^{-1} : V_2 \to V_1$ is automatically bounded. To verify this, we first check that if T is bounded, then the graph of T is closed (easy). Then we observe that the graph of T^{-1} is also closed, since it is obtained from the graph of T by the map $(\phi, \psi) \mapsto (\psi, \phi)$. Thus, the theorem tells us that T^{-1} is bounded.

Theorem A.40 (Principle of Uniform Boundedness) *Suppose* $\{T_\alpha\}$ *is any family of bounded linear maps from a Banach space* V_1 *to a normed space* V_2*. Suppose that for each* $\psi \in V_1$*, there is a constant* C_ψ *such that* $\|T_\alpha \psi\| \leq C_\psi$ *for all* α*. Then there exists a constant* C *such that* $\|T_\alpha\| \leq C$ *for all* α*.*

That is, in contrapositive form, if the family $\{T_\alpha\}$ is unbounded, $\{T_\alpha \psi\}$ must be unbounded on ψ for some $\psi \in V_1$.

Corollary A.41 *Suppose* V *is a Banach space and* E *is a nonempty subset of* V*. Suppose that for all* $\xi \in V^*$ *there exists a constant* C_ξ *such that* $|\xi(\psi)| \leq C_\xi$ *for all* $\psi \in E$*. Then* E *is a bounded set.*

The corollary is obtained by identifying each $\psi \in V$ with the linear map $e_\psi : V^* \to \mathbb{C}$ given by evaluation on ψ; that is, $e_\psi(\xi) = \xi(\psi)$. Note that by Point 2 of Theorem A.38, the norm of e_ψ as an element of V^{**} is equal to the norm of ψ as an element of V.

A.4 Hilbert Spaces and Operators on Them

A.4.1 Inner Product Spaces and Hilbert Spaces

We now introduce a generalization to arbitrary vector spaces over $\mathbb{R}$ or $\mathbb{C}$ of the usual inner product (or dot product) on $\mathbb{R}^n$.

Definition A.42 *An* ***inner product*** *on a vector space over* $\mathbb{F}$ *(*$\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*) is a map* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ *with the following properties.*

1. *For all* $\phi, \psi \in V$*, we have* $\langle \psi, \phi \rangle = \overline{\langle \phi, \psi \rangle}$*.*
2. *For all* $\phi \in V$*,* $\langle \phi, \phi \rangle$ *is real and non-negative, and* $\langle \phi, \phi \rangle = 0$ *only if* $\phi = 0$*.*
3. *For all* $\phi, \psi \in V$ *and* $c \in \mathbb{F}$*, we have* $\langle c\phi, \psi \rangle = \bar{c} \langle \phi, \psi \rangle$ *and* $\langle \phi, c\psi \rangle = c \langle \phi, \psi \rangle$*.*
4. *For all* $\phi, \psi, \chi \in V$*, we have* $\langle \phi + \psi, \chi \rangle = \langle \phi, \chi \rangle + \langle \psi, \chi \rangle$ *and*
$$\langle \phi, \psi + \chi \rangle = \langle \phi, \psi \rangle + \langle \phi, \chi \rangle .$$

Note that we are following the physics convention of taking the complex conjugate in Point 3 of the definition on the *first* factor in the inner product.

Proposition A.43 *If V is an inner product space, then for all $\phi, \psi \in V$, we have the* ***Cauchy–Schwarz inequality****:*

$$|\langle \phi, \psi \rangle|^2 \leq \langle \phi, \phi \rangle \langle \psi, \psi \rangle .$$

Furthermore, if $\|\cdot\| : V \to \mathbb{R}$ is defined by

$$\|\psi\| = \sqrt{\langle \psi, \psi \rangle}, \tag{A.7}$$

then $\|\cdot\|$ is a norm on V.

Definition A.44 *A* ***Hilbert space*** *is a vector space $\mathbf{H}$ over $\mathbb{R}$ or $\mathbb{C}$, equipped with an inner product $\langle \cdot, \cdot \rangle$, such that $\mathbf{H}$ is complete in the norm given by (A.7).*

That is to say, a Hilbert space is a Banach space in which the norm comes from an inner product. In Appendix A.4 only, we allow $\mathbf{H}$ to denote an arbitrary Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. (In the main body of the text, $\mathbf{H}$ denotes a separable complex Hilbert space.)

Definition A.45 *Suppose $\mathbf{H}_j$ is a sequence of separable Hilbert spaces. Then the* ***Hilbert space direct sum****, denoted*

$$\mathbf{H} := \bigoplus_{j=1}^{\infty} \mathbf{H}_j,$$

is the space of sequences $\psi = (\psi_1, \psi_2, \psi_3, \ldots)$ such that $\psi_n \in \mathbf{H}_n$ and such that

$$\|\psi\|^2 := \sum_{j=1}^{\infty} \|\psi_j\|_j^2 < \infty. \tag{A.8}$$

The ***finite direct sum*** *of the $\mathbf{H}_j$'s is the set of $\psi = (\psi_1, \psi_2, \psi_3, \ldots)$ such that $\psi_j = 0$ for all but finitely many values of j.*

We define an inner product on the direct sum by setting

$$\langle \phi, \psi \rangle = \sum_{j=1}^{\infty} \langle \phi_j, \psi_j \rangle \tag{A.9}$$

for all $\phi, \psi \in \mathbf{H}$. This inner product is well defined and $\mathbf{H}$ is complete with respect to this inner product, and hence a Hilbert space.

One important example of a Hilbert space is $L^2(X, \mu)$, where (X, μ) is a measure space.

Definition A.46 *If* (X,μ) *is a measure space, define an inner product on* $L^2(X,\mu)$ *by the formula*

$$\langle \phi, \psi \rangle = \int_X \overline{\phi(x)}\psi(x)\ d\mu(x). \tag{A.10}$$

A standard result in measure theory states that the integral on the right-hand side of (A.10) is absolutely convergent for all ϕ and ψ in $L^2(X,\mu)$. It is then easy to verify that $\langle\cdot,\cdot\rangle$ is indeed an inner product on $L^2(X,\mu)$. Another standard result states that $L^2(X,\mu)$ is complete with respect to the norm associated with the inner product in (A.10); thus, $L^2(X,\mu)$ is a Hilbert space.

A.4.2 Orthogonality

One reason that Hilbert spaces are nicer to work with than general Banach spaces is that we have the concept of orthogonality.

Definition A.47 *Two elements* ϕ *and* ψ *of an inner product space are* ***orthogonal*** *if* $\langle \phi, \psi \rangle = 0$.

Definition A.48 *If* V *is any subspace of* $\mathbf{H}$*, define a subspace* $V^\perp$ *of* $\mathbf{H}$ *by*

$$V^\perp = \{\phi \in \mathbf{H} | \langle \phi, \psi \rangle = 0 \text{ for all } \psi \in V\}.$$

Then $V^\perp$ *is called the* ***orthogonal space*** *of* V.

Proposition A.49

1. *If* V *is a* closed *subspace of* $\mathbf{H}$, *every* $\psi \in \mathbf{H}$ *can be decomposed uniquely as* $\psi = \psi_1 + \psi_2$, *with* $\psi_1 \in V$ *and* $\psi_2 \in V^\perp$.
2. *If* V *is any subspace of* $\mathbf{H}$, *then* $(V^\perp)^\perp = \overline{V}$, *where* $\overline{V}$ *is the closure of* V. *In particular, if* V *is closed, then* $(V^\perp)^\perp = V$.

If V is closed, we call $V^\perp$ the *orthogonal complement* of V.

Definition A.50 *A set* $\{e_j\}$ *of elements of* $\mathbf{H}$, *where* j *ranges over an arbitrary index set, is said to be* ***orthonormal*** *if*

$$\langle e_j, e_k \rangle = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}.$$

An orthonormal set $\{e_j\}$ *is an* ***orthonormal basis*** *for* $\mathbf{H}$ *if the space of finite linear combinations of the* e_j*'s is dense in* $\mathbf{H}$.

If $\mathbf{H} = L^2([-L,L])$, for some positive number L, then the functions,

$$\psi_n = \frac{1}{\sqrt{2L}} e^{2\pi i n x/L}, \quad n \in \mathbb{Z}, \tag{A.11}$$

form an orthonormal basis for $\mathbf{H}$.

Proposition A.51 *Suppose $\{e_j\}$ is an orthonormal basis for $\mathbf{H}$. Then every ψ can be expressed uniquely as a convergent sum*

$$\psi = \sum_j a_j e_j, \tag{A.12}$$

where the coefficients are given by $a_j = \langle e_j, \psi \rangle$. If ψ is as in (A.12), then

$$\|\psi\|^2 = \sum_j |a_j|^2 .$$

Finally, if $\langle a_j \rangle$ is any sequence such that $\sum_j |a_j|^2 < \infty$, there exists a unique $\psi \in \mathbf{H}$ such that $\langle e_j, \psi \rangle = a_j$ for all j.

In the case that the orthonormal basis is the one in (A.11), the resulting series (A.12) is called the *Fourier series* of ψ.

A.4.3 The Riesz Theorem and Adjoints

We let $\mathcal{B}(\mathbf{H})$ denote the space of bounded linear maps of $\mathbf{H}$ to $\mathbf{H}$. It is not hard to show that $\mathcal{B}(\mathbf{H})$ forms a Banach space under the operator norm.

Theorem A.52 (Riesz Theorem) *If $\xi : \mathbf{H} \to \mathbb{C}$ is a bounded linear functional, then there exists a unique $\chi \in \mathbf{H}$ such that*

$$\xi(\psi) = \langle \chi, \psi \rangle$$

for all $\psi \in \mathbf{H}$. Furthermore, the operator norm of ξ as a linear functional is equal to the norm of χ as an element of $\mathbf{H}$.

We now turn to the concept of the *adjoint* of a bounded operator, along with the related concept of *quadratic forms* on $\mathbf{H}$.

Proposition A.53 *For any $A \in \mathcal{B}(\mathbf{H})$, there exists a unique linear operator $A^* : \mathbf{H} \to \mathbf{H}$, called the* ***adjoint*** *of A, such that*

$$\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$$

for all $\phi, \psi \in \mathbf{H}$. For all $A, B \in \mathcal{B}(\mathbf{H})$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned} (A^*)^* &= A \\ (AB)^* &= B^*A^* \\ (\alpha A + \beta B)^* &= \bar{\alpha} A^* + \bar{\beta} B^* \\ I^* &= I. \end{aligned}$$

The operator A^* is bounded and $\|A^*\| = \|A\|$.

Since A is a bounded operator, the map $\psi \mapsto \langle \phi, A\psi \rangle$ is a bounded linear functional for each fixed $\phi \in \mathbf{H}$. The Riesz theorem then tells us that there is a unique $\chi \in \mathbf{H}$ such that $\langle \phi, A\psi \rangle = \langle \chi, \psi \rangle$. The operator A^* is defined by setting $A^*\phi = \chi$. It is not hard to check that this definition makes A^* into a bounded linear operator.

Definition A.54 *An operator $A \in \mathcal{B}(\mathbf{H})$ is said to be* ***self-adjoint*** *if $A^* = A$ and* ***skew-self-adjoint*** *if $A^* = -A$.*

Definition A.55 *An operator U on $\mathbf{H}$ is* ***unitary*** *if U is surjective and preserves inner products, that is, $\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle$ for all $\phi, \psi \in \mathbf{H}$.*

If U is unitary, then U preserves norms ($\|U\psi\| = \|\psi\|$ for all $\psi \in \mathbf{H}$); therefore, U is bounded with $\|U\| = 1$. By the polarization identity (Proposition A.59), if U preserves norms, then it also preserves inner products.

Proposition A.56 *A bounded operator U is unitary if and only if $U^* = U^{-1}$, that is, if and only if $UU^* = U^*U = I$.*

Proposition A.57 *For any closed subspace $V \subset \mathbf{H}$, there is a unique bounded operator P such that $P = I$ on V and $P = 0$ on the orthogonal complement $V^\perp$. This operator is called the* ***orthogonal projection*** *onto V and it satisfies $P^2 = P$ and $P^* = P$.*

Conversely, if P is any bounded operator on $\mathbf{H}$ satisfying $P^2 = P$ and $P^* = P$, then P is the orthogonal projection onto a closed subspace V, where $V = \mathrm{range}(P)$.

A.4.4 Quadratic Forms

In this section, we develop the theory of quadratic forms on Hilbert spaces. Since this is customarily done only for the inner product itself, we include the proofs of the results.

Definition A.58 *A* ***sesquilinear form*** *on $\mathbf{H}$ is a map $L : \mathbf{H} \times \mathbf{H} \to \mathbb{C}$ that is conjugate linear in the first factor and linear in the second factor. A sesquilinear form is* ***bounded*** *if there exists a constant C such that*

$$|L(\phi, \psi)| \leq C \|\phi\| \|\psi\|$$

for all $\phi, \psi \in \mathbf{H}$.

Proposition A.59 *If L is a sesquilinear form on $\mathbf{H}$, L can be recovered from its values on the diagonal (i.e., the value of $L(\psi, \psi)$ for various ψ's) as follows:*

$$\begin{aligned} L(\phi, \psi) &= \frac{1}{2} \left[L(\phi + \psi, \phi + \psi) - L(\phi, \phi) - L(\psi, \psi) \right] \\ &\quad - \frac{i}{2} \left[L(\phi + i\psi, \phi + i\psi) - L(\phi, \phi) - L(i\psi, i\psi) \right]. \end{aligned} \tag{A.13}$$

This formula is known as the ***polarization identity****.*

Note that we do not assume any relationship between $L(\phi,\psi)$ and $L(\psi,\phi)$.
Proof. Direct calculation. ■

Definition A.60 *A* ***quadratic form*** *on a Hilbert space* $\mathbf{H}$ *is a map* $Q:\mathbf{H}\rightarrow\mathbb{C}$ *with the following properties: (1)* $Q(\lambda\psi)=|\lambda|^2\,Q(\psi)$ *for all* $\psi\in\mathbf{H}$ *and* $\lambda\in\mathbb{C}$*, and (2) the map* $L:\mathbf{H}\times\mathbf{H}\rightarrow\mathbb{C}$ *defined by*

$$\begin{aligned}L(\phi,\psi)&=\frac{1}{2}\left[Q(\phi+\psi)-Q(\phi)-Q(\psi)\right]\\&\quad-\frac{i}{2}\left[Q(\phi+i\psi)-Q(\phi)-Q(i\psi)\right]\end{aligned}$$

is a sesquilinear form. A quadratic form Q *is* ***bounded*** *if there exists a constant* C *such that*

$$|Q(\phi)|\leq C\left\|\phi\right\|^2$$

for all $\phi\in\mathbf{H}$*. The smallest such constant* C *is the* ***norm*** *of* Q*.*

Proposition A.61 *If* Q *is a quadratic form on* $\mathbf{H}$ *and* L *is the associated sesquilinear form, we have the following results.*

1. *For all* $\psi\in\mathbf{H}$*, we have* $Q(\psi)=L(\psi,\psi)$*.*
2. *If* Q *is a bounded, then* L *is bounded.*
3. *If* $Q(\psi)$ *belongs to* $\mathbb{R}$ *for all* $\psi\in\mathbf{H}$*, then* L *is conjugate symmetric, that is,*
$$L(\phi,\psi)=\overline{L(\psi,\phi)}$$
for all $\phi,\psi\in\mathbf{H}$*.*

Proof. Point 1 of the proposition is verified by taking $\phi=\psi$ in the expression for $L(\phi,\psi)$ and then using the relation $Q(\lambda\psi)=|\lambda|^2\,Q(\psi)$. For Point 2, suppose $|Q(\psi)|\leq C\left\|\psi\right\|^2$ for all $\psi\in\mathbf{H}$. If $\|\phi\|=\|\psi\|=1$, then $\phi+\psi$ and $\phi+i\psi$ have norm at most 2, and so

$$|L(\phi,\psi)|\leq\frac{1}{2}C\left(4+1+1+4+1+1\right)=6C.$$

Now, for any ϕ and ψ in $\mathbf{H}$, we can find unit vectors $\tilde{\phi}$ and $\tilde{\psi}$ such that $\phi=\|\phi\|\,\tilde{\phi}$ and $\psi=\|\psi\|\,\tilde{\psi}$. Then since L is assumed to be sesquilinear, we have

$$|L(\phi,\psi)|=\|\phi\|\,\|\psi\|\,L\left(\tilde{\phi},\tilde{\psi}\right)\leq 6C\,\|\phi\|\,\|\psi\|\,,$$

showing that L is bounded.

For Point 3, assume that $Q(\psi)$ is real for all $\psi\in\mathbf{H}$ and define a map $M:\mathbf{H}\times\mathbf{H}\rightarrow\mathbb{R}$ by

$$M(\phi,\psi)=\frac{1}{2}\left[Q(\phi+\psi)-Q(\phi)-Q(\psi)\right]=\mathrm{Re}\left[L(\phi,\psi)\right].$$

Then M is real-bilinear (because it is the real part of L) and symmetric (because of the expression for M in terms of Q). Furthermore, $M(i\phi, i\psi) = M(\phi, \psi)$. These properties of M show that $M(\phi, i\psi) = -M(\psi, i\phi)$, and so

$$\begin{aligned} L(\phi, \psi) &= M(\phi, \psi) - iM(\phi, i\psi) \\ &= M(\psi, \phi) + iM(\psi, i\phi) \\ &= \overline{L(\psi, \phi)}, \end{aligned}$$

which is what we wanted to prove. ■

Example A.62 *If A is a bounded operator on $\mathbf{H}$, one can construct a bounded quadratic form Q_A on $\mathbf{H}$ by setting*

$$Q_A(\psi) = \langle \psi, A\psi \rangle, \quad \psi \in \mathbf{H}.$$

The associated sesquilinear form L_A is then given by

$$L_A(\phi, \psi) = \langle \phi, A\psi \rangle, \quad \phi, \psi \in \mathbf{H}.$$

Proposition A.63 *If Q is a bounded quadratic form on $\mathbf{H}$, there is a unique $A \in \mathcal{B}(\mathbf{H})$ such that $Q(\psi) = \langle \psi, A\psi \rangle$ for all $\psi \in \mathbf{H}$. If $Q(\psi)$ belongs to $\mathbb{R}$ for all $\psi \in \mathbf{H}$, then the operator A is self-adjoint.*

Proof. Since Q is bounded, L is also bounded, meaning that there exists a constant C such that $|L(\phi, \psi)| \leq C \|\phi\| \|\psi\|$ for all $\phi, \psi \in \mathbf{H}$. Thus, for any $\phi \in \mathbf{H}$, the linear functional $\psi \mapsto L(\phi, \psi)$ is bounded, with norm at most $C \|\phi\|$. By the Riesz theorem, then, there exists a unique $\chi \in \mathbf{H}$, with $\|\chi\| \leq C \|\phi\|$, such that $L(\phi, \psi) = \langle \chi, \psi \rangle$. We now define a map $B : \mathbf{H} \to \mathbf{H}$ by defining $B\phi = \chi$. Direct calculation shows that B is linear, and the inequality $\|\chi\| \leq C \|\phi\|$ shows that B is bounded. Setting $A = B^*$ establishes the existence of the desired operator. Uniqueness of A follows from the observation that if $\langle \phi, A\psi \rangle = 0$ for all $\phi, \psi \in \mathbf{H}$, then A is the zero operator.

If $Q(\psi)$ is real for all $\psi \in \mathbf{H}$, then by Point 3 of Proposition A.61, L is conjugate symmetric. Thus,

$$\langle \phi, A\psi \rangle = L(\phi, \psi) = \overline{L(\psi, \phi)} = \overline{\langle \psi, A\phi \rangle} = \langle A\phi, \psi \rangle$$

for all $\phi, \psi \in \mathbf{H}$, showing that A is self-adjoint. ■

A.4.5 Tensor Products of Hilbert Spaces

Recall from Appendix A.1 the concept of the tensor product of two vector spaces.

Proposition A.64 *Suppose* V_1 *and* V_2 *are inner product spaces, with inner products* $\langle\cdot,\cdot\rangle_1$ *and* $\langle\cdot,\cdot\rangle_2$. *Then there exists a unique inner product* $\langle\cdot,\cdot\rangle$ *on* $V_1\otimes V_2$ *such that*

$$\langle u_1\otimes v_1, u_2\otimes v_2\rangle = \langle u_1,u_2\rangle_1 \langle v_1\otimes v_2\rangle_2$$

for all $u_1,u_2\in V_1$ *and* $v_1,v_2\in V_2$.

If $\mathbf{H}_1$ and $\mathbf{H}_2$ are Hilbert spaces, then we can equip the tensor product $\mathbf{H}_1\otimes\mathbf{H}_2$ with the inner product in Proposition A.64. If $\mathbf{H}_1$ and $\mathbf{H}_2$ are both infinite dimensional, however, $\mathbf{H}_1\otimes\mathbf{H}_2$ will not be complete with respect to this inner product. Nevertheless, we can complete $\mathbf{H}_1\otimes\mathbf{H}_2$ with respect to this inner product, thus obtaining a new Hilbert space.

Definition A.65 *If* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *are Hilbert spaces, then the* ***Hilbert tensor product*** *of* $\mathbf{H}_1$ *and* $\mathbf{H}_2$, *denoted* $\mathbf{H}_1\hat{\otimes}\mathbf{H}_2$, *is the Hilbert space obtained by completing* $\mathbf{H}_1\otimes\mathbf{H}_2$ *with respect to the inner product in Proposition A.64.*

Proposition A.66 *If* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *are Hilbert spaces with orthonormal bases* $\{e_j\}$ *and* $\{f_k\}$, *respectively, then* $\{e_j\otimes f_k\}$ *is an orthonormal basis for the Hilbert space* $\mathbf{H}_1\hat{\otimes}\mathbf{H}_2$.

Proposition A.67 *If* A *is a bounded operator on* $\mathbf{H}_1$ *and* B *is a bounded operator on* $\mathbf{H}_2$, *then there exists a unique bounded operator on* $\mathbf{H}_1\hat{\otimes}\mathbf{H}_2$, *denoted* $A\otimes B$, *such that*

$$(A\otimes B)(\phi\otimes\psi) = (A\phi)\otimes(B\psi)$$

for all $\phi\in\mathbf{H}_1$ *and* $\psi\in\mathbf{H}_2$.

To see that $A\otimes B$ is bounded, first write $A\otimes B$ as $(A\otimes I)(I\otimes B)$. Then, given any orthonormal basis $\{f_j\}$ for $\mathbf{H}_2$, we can decompose $\mathbf{H}_1\hat{\otimes}\mathbf{H}_2$ as the Hilbert space direct sum of subspaces of the form $\mathbf{H}_1\otimes f_j$. The operator $A\otimes I$ acts on this decomposition as a block-diagonal operator with A in each diagonal block. From this, it is easy to verify that $\|A\otimes I\| = \|A\|$. A similar argument shows that $\|I\otimes B\| = \|B\|$, and so

$$\|A\otimes B\| \leq \|A\otimes I\|\,\|I\otimes B\| = \|A\|\,\|B\|.$$

Meanwhile, by taking a sequence of unit vector $\phi_n\in\mathbf{H}_1$ and $\psi_n\in\mathbf{H}_2$ with $\|A\phi_n\|\to\|A\|$ and $\|B\psi_n\|\to\|B\|$, we see that the reverse inequality holds, and thus that $\|A\otimes B\| = \|A\|\,\|B\|$.

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