

Hints for Some Exercises

Chapter 1

- 1.5 Consider $A(H_n)^2$ and use Exercise 3.
- 1.6 (a) First count the number of sequences $V_{i_0}, V_{i_1}, \dots, V_{i_\ell}$ for which there exists a closed walk with vertices $v_0, v_1, \dots, v_\ell = v_0$ (in that order) such that $v_j \in V_{i_j}$.
- 1.11 Consider the rank of $A(\Gamma)$ and also consider $A(\Gamma)^2$. The answer is very simple and does not involve irrational numbers.
- 1.12 (b) Consider $A(G)^2$.

Chapter 2

- 2.2 See Exercise 9 in Chap. 9.
- 2.5 (c) Mimic the proof for the graph C_n , using the definition

$$\langle \chi_u, \chi_v \rangle = \sum_{w \in \mathbb{Z}_n} \chi_u(w) \overline{\chi_v(w)},$$

where an overhead bar denotes complex conjugation.

Chapter 3

- 3.4 You may find Example 3.1 useful.
- 3.7 It is easier *not* to use linear algebra.
- 3.8 See previous hint.
- 3.10 First show (easy) that if we start at a vertex v and take n steps (using our random walk model), then the probability that we traverse a fixed closed walk W is equal to the probability that we traverse W in reverse order.
- 3.12 See hint for Exercise 7.

Chapter 4

- 4.4** (b) One way to do this is to count in two ways the number of k -tuples (v_1, \dots, v_k) of linearly independent elements from \mathbb{F}_q^m : (1) first choose v_1 , then v_2 , etc., and (2) first choose the subspace W spanned by v_1, \dots, v_k , and then choose v_1, v_2 , etc.
- 4.4** (c) The easiest way is to use (b).

Chapter 5

- 5.5** (a) Show that $N_n \cong B_n/G$ for a suitable group G .
- 5.9** (a) Use Corollary 2.4 with $n = \binom{p}{2}$.
- 5.13** Use Exercise 12.

Chapter 6

- 6.2** (b) Not really a hint, but the result is equivalent [why?] to the case $r = m$, $s = n$, $t = 2$, and $x = 1$ of Exercise 34 in Chap. 8.
- 6.3** Consider $\mu = (8, 8, 4, 4)$.
- 6.5** First consider the case where S has ζ elements equal to 0 (so $\zeta = 0$ or 1), ν elements that are negative, and π elements that are positive, so $\nu + \zeta + \pi = 2m + 1$.

Chapter 7

- 7.16** (a) Use Pólya's theorem.

Chapter 8

- 8.3** Encode a maximal chain by an object that we already know how to enumerate.
- 8.7** Partially order by diagram inclusion the set of all partitions whose diagrams can be covered by nonoverlapping dominos, thereby obtaining a subposet Y_2 of Young's lattice Y . Show that $Y_2 \cong Y \times Y$.
- 8.14** Use induction on n .
- 8.17** (a) One way to do this is to use the generating function $\sum_{n \geq 0} Z_{\mathfrak{S}_n}(z_1, z_2, \dots) x^n$ for the cycle indicator of \mathfrak{S}_n (Theorem 7.13). Another method is to find a recurrence for $B(n+1)$ in terms of $B(0), \dots, B(n)$ and then convert this recurrence into a generating function
- 8.18** Consider the generating function

$$G(q, t) = \sum_{k, n \geq 0} \kappa(n \rightarrow n+k \rightarrow n) \frac{t^k q^n}{(k!)^2}$$

and use (8.25).

- 8.20** (b) Consider the square of the adjacency matrix of $Y_{j-1, j}$.
- 8.24** Use Exercise 14.

Chapter 9

- 9.1** There is a simple proof based on the formula $\kappa(K_p) = p^{p-2}$, avoiding the Matrix-Tree Theorem.
- 9.2** (c) Use the fact that the rows of L sum to 0 and compute the trace of L .
- 9.5** (b) Use Exercise 3 in Chap. 1.
- 9.6** (a) For the most elegant proof, use the fact that commuting $p \times p$ matrices A and B can be simultaneously triangularized, i.e., there exists an invertible matrix X such that both XAX^{-1} and XBX^{-1} are upper triangular.
- 9.6** (d) Use Exercise 8(a).
- 9.7** Let G^* be the full dual graph of G , i.e., the vertices of G^* are the faces of G , including the outside face. For every edge e of G separating two faces R and S of G , there is an edge e^* of G^* connecting the vertices R and S . Thus G^* will have some multiple edges and $\#E(G) = \#E(G^*)$. First show combinatorially that $\kappa(G) = \kappa(G^*)$. (See Corollary 11.19.)
- 9.10** (a) The laplacian matrix $L = L(G)$ acts on the space $\mathbb{R}V(G)$, the real vector space with basis $V(G)$. Consider the subspace W of $\mathbb{R}V(G)$ spanned by the elements $v + \varphi(v)$, $v \in V(G)$.
- 9.11** (a) Let $s(n, q, r)$ be the number of $n \times n$ symmetric matrices of rank r over \mathbb{F}_q . Find a recurrence satisfied by $s(n, q, r)$ and verify that this recurrence is satisfied by

$$s(n, q, r) = \begin{cases} \prod_{i=1}^t \frac{q^{2i}}{q^{2i} - 1} \cdot \prod_{i=0}^{2t-1} (q^{n-i} - 1), & 0 \leq r = 2t \leq n, \\ \prod_{i=1}^t \frac{q^{2i}}{q^{2i} - 1} \cdot \prod_{i=0}^{2t} (q^{n-i} - 1), & 0 \leq r = 2t + 1 \leq n. \end{cases}$$

- 9.12** Any of the three proofs of the Appendix to Chap. 9 can be carried over to the present exercise.

Chapter 10

- 10.3** (b) Use the Perron–Frobenius theorem (Theorem 3.3).
- 10.6** (a) Consider A^ℓ .
- 10.6** (f) There is an example with nine vertices that is not a de Bruijn graph.
- 10.6** (c) Let E be the (column) eigenvector of $A(D)$ corresponding to the largest eigenvalue. Consider AE and $A^t E$, where t denotes transpose.

Chapter 11

- 11.4** Use the unimodularity of the basis matrices C_T and B_T .
- 11.6** (a) Mimic the proof of Theorem 9.8 (the Matrix-Tree Theorem).
- 11.6** (b) Consider ZZ^t .

Chapter 12

- 12.4** The best strategy involves the concept of odd and even permutations.
- 12.5** For the easiest solution, don't use linear algebra but rather use the original Oddtown theorem.
- 12.12** What are the eigenvalues of skew-symmetric matrices?
- 12.15** Consider the incidence matrix M of the sets and their elements. Consider two cases: $\det M = 0$ and $\det M \neq 0$.
- 12.18** Consider the first three rows of H . Another method is to use row operations to factor a large power of 2 from the determinant.
- 12.21** It is easiest to proceed directly and not use the proof of Theorem 12.18.
- 12.23** First find a simple explicit formula for the generating function $\sum_{n \geq 0} f(n)x^n$.
- 12.26** Differentiate with respect to x (12.10) satisfied by y .

Bibliography

1. I. Anderson, *Combinatorics of Finite Sets* (Oxford University Press, Oxford, 1987); Corrected republication by Dover, New York, 2002
2. G.E. Andrews, *The Theory of Partitions* (Addison-Wesley, Reading, 1976)
3. G.E. Andrews, K. Eriksson, *Integer Partitions* (Cambridge University Press, Cambridge, 2004)
4. L. Babai, P. Frankl, *Linear Algebra Methods in Combinatorics*, preliminary version 2 (1992), 216 pp.
5. E.A. Beem, Craig and Irene Schensted don't have a car in the world, in *Maine Times* (1982), pp. 20–21
6. E.A. Bender, D.E. Knuth, Enumeration of plane partitions. *J. Comb. Theor.* **13**, 40–54 (1972)
7. E.R. Berlekamp, On subsets with intersections of even cardinality. *Can. Math. Bull.* **12**, 363–366 (1969)
8. H. Bidkhori, S. Kishore, Counting the spanning trees of a directed line graph. Preprint [arXiv:0910.3442]
9. E.D. Bolker, The finite Radon transform, in *Integral Geometry*, Brunswick, Maine, 1984. Contemporary Mathematics, vol. 63 (American Mathematical Society, Providence, 1987), pp. 27–50
10. C.W. Borchardt, Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante. *J. Reine Angew. Math. (Crelle's J.)* **57**, 111–121 (1860)
11. R.C. Bose, A note on Fisher's inequality for balanced incomplete block designs. *Ann. Math. Stat.* 619–620 (1949)
12. F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, in *Jerusalem Combinatorics '93*. Contemporary Mathematics, vol. 178 (American Mathematical Society, Providence, 1994)
13. A.E. Brouwer, W.H. Haemers, *Spectra of Graphs* (Springer, New York, 2012)
14. W. Burnside, *Theory of Groups of Finite Order* (Cambridge University Press, Cambridge, 1897)
15. W. Burnside, *Theory of Groups of Finite Order*, 2nd edn. (Cambridge University Press, Cambridge, 1911); Reprinted by Dover, New York, 1955
16. Y. Caro, Simple proofs to three parity theorems. *Ars Combin.* **42**, 175–180 (1996)
17. N. Caspard, B. Leclerc, B. Monjardet, in *Finite Ordered Sets*. Encyclopedia of Mathematics and Its Applications, vol. 144 (Cambridge University Press, Cambridge, 2012)
18. A.L. Cauchy, Mémoire sur diverses propriétés remarquables des substitutions régulières ou irrégulières, et des systèmes de substitutions conjuguées (suite). *C. R. Acad. Sci. Paris* **21**, 972–987 (1845); Oeuvres Ser. 1 **9**, 371–387

19. A. Cayley, Note sur une formule pour la réversion des séries. *J. Reine Angew. Math. (Crelle's J.)* **52**, 276–284 (1856)
20. A. Cayley, A theorem on trees. *Q. J. Math.* **23**, 376–378 (1889); *Collected Papers*, vol. 13 (Cambridge University Press, Cambridge, 1897), pp. 26–28
21. E. Curtin, M. Warshauer, The locker puzzle. *Math. Intelligencer* **28**, 28–31 (2006)
22. D.M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs: Theory and Applications*, 3rd edn. (Johann Ambrosius Barth, Heidelberg/Leipzig, 1995)
23. D.M. Cvetković, P. Rowlinson, S. Simić, in *An Introduction to the Theory of Graph Spectra*. London Mathematical Society. Student Texts, vol. 75 (Cambridge University Press, Cambridge, 2010)
24. N.G. de Bruijn, A combinatorial problem. *Proc. Koninklijke Nederlandse Akademie v. Wetenschappen* **49**, 758–764 (1946); *Indagationes Math.* **8**, 461–467 (1946)
25. N.G. de Bruijn, Pólya's theory of counting, in *Applied Combinatorial Mathematics*, ed. by E.F. Beckenbach (Wiley, New York, 1964); Reprinted by Krieger, Malabar, FL, 1981
26. N.G. de Bruijn, Acknowledgement of priority to C. Flye Sainte-Marie on the counting of circular arrangements of 2^n zeros and ones that show each n -letter word exactly once, Technische Hogeschool Eindhoven, T.H.-Report 75-WSK-06, 1975
27. M.R. DeDeo, E. Velasquez, The Radon transform on \mathbb{Z}_n^k . *SIAM J. Discrete Math.* **18**, 472–478 (electronic) (2004/2005)
28. P. Diaconis, R.L. Graham, The Radon transform on \mathbb{Z}_2^k . *Pac. J. Math.* **118**, 323–345 (1985)
29. P. Diaconis, R.L. Graham, *Magical Mathematics* (Princeton University Press, Princeton, 2012)
30. E.B. Dynkin, Some properties of the weight system of a linear representation of a semisimple Lie group (in Russian). *Dokl. Akad. Nauk SSSR (N.S.)* **71**, 221–224 (1950)
31. E.B. Dynkin, The maximal subgroups of the classical groups. *Am. Math. Soc. Transl. Ser. 2* **6**, 245–378 (1957); Translated from *Trudy Moskov. Mat. Obsc.* **1**, 39–166
32. K. Engel, in *Sperner Theory*. Encyclopedia of Mathematics and Its Applications, vol. 65 (Cambridge University Press, Cambridge, 1997)
33. P. Fishburn, *Interval Orders and Interval Graphs: A Study of Partially Ordered Sets* (Wiley, New York, 1985)
34. R.A. Fisher, An examination of the different possible solutions of a problem in incomplete blocks. *Ann. Eugen.* **10**, 52–75 (1940)
35. C. Flye Sainte-Marie, Solution to question nr. 48. *l'Intermédiaire des Mathématiciens* **1**, 107–110 (1894)
36. S. Fomin, Duality of graded graphs. *J. Algebr. Combin.* **3**, 357–404 (1994)
37. S. Fomin, Schensted algorithms for dual graded graphs. *J. Algebr. Combin.* **4**, 5–45 (1995)
38. J.S. Frame, G. de B. Robinson, R.M. Thrall, The hook graphs of S_n . *Can. J. Math.* **6** 316–324 (1954)
39. D.S. Franzblau, D. Zeilberger, A bijective proof of the hook-length formula. *J. Algorithms* **3**, 317–343 (1982)
40. F.G. Frobenius, Über die Congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul. *J. Reine Angew. Math. (Crelle's J.)* **101**, 273–299 (1887); Reprinted in *Gesammelte Abhandlungen*, vol. 2 (Springer, Heidelberg, 1988), pp. 304–330
41. F.G. Frobenius, Über die Charaktere der symmetrischen Gruppe, in *Sitzungsber. Kön. Preuss. Akad. Wissen. Berlin* (1900), pp. 516–534; *Gesammelte Abh. III*, ed. by J.-P. Serre (Springer, Berlin, 1968), pp. 148–166
42. W.E. Fulton, *Young Tableaux*. Student Texts, vol. 35 (London Mathematical Society/Cambridge University Press, Cambridge, 1997)
43. A. Gál, P.B. Miltersen, The cell probe complexity of succinct data structures, in *Proceedings of the 30th International Colloquium on Automata, Languages and Programming (ICALP)* (2003), pp. 332–344
44. M. Gardner, Squaring the square, in *The 2nd Scientific American Book of Mathematical Puzzles and Diversions* (Simon and Schuster, New York, 1961)
45. I.J. Good, Normally recurring decimals. *J. Lond. Math. Soc.* **21**, 167–169 (1947)

46. R.L. Graham, H.O. Pollak, On the addressing problem for loop switching. *Bell Syst. Tech. J.* **50**, 2495–2519 (1971)
47. R.L. Graham, H.O. Pollak, On embedding graphs in squashed cubes, in *Lecture Notes in Mathematics*, vol. 303 (Springer, New York, 1973), pp. 99–110
48. C. Greene, A. Nijenhuis, H.S. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape. *Adv. Math.* **31**, 104–109 (1979)
49. J.I. Hall, E.M. Palmer, R.W. Robinson, Redfield's lost paper in a modern context. *J. Graph Theor.* **8**, 225–240 (1984)
50. F. Harary, E.M. Palmer, *Graphical Enumeration* (Academic, New York, 1973)
51. F. Harary, R.W. Robinson, The rediscovery of Redfield's papers. *J. Graph Theory* **8**, 191–192 (1984)
52. G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, 2nd edn. (Cambridge University Press, Cambridge, 1952)
53. L.H. Harper, Morphisms for the strong Sperner property of Stanley and Griggs. *Linear Multilinear Algebra* **16**, 323–337 (1984)
54. T.W. Hawkins, The origins of the theory of group characters. *Arch. Hist. Exact Sci.* **7**, 142–170 (1970/1971)
55. T.W. Hawkins, Hypercomplex numbers, Lie groups, and the creation of group representation theory. *Arch. Hist. Exact Sci.* **8**, 243–287 (1971/1972)
56. T.W. Hawkins, New light on Frobenius' creation of the theory of group characters. *Arch. Hist. Exact Sci.* **12**, 217–243 (1974)
57. A.P. Hillman, R.M. Grassl, Reverse plane partitions and tableaux hook numbers. *J. Comb. Theor. A* **21**, 216–221 (1976)
58. R.A. Horn, C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1985)
59. J.W.B. Hughes, Lie algebraic proofs of some theorems on partitions, in *Number Theory and Algebra*, ed. by H. Zassenhaus (Academic, New York, 1977), pp. 135–155
60. A. Hurwitz, Über die Anzahl der Riemannschen Flächen mit gegebenen Verzweigungspunkten. *Math. Ann.* **55**, 53–66 (1902)
61. A. Joyal, Une théorie combinatoire des séries formelles. *Adv. Math.* **42**, 1–82 (1981)
62. N.D. Kazarinoff, R. Weitzenkamp, Squaring rectangles and squares. *Am. Math. Mon.* **80**, 877–888 (1973)
63. G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der Linearen Vertheilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.* **72**, 497–508 (1847)
64. C. Krattenthaler, Bijective proofs of the hook formulas for the number of standard Young tableaux, ordinary and shifted. *Electronic J. Combin.* **2**, R13, 9 pp. (1995)
65. D.E. Knuth, Permutations, matrices, and generalized Young tableaux. *Pac. J. Math.* **34**, 709–727 (1970)
66. I. Krasikov, Y. Roditty, Balance equations for reconstruction problems. *Arch. Math. (Basel)* **48**, 458–464 (1987)
67. J.P.S. Kung, Radon transforms in combinatorics and lattice theory, in *Combinatorics and Ordered Sets*, Arcata, CA, 1985. *Contemporary Mathematics*, vol. 57 (American Mathematical Society, Providence, 1986), pp. 33–74
68. K.H. Leung, B. Schmidt, New restrictions on possible orders of circulant Hadamard matrices. *Designs Codes Cryptogr.* **64**, 143–151 (2012)
69. E.K. Lloyd, J. Howard Redfield: 1879–1944. *J. Graph Theor.* **8**, 195–203 (1984)
70. L. Lovász, A note on the line reconstruction problem. *J. Comb. Theor. Ser. B* **13**, 309–310 (1972)
71. L. Lovász, Random walks on graphs: a survey, in *Combinatorics. Paul Erdős is Eighty*, vol. 2, Bolyai Society Mathematical Studies, vol. 2 (Keszthely, Hungary, 1993), pp. 1–46
72. D. Lubell, A short proof of Sperner's lemma. *J. Comb. Theor.* **1**, 299 (1966)
73. P.A. MacMahon, Memoir on the theory of the partitions of numbers — Part I. *Philos. Trans. R. Soc. Lond. A* **187**, 619–673 (1897); *Collected Works*, vol. 1, ed. by G.E. Andrews (MIT, Cambridge, 1978), pp. 1026–1080

74. P.A. MacMahon, Memoir on the theory of the partitions of numbers — Part IV. *Philos. Trans. R. Soc. Lond. A* **209**, 153–175 (1909); *Collected Works*, vol. 1, ed. by G.E. Andrews (MIT, Cambridge, 1978), pp. 1292–1314
75. P.A. MacMahon, *Combinatory Analysis*, vols. 1, 2 (Cambridge University Press, Cambridge, 1915/1916); Reprinted in one volume by Chelsea, New York, 1960
76. J. Matoušek, *Thirty-Three Miniatures* (American Mathematical Society, Providence, 2010)
77. J.W. Moon, in *Counting Labelled Trees*. Canadian Mathematical Monographs, vol. 1 (Canadian Mathematical Congress, 1970)
78. V. Müller, The edge reconstruction hypothesis is true for graphs with more than $n \log_2 n$ edges. *J. Comb. Theor. Ser. B* **22**, 281–283 (1977)
79. P.M. Neumann, A lemma that is not Burnside’s. *Math. Sci.* **4**, 133–141 (1979)
80. J.-C. Novelli, I. Pak, A.V. Stoyanovskii, A new proof of the hook-length formula. *Discrete Math. Theor. Comput. Sci.* **1**, 053–067 (1997)
81. K.M. O’Hara, Unimodality of Gaussian coefficients: a constructive proof. *J. Comb. Theor. Ser. A* **53**, 29–52 (1990)
82. W.V. Parker, The matrices AB and BA . *Am. Math. Mon.* **60**, 316 (1953); Reprinted in *Selected Papers on Algebra*, ed. by S. Montgomery et al. (Mathematical Association of America, Washington), pp. 331–332
83. J. Pitman, Coalescent random forests. *J. Comb. Theor. Ser. A* **85**, 165–193 (1999)
84. G. Pólya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Math.* **68**, 145–254 (1937)
85. G. Pólya, R.C. Read, *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds* (Springer, New York, 1987)
86. M. Pouzet, Application d’une propriété combinatoire des parties d’un ensemble aux groupes et aux relations. *Math. Zeit.* **150**, 117–134 (1976)
87. M. Pouzet, I.G. Rosenberg, Sperner properties for groups and relations. *Eur. J. Combin.* **7**, 349–370 (1986)
88. R.A. Proctor, A solution of two difficult combinatorial problems with linear algebra. *Am. Math. Mon.* **89**, 721–734 (1982)
89. H. Prüfer, Neuer Beweis eines Satzes über Permutationen. *Arch. Math. Phys.* **27**, 742–744 (1918)
90. J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig* **69**, 262–277 (1917); Translation by P.C. Parks, On the determination of functions from their integral values along certain manifolds. *IEEE Trans. Med. Imaging* **5**, 170–176 (1986)
91. J.H. Redfield, The theory of group reduced distributions. *Am. J. Math.* **49**, 433–455 (1927)
92. J.H. Redfield, Enumeration by frame group and range groups. *J. Graph Theor.* **8**, 205–223 (1984)
93. J.B. Remmel, Bijective proofs of formulae for the number of standard Young tableaux. *Linear Multilinear Algebra* **11**, 45–100 (1982)
94. G. de B. Robinson, On the representations of S_n . *Am. J. Math.* **60**, 745–760 (1938)
95. H.J. Ryser, *Combinatorial Mathematics* (Mathematical Association of America, Washington, 1963)
96. B.E. Sagan, *The Symmetric Group*, 2nd edn. (Springer, New York, 2001)
97. C.E. Schensted, Longest increasing and decreasing subsequences. *Can. J. Math.* **13**, 179–191 (1961)
98. J. Schmid, A remark on characteristic polynomials. *Am. Math. Mon.* **77**, 998–999 (1970); Reprinted in *Selected Papers on Algebra*, ed. by S. Montgomery et al. (Mathematical Association of America, Washington), pp. 332–333
99. C.A.B. Smith, W.T. Tutte, On unicursal paths in a network of degree 4. *Am. Math. Mon.* **48**, 233–237 (1941)
100. E. Sperner, Ein Satz über Untermengen einer endlichen Menge. *Math. Z.* **27**(1), 544–548 (1928)

101. R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebr. Discrete Meth.* **1**, 168–184 (1980)
102. R. Stanley, Differentiably finite power series. *Eur. J. Combin.* **1**, 175–188 (1980)
103. R. Stanley, Quotients of Peck posets. *Order* **1**, 29–34 (1984)
104. R. Stanley, Differential posets. *J. Am. Math. Soc.* **1**, 919–961 (1988)
105. R. Stanley, Unimodal and log-concave sequences in algebra, combinatorics, and geometry, in *Graph Theory and Its Applications: East and West*. *Annals of the New York Academy of Sciences*, vol. 576 (1989), pp. 500–535
106. R. Stanley, Variations on differential posets, in *Invariant Theory and Tableaux*, ed. by D. Stanton. *The IMA Volumes in Mathematics and Its Applications*, vol. 19 (Springer, New York, 1990), pp. 145–165
107. R. Stanley, *Enumerative Combinatorics*, vol. 1, 2nd edn. (Cambridge University Press, Cambridge, 2012)
108. R. Stanley, *Enumerative Combinatorics*, vol. 2 (Cambridge University Press, New York, 1999)
109. B. Stevens, G. Hurlbert, B. Jackson (eds.), Special issue on “Generalizations of de Bruijn cycles and Gray codes”. *Discrete Math.* **309** (2009)
110. K. Sutner, Linear cellular automata and the Garden-of-Eden. *Math. Intelligencer* **11**, 49–53 (1989)
111. J.J. Sylvester, On the change of systems of independent variables. *Q. J. Math.* **1**, 42–56 (1857); *Collected Mathematical Papers*, vol. 2 (Cambridge, 1908), pp. 65–85
112. J.J. Sylvester, Proof of the hitherto undemonstrated fundamental theorem of invariants. *Philos. Mag.* **5**, 178–188 (1878); *Collected Mathematical Papers*, vol. 3 (Chelsea, New York, 1973), pp. 117–126
113. W.T. Trotter, in *Combinatorics and Partially Ordered Sets: Dimension Theory*. *Johns Hopkins Studies in the Mathematical Sciences*, vol. 6 (Johns Hopkins University Press, Baltimore, 1992)
114. R.J. Turyn, Character sums and difference sets. *Pac. J. Math.* **15**, 319–346 (1965)
115. R.J. Turyn, Sequences with small correlation, in *Error Correcting Codes*, ed. by H.B. Mann (Wiley, New York, 1969), pp. 195–228
116. W.T. Tutte, The dissection of equilateral triangles into equilateral triangles. *Proc. Camb. Philos. Soc.* **44**, 463–482 (1948)
117. W.T. Tutte, Lectures on matroids. *J. Res. Natl. Bur. Stand. Sect. B* **69**, 1–47 (1965)
118. W.T. Tutte, The quest of the perfect square. *Am. Math. Mon.* **72**, 29–35 (1965)
119. T. van Aardenne-Ehrenfest, N.G. de Bruijn, Circuits and trees in oriented linear graphs. *Simon Stevin (Bull. Belgian Math. Soc.)* **28**, 203–217 (1951)
120. M.A.A. van Leeuwen, The Robinson-Schensted and Schützenberger algorithms, Part 1: new combinatorial proofs, Preprint no. AM-R9208 1992, Centrum voor Wiskunde en Informatica, 1992
121. E.M. Wright, Burnside’s lemma: a historical note. *J. Comb. Theor. B* **30**, 89–90 (1981)
122. A. Young, Qualitative substitutional analysis (third paper). *Proc. Lond. Math. Soc. (2)* **28**, 255–292 (1927)
123. D. Zeilberger, Kathy O’Hara’s constructive proof of the unimodality of the Gaussian polynomials. *Am. Math. Mon.* **96**, 590–602 (1989)

Index

Numbers

4483130665195087, 32, 40

A

van Aardenne-Ehrenfest, Tanya, 159

access time, 23

acts on (by a group), 43

acyclic (set of edges), 168

adjacency matrix, 1

adjacent (vertices), 1

adjoint (operator), 36

algebraic (Laurent series), 206

algebraic integer, 197

Anderson, Ian, 40

Andrews, George W. Eyre, 72

antichain, 33

antisymmetry, 31

automorphism

of a graph, 98

of a poset, 44

automorphism group

of a graph, 98

of a poset, 44

B

Babai, László, 203

balanced digraph, 151

balanced incomplete block design, 191

basis matrix, 168

Bender, Edward Anton, 125

Berlekamp, Elwyn Ralph, 203

Bernardi, Olivier, 143

BEST theorem, 159

BIBD, 191

Bidkhor, Hoda, 159

binary de Bruijn sequence, *see* de Bruijn sequence

binary sequence, 156

Binet-Cauchy theorem, 136, 137

binomial moment, 29

bipartite graph, 7

bipartition, 7

block (of a block design), 191

block design, 191

Bolker, Ethan David, 17

bond, 165

bond space, 165

boolean algebra, 31

Borchardt, Carl Wilhelm, 147

Bose, Raj Chandra, 192, 203

Brenti, Francesco, 53

de Bruijn, Nicolaas Govert, 97, 159

de Bruijn sequence, 156

de Bruijn graph, 156

bump

in RSK algorithm for CSPP, 117

in RSK algorithm for SYT, 113

Burnside's lemma, 80, 97

Burnside, William, 97

C

\mathbb{C} -algebra, 200

Caro, Yair, 203

Caspard, Nathalie, 40

Cauchy, Augustin Louis, 97

Cauchy-Binet theorem, 136, 137

Cauchy-Frobenius lemma, 80, 97

Cayley, Arthur, 146, 147

chain (in a poset), 32

characteristic polynomial, 7

characteristic vector (of a set), 189

circuit, 163
 circulant (matrix), 195
 circulation, 163
 closed walk, 4
 coboundary, 164
 Collatz, Lothar, 7
 coloring, 76
 column-strict plane partition, 117
 commutative diagram, 48
 complementary graph, 148
 complete bipartite graph, 8, 147
 complete graph, 4
 complete p -partite graph, 8
 complexity (of a graph), 135
 conjugate
 of an algebraic number, 198
 partition, 59
 connected graph, 135
 cotree, 170
 covers (in a poset), 31
 CSPP, 117
 cube (graph), 11
 Curtin, Eugene, 203
 Cvetković, Dragoš M., 7
 cycle index polynomial, 83
 cycle indicator, 97
 of a group of permutations, 83
 of a permutation, 83
 cycle space, 163
 cyclic group, 18
 cyclomatic number, 170
 cyclotomic polynomial, 199

D

Dedekind, Julius Wilhelm Richard, 17
 DeDeo, Michelle Rose, 18
 degree (of a vertex), 8, 21
 deleted neighborhood (of a vertex), 204
 D -finite, 206
 Diaconis, Persi Warren, 18, 159
 diagonal (of a power series), 207
 diagram (of a partition), 58
 differentially finite, 206
 digraph, 151
 of a permutation, 144
 dihedral necklace, 86
 direct product (of posets), 73
 directed graph, 151
 Doob, Michael, 7
 doubly-rooted tree, 144
 down (linear transformation), 36
 dual (of a planar graph), 178
 Dynkin, Eugene (Evgenii) Borisovitch, 72

E

edge reconstruction conjecture, 50
 edge set (of a graph), 1
 eigenvalues of a graph, 3
 elementary cycle, 163
 Engel, Konrad, 40
 equivalent colorings, 76, 77
 Erdős-Moser conjecture, 71
 weak, 71
 Eriksson, Kimmo, 72
 Euler phi-function, 85
 Euler's constant, 188
 Eulerian cycle (in a graph), 151
 Eulerian digraph, 151
 Eulerian graph, 151
 Eulerian tour
 in a digraph, 151
 in a graph, 151
 extended Smith diagram, 182

F

face (of a planar embedding), 178
 faithful action, 43
 Fermat's Last Theorem, 196
 Ferrers diagram, 58
 Fibonacci number, 130, 147, 200
 final vertex (of an edge)
 in a digraph, 151
 in an orientation, 139
 Fishburn, Peter, 40
 Fisher, Ronald Aylmer, 192, 203
 Flye Sainte-Marie, Camille, 159
 Fomin, Sergey Vladimirovich, 124
 forest, 145, 170
 Forsyth, Andrew Russell, 125
 Frame, James Sutherland, 123
 Frame–Robinson–Thrall, 105
 Frankl, Peter, 203
 Franzblau, Deborah Sharon, 124
 Frobenius, Ferdinand Georg, 17, 97, 98, 123
 Fulton, William Edgar, 125
 Fundamental Theorem of Algebra, 52

G

Gál, Anna, 203
 Gardner, Martin, 183
 Gauss' lemma, 196
 Gaussian coefficient, 61
 generalized ballot sequence, 123
 generating function, 6, 79
 G -equivalent, 44
 colorings, 77
 germ, 200

Good, Irving John, 159
 graded poset, 32
 Graham, Ronald Lewis, 18, 159, 203
 graph, 1
 Grassl, Richard, 124
 Greene, Curtis, 124
 group determinant, 17
 group reduction function, 97

H

Hadamard matrix, 194
 Hamiltonian cycle, 18
 Hamming weight, 14
 Harary, Frank, 97
 Hardy, Godfrey Harold, 53
 Harper, Lawrence Hueston, 53
 Hasse diagram, 31
 Hasse walk, 104
 Hawkins, Thomas W., 18
 Hillman, Abraham, 124
 hitting time, 23
 hook length formula, 105, 123
 Horn, Roger Alan, 27
 Hughes, J. W. B., 72
 Hurwitz, Adolf, 98

I

incidence matrix
 Oddtown, 189
 of a digraph, 166
 of a graph, 139
 incident, 1
 indegree (of a vertex), 151
 induced subgraph, 194
 initial vertex (of an edge)
 in a digraph, 151
 in an orientation, 139
 internal zero, 51
 inverse bump (in RSK algorithm), 115
 isolated vertex, 23
 isomorphic
 graphs, 49
 posets, 32
 isomorphism class (of simple graphs), 49

J

Johnson, Charles Royal, 27
 Joyal, André, 144, 147

K

Kazarinoff, Nicholas D., 183
 Kirchhoff's laws, 173

Kirchhoff, Gustav Robert, 146, 183
 Kishore, Shaunak, 159
 Knuth, Donald Ervin, 110, 124, 125
 Krasikov, Ilia, 53
 Krattenthaler, Christian Friedrich, 124
 Kronecker, Leopold, 198
 Kummer, Ernst Eduard, 196
 Kung, Joseph PeeSin, 18

L

laplacian matrix, 139
 lattice, 57
 lattice permutation, 123
 Laurent series, 206
 Leclerc, Bruno, 40
 van Leeuwen, Marc A. A., 124
 length
 of a chain, 32
 of a necklace, 54, 85
 of a walk, 2
 Leung, Ka Hin, 203
 level (of a ranked poset), 32
 Lights Out Puzzle, 203
 Littlewood, Dudley Ernest, 124
 Littlewood, John Edensor, 53
 Littlewood–Richardson rule, 124
 log-concave, 51
 polynomial, 55
 logarithmically concave, 51
 loop
 in a digraph, 151
 in a graph, 1
 Lovász, László, 27, 53
 Lubell, David, 33, 38, 40

M

MacMahon, Percy Alexander, 123–125
 mail carrier, 155
 Markov chain, 21
 Matoušek, Jiří, 203
 matrix
 irreducible, 23
 nonnegative, 23
 permutation, 23
 matrix analysis, 27
 Matrix-Tree Theorem, 141, 147
 matroid, 180
 maximal chain, 32
 Miltersen, Peter Bro, 203
 Möbius function, 98
 Monjardet, Bernard, 40
 Moon, John W., 147

Müller, Vladimír, 53
 multiple edge, 1
 multiset, 1

N

n -cycle, 18
 necklace, 54, 85
 neighborhood (of a vertex), 193
 Newton, Isaac, 51, 53
 Nijenhuis, Albert, 124
 \mathbb{N} -matrix, 118
 no internal zero, 51
 nonuniform Fisher inequality, 192
 Novelli, Jean-Christophe, 124

O

Oddtown, 189
 O'Hara, Kathleen Marie, 65, 72
 Ohm's law, 173
 orbit, 44
 order (of a poset), 32
 order-matching, 34
 explicit for B_n , 39
 order-raising operator, 35
 orientation (of a graph), 138
 orthogonal complement, 167
 orthogonal Lie algebra, 71
 orthogonal subspace, 193
 outdegree (of a vertex), 151

P

Pak, Igor M., 124
 Palmer, Edgar Milan, 97
 parallel connection, 173
 Parker, William Vann, 72
 part (of a partition of n), 57
 partially ordered set, 31
 partition
 of a set X , 44
 of an integer n , 57
 Pascal's triangle, 33, 62
 q -analogue, 62
 path (in a graph), 135
 closed, 18
 perfect squared rectangle, 180
 Perron–Frobenius theorem, 23
 physical intuition, 173
 Pitman, James William, 145, 147
 planar embedding, 178
 planar graph, 178
 plane partition, 115
 history of, 124

planted forest, 145
 pole (of a Smith diagram), 180
 Pollak, Henry Otto, 203
 Pólya, George (György), 53, 75, 97
 Pólya theory, 75
 polynomially recursive function, 200
 poset, 31
 positive definite, 192
 positive semidefinite, 37, 192
 potential, 164
 potential difference, 165
 Pouzet, Maurice André, 40, 53
 P -recursive function, 200
 primitive necklace, 98
 probability matrix, 21, 27
 Proctor, Robert Alan, 72
 Prüfer sequence, 143
 Prüfer, Ernst Paul Heinz, 143, 147

Q

q -binomial coefficient, 41, 61
 quantum order-matching, 36
 quotient poset, 45

R

Radon transform, 13
 Radon, Johann Karl August, 17
 rank
 of a boolean algebra, 31
 of a graded poset, 32
 of a poset element, 32
 rank-generating function, 33
 rank-symmetric, 33
 rank-unimodal, 33
 reciprocity theorem, 99
 Redfield, John Howard, 75, 97
 reduced incidence matrix, 140
 reflexivity, 31
 region (of a planar embedding), 178
 regular graph, 22
 Remmel, Jeffrey Brian, 124
 Robinson, Gilbert de Beauregard, 109, 123,
 124
 Robinson–Schensted correspondence, 110
 Roditty, Yehuda, 53
 Rolle's theorem, 52
 root (of a tree), 145
 Rosenberg, Ivo G., 40, 53
 row insertion, 113, 117
 Rowlinson, Peter, 7
 RSK algorithm, 109, 124
 Ryser, Herbert John, 203

S

Sachs, Horst, 7
 Sagan, Bruce Eli, 125
 Schensted, Craige Eugene, 109, 124
 Schmid, Josef, 72
 Schmidt, Bernard, 203
 Schützenberger, Marcel-Paul, 124
 semidefinite, 37, 192
 series connection, 173
 series-parallel network, 174
 shape (of a CSPP), 117
 shifted Ferrers diagram, 119
 Simić, Slobodan, 7
 simple (squared square), 183
 simple graph, 1
 simple group, 99
 Sinogowitz, Ulrich, 7
 Smith diagram, 180
 Smith, Cedric Austen Bardell, 159
 solid partition, 123
 spanning subgraph, 49, 135
 spectral graph theory, 7
 Sperner poset, 33
 Sperner property, 33
 Sperner's theorem, 33
 Sperner, Emanuel, 40
 squared rectangle, 180
 stabilizer, 46
 standard Young tableau, 104
 Stanley, Richard Peter, 40, 53, 72, 98, 124, 125, 159, 203
 stationary distribution, 28
 von Staudt, Karl Georg Christian, 146
 Stirling number, signless of the first kind, 88
 Stoyanovskii, Alexander V., 124
 strongly log-concave, 51
 sum (of vector spaces), 201
 support (of a function), 168
 Sutner, Klaus, 193, 203
 switching (at a vertex), 54
 switching reconstructible, 55
 Sylvester, James Joseph, 65, 72, 147
 symmetric chain decomposition, 40
 symmetric function, 79
 symmetric plane partition, 132
 symmetric sequence, 33
 SYT, 104

T

tensor product (of vector spaces), 202
 Thrall, Robert McDowell, 123
 total resistance, 174
 totient function, 85

tour (in a digraph), 151
 trace (of a plane partition), 132
 transitive (group action), 55, 80
 transitivity, 31
 transport, 48
 transposition, 99
 tree, 135
 Trotter, William Thomas, Jr., 40
 Turyn, Richard Joseph, 195, 203
 Tutte, William Thomas, 159, 183
 two-line array, 118
 type
 of a Hasse walk, 104
 of a permutation, 82

U

unimodal sequence, 33
 unimodular (matrix), 171
 universal cycle for \mathfrak{S}_n , 161
 universality (of tensor products), 202
 up (linear transformation), 35

V

valid λ -word, 106
 Velasquez, Elinor Laura, 18
 Venkataramana, Praveen, 147
 vertex bipartition, 7
 vertex reconstruction conjecture, 50
 vertex set, 1

W

walk, 2
 Warshauer, Max, 203
 weakly switching-reconstructible, 55
 weight
 of a binary vector, 14
 of a necklace, 54
 Weitzenkamp, Roger, 183
 Wheatstone bridge, 174
 Wilf, Herbert Saul, 124
 wreath product, 64

Y

Young diagram, 58
 Young, Alfred, 57, 123, 124
 Young's lattice, 57

Z

Zeilberger, Doron, 72, 124
 Zyklus, 83