Appendix

A.1 The Aldous–Hoover Representation II

In this section, we will prove the Aldous–Hoover representation for exchangeable arrays stated in Theorem 1.4 using a different approach based on the ideas of Lovász and Szegedy in the framework of limits of dense graph sequences. For convenience, let us recall that an infinite random array $s = (s_{l,l'})_{l,l'\geq 1}$ is called an *exchangeable array* if for any permutations π and ρ of finitely many indices we have equality in distribution

$$\big(s_{\pi(l),\rho(l')}\big)_{l,l'\geq 1} \stackrel{d}{=} \big(s_{l,l'}\big)_{l,l'\geq 1}. \tag{A.1}$$

We will prove the following.

Theorem A.1 (Aldous–Hoover). *For any infinite exchangeable array* $(s_{l,l'})_{l,l'\geq 1}$,

$$\big(s_{l,l'}\big)_{l,l'\geq 1} \stackrel{d}{=} \big(\sigma(w,u_l,v_{l'},x_{l,l'})\big)_{l,l'\geq 1} \tag{A.2}$$

for some measurable function $\sigma : [0,1]^4 \to \mathbb{R}$ *and i.i.d. random variables* w, (u_l), $(v_{l'})$, *and* $(x_{l,l'})$ *that have the uniform distribution on* $[0,1]$.

As was explained in Sect. 1.4, we only need to consider the case when the entries $s_{l,l'}$ take values in $[0,1]$. In fact, on first reading, one can even suppose that the entries take two values 0 and 1, since this case already illustrates all the main ideas, while simplifying the argument. Since two such arrays are equal in distribution if and only if all the joint moments of their entries are equal, we would like to show that for some function σ, for any $n\geq 1$ and any integer $k_{l,l'}\geq 0$,

$$\begin{aligned}\mathbb{E}\prod_{l,l'\leq n} s_{l,l'}^{k_{l,l'}} &= \mathbb{E}\prod_{l,l'\leq n}\sigma(w,u_l,v_{l'},x_{l,l'})^{k_{l,l'}}\\ &= \mathbb{E}\prod_{l,l'\leq n}\sigma^{(k_{l,l'})}(w,u_l,v_{l'}),\end{aligned} \tag{A.3}$$

D. Panchenko, *The Sherrington-Kirkpatrick Model*, Springer Monographs in Mathematics,
DOI 10.1007/978-1-4614-6289-7,

where we introduced the notation

$$\sigma^{(k)}(w,u,v) = \int_0^1 \sigma(w,u,v,x)^k \, dx. \tag{A.4}$$

When $s_{l,l'} \in \{0,1\}$, it is enough to consider only $k = 0,1$. We will construct such function σ from the array $(s_{l,l'})$ itself, using the exchangeability condition (A.1) in the following way. We know that the joint distribution of the entries of $(s_{l,l'})$ is not affected by permutations and

$$\mathbb{E} \prod_{l,l' \le n} s_{l,l'}^{k_{l,l'}} = \mathbb{E} \prod_{l,l' \le n} s_{\pi(l),\rho(l')}^{k_{l,l'}}.$$

Then, this equality also holds for random permutations and we can take π and ρ to be independent uniform random permutations of $\{1,\ldots,N\}$ for $N \ge n$ keeping all other indices fixed. When N is much larger than n, the random indices $\pi(l)$ and $\rho(l)$ for $l \le n$ behave almost like independent uniform random variables on $\{1,\ldots,N\}$. More precisely, if we redefine $\pi(l)$ and $\rho(l)$ for $l \le n$ to be independent and uniform on $\{1,\ldots,N\}$ then, conditionally on the event

$$\big\{\pi(l) \ne \pi(l'), \rho(l) \ne \rho(l') \text{ for } 1 \le l \ne l' \le n\big\},$$

they will have the same distribution as the first n coordinates of a random permutation. Since, for large N, this event has overwhelming probability,

$$\mathbb{E} \prod_{l,l' \le n} s_{l,l'}^{k_{l,l'}} = \lim_{N\to\infty} \mathbb{E} \prod_{l,l' \le n} s_{\pi(l),\rho(l')}^{k_{l,l'}}.$$

This already resembles Eq. (A.3) if we represent the random variables $\pi(l)$ and $\rho(l')$ as functions of i.i.d. random variables u_l and $v_{l'}$, uniform on $[0,1]$. If we generate the random array $s = s(w)$ as a function of a uniform random variable w on $[0,1]$ and define

$$\sigma_N(w,u,v) = s_{l,l'}(w) \text{ when } u \in \Big[\frac{l-1}{N}, \frac{l}{N}\Big), v \in \Big[\frac{l'-1}{N}, \frac{l'}{N}\Big) \tag{A.5}$$

for $l,l' \in \{1,\ldots,N\}$, then we can rewrite the equation above as

$$\mathbb{E} \prod_{l,l' \le n} s_{l,l'}^{k_{l,l'}} = \lim_{N\to\infty} \mathbb{E} \prod_{l,l' \le n} \sigma_N(w,u_l,v_{l'})^{k_{l,l'}}. \tag{A.6}$$

Exchangeability had played its role. Now, a difficult question arises, how to extract a function σ from a (sub)sequence of functions σ_N, so that the limit (A.6) can be written as Eq. (A.3). The key idea is to think of $\sigma_N(w,\cdot,\cdot)$ as a (random) function on $[0,1]^2$ and to define a distance between such functions that is weak enough to allow us to find a converging subsequence but, at the same time, strong enough to imply the convergence of the joint moments in Eq. (A.6). Another subtle point is that the limiting function σ depends on one more coordinate $x \in [0,1]$, so we will, in fact, be defining simultaneous convergence of the powers $\sigma_N(w,\cdot,\cdot)^k$ to some functions $\sigma^{(k)}(w,\cdot,\cdot)$ for all $k \geq 1$ that will then be related to one function σ via the Eq. (A.4). To motivate the definition of a distance, let us see how we can control the difference between the joint moments

$$\mathbb{E} \prod_{l,l' \leq n} \sigma_N(w, u_l, v_{l'})^{k_{l,l'}} - \mathbb{E} \prod_{l,l' \leq n} \sigma^{(k_{l,l'})}(w, u_l, v_{l'}) \tag{A.7}$$

in terms of some distance between the functions σ_N^k and $\sigma^{(k)}$. If we consider a pair $T = (T_1, T_2)$ of measurable functions $T_j : [0,1]^2 \to [0,1]$ such that for all $w \in [0,1]$,

$$T_1(w,\cdot), T_2(w,\cdot) : [0,1] \to [0,1]$$

are (Lebesgue) measure-preserving transformations, then, conditionally on w, the sequences $(T_1(w,u_l))_{l\geq 1}$, $(T_2(w,v_l))_{l\geq 1}$ are i.i.d. with the uniform distribution on $[0,1]$ and, therefore, the difference in Eq. (A.7) equals

$$\mathbb{E} \prod_{l,l' \leq n} \sigma_N(w, T_1(w,u_l), T_2(w,v_{l'}))^{k_{l,l'}} - \mathbb{E} \prod_{l,l' \leq n} \sigma^{(k_{l,l'})}(w, u_l, v_{l'}).$$

In other words, from the point of view of the joint moments, we have a freedom to redefine how the function σ_N depends on the uniform random variables u and v on $[0,1]$. For simplicity of notation, given a function $f : [0,1]^3 \to \mathbb{R}$, let us write

$$f_T(w,u,v) = f\big(w, T_1(w,u), T_2(w,v)\big). \tag{A.8}$$

The next idea is crucial. If we arrange all pairs (l,l') for $1 \leq l,l' \leq n$ in some linear order $\leq$ and denote

$$f_{l,l'} = \prod_{(j,j')<(l,l')} \sigma_{N,T}(w,u_j,v_{j'})^{k_{j,j'}} \prod_{(j,j')>(l,l')} \sigma^{(k_{j,j'})}(w,u_j,v_{j'})$$

then the above difference can be written as a telescopic sum:

$$\sum_{l,l'} \mathbb{E}\big(\sigma_{N,T}(w,u_l,v_{l'})^{k_{l,l'}} - \sigma^{(k_{l,l'})}(w,u_l,v_{l'})\big) f_{l,l'}. \tag{A.9}$$

Let us first integrate one term in u_l and $v_{l'}$, conditionally on all the other random variables. Notice that each factor in $f_{l,l'}$ depends only on u_l, or $v_{l'}$, or neither, but not on both, and all factors are bounded in absolute value by 1. Therefore, if we define a distance between two functions $\sigma_1, \sigma_2 : [0,1]^2 \to \mathbb{R}$ by

$$\|\sigma_1 - \sigma_2\|_\square = \sup_{\|f\|_\infty, \|g\|_\infty \leq 1} \left| \iint_{[0,1]^2} \big(\sigma_1(u,v) - \sigma_2(u,v)\big) f(u) g(v)\, du dv \right|, \tag{A.10}$$

then the sum in Eq. (A.9) can be bounded in absolute value by

$$\sum_{l,l' \leq n} \mathbb{E}\big\|\sigma_{N,T}(w,\cdot,\cdot)^{k_{l,l'}} - \sigma^{(k_{l,l'})}(w,\cdot,\cdot)\big\|_\square. \tag{A.11}$$

Since the difference in Eq. (A.7) does not depend on T, we can minimize the bound (A.11) over T, which means that, in order to prove Eq. (A.3) and the Aldous–Hoover representation, it is enough to show the following.

Theorem A.2 (Lovász–Szegedy). *For any sequence of functions $\sigma_N : [0,1]^3 \to [0,1]$, there exists a function $\sigma : [0,1]^4 \to [0,1]$ such that*

$$\liminf_{N\to\infty} \inf_T \sum_{k\geq 1} 2^{-k}\mathbb{E}\Big\|\sigma_{N,T}(w,\cdot,\cdot)^k - \sigma^{(k)}(w,\cdot,\cdot)\Big\|_{\square} = 0, \tag{A.12}$$

where the functions $\sigma^{(k)}$ are defined in Eq. (A.4).

The distance $\|\cdot\|_{\square}$ is called the *cut norm* or *rectangle norm*. It also equals to

$$\|\sigma_1 - \sigma_2\|_{\square} = \sup_{A_1,A_2\subseteq[0,1]}\Big|\iint_{A_1\times A_2}\big(\sigma_1(u,v)-\sigma_2(u,v)\big)\,du\,dv\Big|, \tag{A.13}$$

which is easy to see by approximating the functions $f(u)$, $g(v)$ in Eq. (A.10) by the symmetric convex combinations of indicators of measurable sets in $[0,1]$. The distance (A.13) possesses good properties that will allow us to extract a limiting object σ along some subsequence of (σ_N) in the sense of Eq. (A.12). Its main property will be expressed in the regularity lemma below which will be based on the following.

Lemma A.1. *Let H be a Hilbert space and let $(H_n)_{n\geq 1}$ be a sequence of arbitrary subsets $H_n \subseteq H$. For any $h\in H$ and $\varepsilon>0$, there exist $n\leq 1+\varepsilon^{-2}$ and $h_i\in H_i, \lambda_i\in\mathbb{R}$ for $i\leq n$ such that for all $h'\in H_{n+1}$,*

$$\big|\big\langle h', h - \sum_{i\leq n}\lambda_i h_i\big\rangle\big| \leq \varepsilon\big\|h'\big\|\big\|h\big\|. \tag{A.14}$$

Proof. Given $h\in H$ and $n\geq 1$, let us define

$$\delta_n = \inf\Big\{\|h-h^*\|^2 : h^* = \sum_{i\leq n}\lambda_i h_i, h_i\in H_i, \lambda_i\in\mathbb{R}\Big\}.$$

Since $\|h\|^2 \geq \delta_1\geq\delta_2\geq\ldots\geq 0$, this implies that

$$\delta_n < \delta_{n+1} + \varepsilon^2\|h\|^2$$

for some $n\leq 1+\varepsilon^{-2}$, because, otherwise, for $\varepsilon^{-2} < n_0 \leq 1+\varepsilon^{-2}$,

$$\|h\|^2 \geq \delta_1 \geq \sum_{1\leq i\leq n_0}(\delta_i - \delta_{i+1}) \geq n_0\varepsilon^2\|h\|^2 > \|h\|^2.$$

For such n, let us choose $h^* = \sum_{i\leq n}\lambda_i h_i$ such that $\|h-h^*\|^2 \leq \delta_{n+1}+\varepsilon^2\|h\|^2$. Then, by the definition of δ_{n+1}, for any $h'\in H_{n+1}$ and $t\in\mathbb{R}$,

$$\|h-h^*\|^2 \leq \|h-h^*-th'\|^2 + \varepsilon^2\|h\|^2,$$

and it is easy to check that this can hold only if Eq. (A.14) holds. □

Lemma A.1 applied to the Hilbert space $H = L^2([0,1]^2, dudv)$ will yield a very useful approximation property with respect to the norm $\|\cdot\|_\square$. We will say that $\mathscr{D}$ is a *finite product σ-algebra* on $[0,1]^2$ if there exist two finite measurable partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of $[0,1]$ such that $\mathscr{D}$ is generated by the rectangles $P_1 \times P_2$ such that $P_1 \in \mathscr{P}_1$ and $P_2 \in \mathscr{P}_2$. Let us define the size

$$|\mathscr{D}| = \max(|\mathscr{P}_1|, |\mathscr{P}_2|) \tag{A.15}$$

of the partition $\mathscr{D}$ by the largest of the cardinalities of the partitions $\mathscr{P}_1$, $\mathscr{P}_2$. Given $h \in H$, let us denote by $h_{\mathscr{D}}$ the conditional expectation $\mathbb{E}(h|\mathscr{D})$ of h given $\mathscr{D}$ with respect to the Lebesgue measure,

$$h_{\mathscr{D}}(u,v) = \frac{1}{|P_1||P_2|} \iint_{P_1 \times P_2} h(x,y)\,dxdy, \tag{A.16}$$

for $(u,v) \in P_1 \times P_2$ whenever $|P_1||P_2| > 0$; otherwise, $h_{\mathscr{D}}(u,v) = 0$. The following lemma describes a property of the rectangle norm $\|\cdot\|_\square$ which will be the key to proving Eq. (A.12). Let $\|\cdot\|$ denote the L^2-norm on H.

Lemma A.2 (Regularity Lemma). *For any $\varepsilon > 0$ and any function $h \in H$, there exists a finite product σ-algebra $\mathscr{D}$ on $[0,1]^2$ such that*

$$\|h - h_{\mathscr{D}}\|_\square \le 2\varepsilon \|h\| \tag{A.17}$$

and the size $|\mathscr{D}| \le 2^{1+1/\varepsilon^2}$.

This means that any h can be approximated within $\varepsilon\|h\|$ in the cut norm by a simple step function $h_{\mathscr{D}}$ with the number of steps $|\mathscr{D}|$ controlled uniformly over all h. Of course, $\mathscr{D}$ itself can depend on h, but its size (A.15) is bounded independently of h. To appreciate this result, let us notice that we can, of course, approximate any $h \in H$ by a step function even in the stronger L^2-norm, but the number of steps, in general, will not be independent of h.

Proof (Lemma A.2). Given a rectangle $A = A_1 \times A_2 \subseteq [0,1]^2$, let us denote by $\mathscr{D}(A)$ a finite product σ-algebra generated by the rectangles $A_1 \times [0,1]$ and $[0,1] \times A_2$. Let

$$H_1 = \big\{h \in H : h \text{ is } \mathscr{D}(A)\text{-measurable for some rectangle } A\big\}.$$

Here, the choice of a rectangle A can depend on h. Let $H_n := H_1$ for all $n \ge 2$. By Lemma A.1, for any $h \in H$, we can find $n \le 1 + \varepsilon^{-2}$ and a linear combination

$$h^* = \sum_{i \le n} \lambda_i h_i$$

with some $h_i \in H_i, \lambda_i \in \mathbb{R}$ for $i \le n$, such that for all $h' \in H_{n+1}$,

$$\big|\langle h', h - h^* \rangle\big| \le \varepsilon \big\|h'\big\|\big\|h\big\|.$$

Since, for any rectangle $A \subseteq [0,1]^2$, its indicator $h' = I_A$ belongs to $H_{n+1} = H_1$, the definition (A.13) implies that

$$\|h - h^*\|_\square = \sup_A \left|\left\langle I_A, h - h^* \right\rangle\right| \le \varepsilon \|h\|.$$

If $A^i = A^i_1 \times A^i_2$ is the rectangle corresponding to $h_i \in H_i = H_1$, then the linear combination h^* is, obviously, measurable with respect to the σ-algebra $\mathscr{D}$ generated by $\{A^i : i \le n\}$. If for $j = 1,2$, $\mathscr{P}_j$ is the partition of $[0,1]$ generated by the sets $(A^i_j)_{i \le n}$ then $\mathscr{D}$ is obviously a finite product σ-algebra generated by the partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ and its size $|\mathscr{D}|$, defined in Eq. (A.15), is bounded by 2^n. Let us now consider two functions $f = f(u)$ and $g = g(v)$ such that $\|f\|_\infty, \|g\|_\infty \le 1$. Since Eq. (A.16) implies that $(fg)_\mathscr{D} = f_\mathscr{D} g_\mathscr{D}$ and since both $h_\mathscr{D}$ and h^* are $\mathscr{D}$-measurable,

$$\begin{aligned} \left|\iint (h^* - h_\mathscr{D}) fg \, du dv\right| &= \left|\iint (h^* - h_\mathscr{D}) f_\mathscr{D} g_\mathscr{D} \, du dv\right| \\ &= \left|\iint (h^* - h) f_\mathscr{D} g_\mathscr{D} \, du dv\right| \le \|h^* - h\|_\square \le \varepsilon \|h\|. \end{aligned}$$

Therefore, by Eq. (A.10), $\|h^* - h_\mathscr{D}\|_\square \le \varepsilon \|h\|$ and $\|h - h_\mathscr{D}\|_\square \le 2\varepsilon \|h\|$. □

It is worth writing down a simple observation at the end of the above proof as a separate statement.

Lemma A.3. *If h^* is measurable with respect to some finite product σ-algebra $\mathscr{D}$ then $\|h - h_\mathscr{D}\|_\square \le 2\|h - h^*\|_\square$.*

Suppose that, using the regularity lemma, we found a finite product σ-algebra $\mathscr{D}$ such that $\|h - h_\mathscr{D}\|_\square \le \varepsilon$ and $\mathscr{D}$ is generated by some finite partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of $[0,1]$. Since in Eq. (A.12) we have the freedom of applying a measure-preserving transformation to the coordinates u and v, before we proceed to prove Theorem A.2, let us first observe that we can "transfer" measurable sets in the partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ into proper intervals by some measure-preserving transformation without losing control of $\|h - h_\mathscr{D}\|_\square$.

Lemma A.4. *Suppose we are given two partitions of $[0,1]$ into measurable sets $\{P_1, \ldots, P_k\}$ and intervals $\{I_1, \ldots, I_k\}$ such that their measures $|P_j| = |I_j|$ for $j \le k$. Then, there exists a measure-preserving transformation T such that $T(I_j) \subseteq P_j$ for $j \le k$ and T is one-to-one outside a set of measure zero.*

Proof. Since this is a very basic exercise in probability, we will only give a brief sketch. For each $j \le k$, the restriction of the uniform distribution on $[0,1]$ to P_j can be realized by the quantile transformation on the interval I_j of the same measure. Since this restriction is a continuous measure, the quantile will be one-to-one. It is not necessarily into P_j, but the measure of points mapped outside of P_j will, obviously, be zero and we can redefine the transformation to map them into a fixed point in P_j. Define T on I_j by this redefined quantile transformation. □

Let T_1, T_2 be two such measure-preserving transformations that transfer the partitions $\mathscr{P}_1, \mathscr{P}_2$ that generate $\mathscr{D}$ in Lemma A.2 into interval partitions $\mathscr{I}_1, \mathscr{I}_2$. Let $\mathscr{F}$ be the finite product σ-algebra generated by $\mathscr{I}_1$ and $\mathscr{I}_2$. Similarly to Eq. (A.8), for a function $f : [0,1]^2 \to \mathbb{R}$ let us define

$$f_T(u,v) = f(T_1(u), T_2(v)). \tag{A.18}$$

Notice that $h_{\mathscr{D},T} := (h_{\mathscr{D}})_T$ is now measurable with respect to $\mathscr{F}$, since its value on the rectangle $I_1 \times I_2$ is given by the value of $h_{\mathscr{D}}$ on the rectangle $P_1 \times P_2$ such that $T_1(I_1) \subseteq P_1$ and $T_2(I_2) \subseteq P_2$, i.e.

$$h_{\mathscr{D},T}(u,v) = \frac{1}{|P_1||P_2|} \iint_{P_1 \times P_2} h(x,y)\,dxdy$$

for $(u,v) \in I_1 \times I_2$. This means that $h_{\mathscr{D},T}$ is a step function with steps given by "geometric" and not just measurable rectangles, which will be helpful in the proof of Theorem A.2 below. On the other hand, $h_{\mathscr{D},T}$ approximates h_T just as well as $h_{\mathscr{D}}$ approximated h.

Lemma A.5. *If the measure-preserving transformations T_1 and T_2 are one-to-one outside a set of measure zero then*

$$\|h_T - f_T\|_\square = \|h - f\|_\square. \tag{A.19}$$

Proof. By the definition (A.18), for any measurable sets $A_1, A_2 \subseteq [0,1]$,

$$\iint\limits_{A_1 \times A_2} (h_T - f_T)\,dudv = \iint\limits_{A_1 \times A_2} \big(h(T_1(u),T_2(v)) - f(T_1(u),T_2(v))\big)\,dudv.$$

Since the map T_1 is one-to-one outside a set of measure zero, the sets $\{u \in A_1\}$ and $\{T_1(u) \in T_1(A_1)\}$ differ by a set of measure zero and in the above integral we can replace A_1 by $T_1^{-1}(T_1(A_1))$ and, similarly, A_2 by $T_2^{-1}(T_2(A_2))$. Then, making the change of variables, we get

$$\iint\limits_{A_1 \times A_2} (h_T - f_T)\,dudv = \iint\limits_{T_1(A_1) \times T_2(A_2)} \big(h(u,v) - f(u,v)\big)\,dudv.$$

By the definition (A.13), this implies $\|h_T - f_T\|_\square \le \|h - f\|_\square$. To obtain the opposite inequality in Eq. (A.19), we start with the sets $T_1^{-1}(A_1)$ and $T_2^{-1}(A_2)$ to get

$$\iint\limits_{T_1^{-1}(A_1) \times T_2^{-1}(A_2)} (h_T - f_T)\,dudv = \iint\limits_{A_1 \times A_2} \big(h(u,v) - f(u,v)\big)\,dudv.$$

By the definition (A.13), this implies $\|h_T - f_T\|_\square \ge \|h - f\|_\square$. ☐

On first reading, one can assume that functions σ, σ_N in the proof below take values in $\{0,1\}$, in which case we only need to consider $k = 1$.

Proof (Theorem A.2). First, consider a function $\sigma : [0,1]^2 \to [0,1]$. Using the regularity Lemma A.2, for each $m \geq 1$ and each $k \leq m$, let us find a finite product σ-algebra $\mathscr{D}_{mk}$ such that

$$\left\| \sigma^k - (\sigma^k)_{\mathscr{D}_{mk}} \right\|_\square \leq \frac{1}{2m},$$

where σ^k is the kth power of σ. For each $m \geq 1$, let $\mathscr{D}_m = \vee_{n \leq m} \vee_{k \leq n} \mathscr{D}_{nk}$. This definition implies that

$$\mathscr{D}_1 \subseteq \mathscr{D}_2 \subseteq \ldots \subseteq \mathscr{D}_m \subseteq \ldots \tag{A.20}$$

and, since $\mathscr{D}_m$ is a refinement of $\mathscr{D}_{mk}$ for $k \leq m$, Lemma A.3 implies that

$$\left\| \sigma^k - (\sigma^k)_{\mathscr{D}_m} \right\|_\square \leq \frac{1}{m} \text{ for } k \leq m.$$

Finally, for each $m \geq 1$, let us utilize Lemma A.4 and find a pair of measure-preserving transformations $T_m = (T_{m1}, T_{m2})$ that transfer $\mathscr{D}_m$ into a finite product σ-algebra $\mathscr{F}_m$ generated by some partitions into intervals $\mathscr{I}_1^m$ and $\mathscr{I}_2^m$. By Eq. (A.20), partitions generating $(\mathscr{D}_m)$ are being refined as m increases and when we define (T_m) that transfer sets in these partitions into intervals we can define (T_m) recursively in a way that preserves the inclusion relation. In other words, if $\mathscr{D}_m$ is generated by the partitions $\mathscr{P}_1^m$ and $\mathscr{P}_2^m$ and if an element, say $P \in \mathscr{P}_1^m$, is broken into a disjoint union of sets at the next step $m+1$,

$$P = P_1 \cup \ldots \cup P_r \text{ for some } P_1, \ldots, P_r \in \mathscr{P}_1^{m+1},$$

then, if P is transferred into an interval I by the transformation T_{m1}, $T_{m1}(I) \subseteq P$, then $P_1, \ldots, P_r$ are transferred into some intervals $I_1, \ldots, I_r$ by the transformation $T_{(m+1)1}$ such that I is the disjoint union of $I_1, \ldots, I_r$. The inclusion structure of the partitions generating Eq. (A.20) should be preserved after we transfer them into intervals. In particular,

$$\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots \subseteq \mathscr{F}_m \subseteq \ldots . \tag{A.21}$$

Let us recall that each function $(\sigma^k)_{\mathscr{D}_m, T_m}$ is now measurable with respect to $\mathscr{F}_m$ and, by Eq. (A.19), we will have

$$\left\| (\sigma^k)_{T_m} - (\sigma^k)_{\mathscr{D}_m, T_m} \right\|_\square \leq \frac{1}{m} \text{ for } k \leq m.$$

Notice that, for each $k \geq 1$, the sequence $((\sigma^k)_{\mathscr{D}_m}, \mathscr{D}_m)_{m \geq 1}$ is a martingale and, since the transformations (T_m) preserve the inclusion structure and the measure of the elements of the partitions, the sequence $((\sigma^k)_{\mathscr{D}_m, T_m}, \mathscr{F}_m)_{m \geq 1}$ is also a martingale. To simplify the notation, let us now denote

$$\sigma_{mk} = (\sigma^k)_{\mathscr{D}_m, T_m}. \tag{A.22}$$

To summarize, given a function $\sigma : [0,1]^2 \to [0,1]$, we constructed a sequence of pairs of measure-preserving transformations (T_m), a sequence $(\mathscr{F}_m)$ of finite product σ-algebras generated by partitions into intervals and, for each $k \geq 1$, a martingale sequence $(\sigma_{mk}, \mathscr{F}_m)_{m\geq 1}$ such that

$$\big\|(\sigma^k)_{T_m} - \sigma_{mk}\big\|_\square \leq \frac{1}{m} \text{ for } k \leq m. \tag{A.23}$$

In particular, this implies that

$$\inf_T \sum_{1\leq k\leq m} 2^{-k}\big\|(\sigma^k)_T - \sigma_{mk}\big\|_\square \leq \frac{1}{m}. \tag{A.24}$$

Another important feature of the sequence (σ_{mk}) is that, for any fixed m and any $u,v \in [0,1]$, the sequence $\sigma_{mk}(u,v)$ for $k \geq 1$ is a sequence of moments of some probability distribution on $[0,1]$. This is because for any $(u,v) \in P_1 \times P_2$ in some measurable rectangle in $\mathscr{D}_m$,

$$(\sigma^k)_{\mathscr{D}_m}(u,v) = \frac{1}{|P_1||P_2|} \iint_{P_1\times P_2} \sigma^k(x,y)\,dxdy \tag{A.25}$$

is the sequence of moments of σ viewed as a random variable on $P_1 \times P_2$ with the uniform distribution, and σ_{mk} are obtained from $(\sigma^k)_{\mathscr{D}_m}$ for all $k \geq 1$ by the same change of variables (A.22). Let us observe a simple fact that will be used below that, if for each $N \geq 1$, $(\mu_k^N)_{k\geq 0}$ is a sequence of moments of some probability distribution on $[0,1]$ and it converges to $(\mu_k)_{k\geq 0}$ as $N \to \infty$ then, by the selection theorem, the limiting sequence is also a sequence of moments of some probability distribution on $[0,1]$. Finally, we come to the main argument. Given a function $\sigma_N : [0,1]^3 \to [0,1]$, for each fixed $w \in [0,1]$, we can think of $\sigma_N(w,\cdot,\cdot)$ as a function on $[0,1]^2$ and, as above, find a sequence of functions $\sigma_{N,mk}(w,\cdot,\cdot)$ such that

$$\inf_T \sum_{1\leq k\leq m} 2^{-k}\big\|\sigma^k_{N,T}(w,\cdot,\cdot) - \sigma_{N,mk}(w,\cdot,\cdot)\big\|_\square \leq \frac{1}{m}. \tag{A.26}$$

Of course, the sequence $(\mathscr{F}_{N,m}(w))$ of corresponding finite product σ-algebras will now depend on w but, otherwise, $(\sigma_{N,mk}(w,\cdot,\cdot))$ will satisfy the same properties as (σ_{mk}) for each $N \geq 1$ and all $w \in [0,1]$. (Remark: There is a minor issue here of whether the functions $\sigma_{N,mk}$ are jointly measurable. However, this issue can be avoided if we first approximate σ_N arbitrarily well in the L^2-norm by a step function on $[0,1]^3$ and then apply the above construction to this discretized approximation.) Even though the sequence $(\mathscr{F}_{N,m}(w))$ depends on w, the size of each σ-algebra $\mathscr{F}_{N,m}(w)$, in the sense of the definition (A.15), is bounded by a function of m only, which was the key point of the regularity Lemma A.2. Therefore, the σ-algebra $\mathscr{F}_{N,m}(w)$ can be described by a finite collection of random variables (functions of w) that encode the number of intervals in the partitions $\mathscr{I}_1, \mathscr{I}_2$ generating this finite product σ-algebra and the Lebesgue measures of all these intervals. The functions $\sigma_{N,mk}(w,\cdot,\cdot)$ can be similarly described by random variables given by their values on the rectangles that generate $\mathscr{F}_{N,m}(w)$. We can find a subsequence (N_j) along which all these random variables, for all $k,m \geq 1$, converge almost surely and, since for each $m \geq 1$ the number of intervals in the partitions stays bounded, the limiting random variables can be used in a natural way to define a sequence of finite product σ-algebras $(\mathscr{F}_m(w))$ and functions $(\sigma_{mk}(w,\cdot,\cdot))$. It is clear from this construction that, for all $m,k \geq 1$,

$$\lim_{j\to\infty} \big\| \sigma_{N_j,mk}(w,\cdot,\cdot) - \sigma_{mk}(w,\cdot,\cdot) \big\|_{\square} = 0$$

almost surely (in fact, here we could use a stronger L^2-norm) and Eq. (A.26) implies that

$$\limsup_{j\to\infty} \inf_T \sum_{1\leq k\leq m} 2^{-k} \big\| \sigma^k_{N_j,T}(w,\cdot,\cdot) - \sigma_{mk}(w,\cdot,\cdot) \big\|_{\square} \leq \frac{1}{m} \tag{A.27}$$

almost surely. It is also clear from the construction that the limiting sequence σ_{mk} will satisfy the same properties as $\sigma_{N,mk}$ for almost all w. First of all, for each $k \geq 1$, the sequence

$$\big(\sigma_{mk}(w,\cdot,\cdot), \mathscr{F}_m(w)\big)_{m\geq 1}$$

is again a martingale, since, in the limit, a value of $\sigma_{mk}(w,\cdot,\cdot)$ on any rectangle in the σ-algebra $\mathscr{F}_m(w)$ will still be obtained by averaging the values of $\sigma_{(m+1)k}(w,\cdot,\cdot)$ over sub-rectangles from the σ-algebra $\mathscr{F}_{m+1}(w)$. Moreover, for each $m \geq 1$ and almost all $w,u,v \in [0,1]$, the sequence $(\sigma_{mk}(w,u,v))_{k\geq 1}$ is a sequence of moments of a probability distribution on $[0,1]$, since such sequences are closed under taking limits, as we mentioned above. By the martingale convergence theorem, for all such w, the limits

$$\sigma^{(k)}(w,\cdot,\cdot) = \lim_{m\to\infty} \sigma_{mk}(w,\cdot,\cdot) \tag{A.28}$$

exist almost surely and in $L^2([0,1]^2, dudv)$ for all $k \geq 1$ and, therefore,

$$\lim_{m\to\infty} \sum_{1\leq k\leq m} 2^{-k} \big\| \sigma_{mk}(w,\cdot,\cdot) - \sigma^{(k)}(w,\cdot,\cdot) \big\|_{\square} = 0.$$

Combining with Eq. (A.27), we obtain that

$$\lim_{j\to\infty} \inf_T \sum_{k\geq 1} 2^{-k} \| \sigma^k_{N_j,T}(w,\cdot,\cdot) - \sigma^{(k)}(w,\cdot,\cdot) \|_{\square} = 0. \tag{A.29}$$

This holds almost surely and, hence, on average over w. Finally, by Eq. (A.28), for almost all (w,u,v), the sequence $(\sigma^{(k)}(w,u,v))_{k\geq 1}$ is again a sequence of moments of a probability distribution on $[0,1]$, and we can redefine these limits on the set of measure zero to have this property for all (w,u,v). This precisely means that there exists a function $\sigma : [0,1]^4 \to [0,1]$ such that Eq. (A.4) holds, which finishes the proof. (Remark: Again, there is a minor issue of measurability of the function σ on $[0,1]^4$. First of all, all moments $\sigma^{(k)}$ are measurable on $[0,1]^3$ as the limits of measurable functions, and there is a standard way to approximate a distribution on $[0,1]$ and its quantile in terms of finitely many moments at a time (see, e.g., Hausdorff moment problem in Sect. 7.3 in [24]). It is easy to check that, using such approximation procedure, one can define σ in terms of the sequence $\sigma^{(k)}$ in a measurable way.) $\square$

Bibliography

Notes and Comments

Chapter 1. As we mentioned in the preface, the Sherrington–Kirkpatrick model was introduced by Sherrington and Kirkpatrick [58] in 1975 and the formula for the free energy was invented in a celebrated work of Parisi [51, 52].

The concentration (self-averaging) of the free energy type functionals in Theorem 1.2 is just one example of a general Gaussian concentration phenomenon (see, e.g., [33]). The concentration of the free energy in the SK model given in Eq. (1.42) was first observed by Pastur and Shcherbina in [55]. Guerra and Toninelli [26] proved the existence of the limit of the free energy in the Sherrington–Kirkpatrick model in Theorem 1.1. Aizenman et al. [2] invented the scheme leading to the representation (1.58) in Sect. 1.3. The proof of Theorem 1.3 is from Panchenko [48], but it was shown to me earlier by Talagrand. The Aldous–Hoover representation in Sect. 1.4 was proved by Aldous [3] and Hoover [29] (see also [4, 30]). A very nice modern overview of the topic of exchangeability can be found in Austin [7], and the proof of the Aldous–Hoover representation for weakly exchangeable arrays in Sect. 1.4 essentially follows [7]. Dovbysh and Sudakov [22] proved the representation result for Gram-de Finetti arrays that appears in Sect. 1.5 and another proof can be found in Panchenko [44], where some of the ideas are taken from Hestir [28]. The short and elegant proof we give here was discovered by Tim Austin in [8]. The relevance of these representation results in the analysis of the Sherrington-Kirkpatrick model was first brought to light by Arguin and Aizenman [5].

Chapter 2. The Ruelle probability cascades were introduced by Ruelle in [56] and, as the title of [56] suggests, their definition was motivated by Derrida's random energy model (REM) [16, 17] and the generalized random energy model (GREM) [18, 19], which are two simplified mean-field models of spin glasses. As we mentioned in the preface, the random measures in Sect. 2.2 describe to the asymptotic Gibbs measures of the REM (this is explained very well in Sect. 1.2 in [62]) and the random measures in Sect. 2.3 (the Ruelle probability cascades) describe to the asymptotic Gibbs measures of the GREM, which was rigorously proved by Bovier

D. Panchenko, *The Sherrington-Kirkpatrick Model*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-6289-7, © Springer Science+Business Media New York 2013

and Kurkova [11]. An overview of the Poisson processes in Sect. 2.1 follows the classical text by Kingman [32]. Theorem 2.6, which plays a key role in the proof of all the invariance properties in this chapter, as well as Theorem 2.12, was proved in the seminal work of Bolthausen and Sznitman [10]. Talagrand [62] first discovered the identities (2.39), which express the Ghirlanda–Guerra identities in the setting of the Poisson–Dirichlet processes. Following a similar idea, the Ghirlanda–Guerra identities in the setting of the Ruelle probability cascades were proved by Bovier and Kurkova [11]. The current approach to both results in Theorems 2.8 and 2.10 is due to Talagrand [66]. The material in Sect. 2.4—the result in Theorem 2.13 that the Ghirlanda–Guerra identities and ultrametricity together uniquely determine the distribution of the overlap array—was well known following the proof of the Ghirlanda–Guerra identities in [25] and can be found, for example, in Baffioni and Rosati [9], Bovier and Kurkova [11], or Talagrand [66]. Theorem 2.15 is from Panchenko [43]. The fact that the Ghirlanda–Guerra identities imply ultrametricity of the overlaps, Theorem 2.14, was proved in Panchenko [50]. A partial result, which jump-started most of the research in this direction, was proved in a paper of Arguin and Aizenman [5], where, instead of the Ghirlanda–Guerra identities, the authors used the stochastic stability property discovered by Aizenman and Contucci [1]. Several other partial results, based on the Ghirlanda–Guerra identities, can be found in Panchenko [43, 47], Parisi and Talagrand [54] and Talagrand [65].

Chapter 3. The Parisi formula in Theorem 3.1 was first proved in the setting of the mixed even p-spin models in the famous work of Talagrand [64], following the breakthrough invention of the replica symmetry breaking interpolation by Guerra in [27]. The proof of the general case presented in Chap. 3, which also covers the case of odd p-spin interactions, follows the argument in Panchenko [48]; however, besides the main result in [50], all other ideas of the proof were well known to many people; see, e.g., Arguin and Chatterjee [6]. Ghirlanda and Guerra [25] proved the fundamental identities described in Sect. 3.2. Even though we do not use it here, a related property, called the stochastic stability, was discovered by Aizenman and Contucci around the same time in [1] and played an important role in the development of the area, for example, in the work of Arguin and Aizenman [5] that we mentioned above. Various proofs of the stochastic stability can be found in Contucci and Giardina [14] and Talagrand [65]. A stability property which unifies the Ghirlanda–Guerra identities and Aizenman–Contucci stochastic stability was proved in Panchenko [49] and, in some sense, it is an analogue of the Bolthausen–Sznitman invariance property (2.84) in the setting of the Ruelle probability cascades. The positivity principle, which appears in Theorems 2.16 under the exact Ghirlanda–Guerra identities and in Theorem 3.4 under the approximate Ghirlanda–Guerra identities, was originally proved by Talagrand in [62]. The current proof is new and much simplified, and it was also pointed out to me by Talagrand. As we mentioned above, the replica symmetry breaking interpolation in Sect. 3.4 was first invented by Guerra in [27], but the original proof did not utilize the Ruelle probability cascades. The general idea of the current proof in terms of the Ruelle probability cascades, which allows to simplify the computations, appears in Aizenman et al. [2]. The role of the positivity principle in the Guerra replica symmetry breaking interpolation, in the case when odd p-spin terms are present, was observed by Talagrand in [60], and the presentation here follows Panchenko [41]. We already mentioned that the cavity computation scheme in Sect. 3.5 was discovered by Aizenman, Sims, and Starr in [2]. The differentiability of the Parisi formula in Theorem 3.7 was proved by Talagrand in [61, 63], and the proof we give here is a simplification of the argument in Panchenko [42]. Strong version of the Ghirlanda–Guerra identities for the mixed p-spin models in Theorem 3.8 is from Panchenko [45]. Arguin and Chatterjee [6] proved a similar strong version of the Aizenman–Contucci stochastic stability. An application of the strong version of the Ghirlanda–Guerra identities to the chaos problem can be found in Chen and Panchenko [13]. Carmona and Hu [12] proved universality in the disorder result in Sect. 3.8, generalizing an earlier result of Talagrand [59] in the case of the Bernoulli disorder.

Chapter 4. This chapter covers about half of the material in Panchenko [46]. The other half, which concerns diluted models, is not considered here and is the main motivation to study the asymptotic distributions of all spins. In [46] one can also find the details of how the generalized Parisi ansatz is related to the computation of the free energy in some family of diluted models.

Appendix. The proof of the Aldous–Hoover representation for exchangeable arrays presented in Sect. A.1 follows the ideas of Lovász and Szegedy [34, 35, 36] in the framework of limits of dense graph sequences. A connection between the Aldous–Hoover representation and the Lovász–Szegedy representation was first explained by Diaconis and Janson in [21].

References

1. Aizenman, M., Contucci, P.: On the stability of the quenched state in mean-field spin-glass models. J. Statist. Phys. **92**(5–6), 765–783 (1998)
2. Aizenman, M., Sims, R., Starr, S.L.: An extended variational principle for the SK spin-glass model. Phys. Rev. B **68**, 214403 (2003)
3. Aldous, D.: Representations for partially exchangeable arrays of random variables. J. Multivariate Anal. **11**(4), 581–598 (1981)
4. Aldous, D.: Exchangeability and related topics. École d'été Probabilités de Saint-Flour, XIII-1983, Lecture Notes in Math., vol. 1117, pp. 1–198. Springer, Berlin (1985)
5. Arguin, L.-P., Aizenman, M.: On the structure of quasi-stationary competing particles systems. Ann. Probab. **37**(3), 1080–1113 (2009)
6. Arguin, L.-P., Chatterjee, S.: Random overlap structures: properties and applications to spin glasses. Probab. Theory Related Fields (2012) doi: 10.1007/s00440-012-0431-6
7. Austin, T.: On exchangeable random variables and the statistics of large graphs and hypergraphs. Probab. Surv. **5**, 80–145 (2008)
8. Austin, T.: Exchangeable random measures. Preprint (2013)
9. Baffioni, F., Rosati, F.: Some exact results on the ultrametric overlap distribution in mean field spin glass models. Eur. Phys. J. B **17**, 439–447 (2000)
10. Bolthausen, E., Sznitman, A.-S.: On Ruelle's probability cascades and an abstract cavity method. Comm. Math. Phys. **197**(2), 247–276 (1998)
11. Bovier, A., Kurkova, I.: Derrida's generalized random energy models I. Models with finitely many hierarchies. Ann. Inst. H. Poincaré Probab. Statist. **40**(4), 439–480 (2004)

12. Carmona, P., Hu, Y.: Universality in Sherrington-Kirkpatrick's spin glass model. Ann. Inst. H. Poincaré Probab. Statist. **42**(2), 215–222 (2006)
13. Chen, W.-K., Panchenko, D.: An approach to chaos in some mixed p-spin models. Probab. Theory Related Fields (2012) doi: 10.1007/s00440-012-0460-1
14. Contucci, P., Giardina, C.: Spin-glass stochastic stability: a rigorous proof. Ann. Henri Poincaré **6**(5), 915–923 (2005)
15. de Dominicis, C., Hilhorst, H.: Random (free) energies in spin glasses. J. Phys. Lett. **46**, L909–L914 (1985)
16. Derrida, B.: Random-energy model: limit of a family of disordered models. Phys. Rev. Lett. **45**(2), 79–82 (1980)
17. Derrida, B.: Random-energy model: an exactly solvable model of disordered systems. Phys. Rev. B (3) **24**(5), 2613–2626 (1981)
18. Derrida, B.: A generalization of the random energy model that includes correlations between the energies. J. Phys. Lett. **46**, 401–407 (1985)
19. Derrida, B., Gardner, E.: Solution of the generalised random energy model. J. Phys. C **19**, 2253–2274 (1986)
20. Derrida, B., Toulouse, G.: Sample to sample fluctuations in the random energy model. J. Phys. Lett. **46**, L223–L228 (1985)
21. Diaconis, P., Janson, S.: Graph limits and exchangeable random graphs. Rend. Mat. Appl. (7) **28**(1), 33–61 (2008)
22. Dovbysh, L.N., Sudakov, V.N.: Gram-de Finetti matrices. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. **119**, 77–86 (1982)
23. Dudley, R.M.: Real analysis and probability. In: Cambridge Studies in Advanced Mathematics, vol. 74. Cambridge University Press, Cambridge (2002)
24. Feller, W.: An Introduction to Probability Theory and Its Applications, vol. II. Wiley, New York (1966)
25. Ghirlanda, S., Guerra, F.: General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity. J. Phys. A **31**(46), 9149–9155 (1998)
26. Guerra, F., Toninelli, F.L.: The thermodynamic limit in mean field spin glass models. Comm. Math. Phys. **230**(1), 71–79 (2002)
27. Guerra, F.: Broken replica symmetry bounds in the mean field spin glass model. Comm. Math. Phys. **233**(1), 1–12 (2003)
28. Hestir, K.: A representation theorem applied to weakly exchangeable nonnegative definite arrays. J. Math. Anal. Appl. **142**(2), 390–402 (1989)
29. Hoover, D.N.: Relations on probability spaces. Preprint (1979)
30. Hoover, D.N.: Row-column exchangeability and a generalized model for probability. In: Exchangeability in Probability and Statistics (Rome, 1981), pp. 281–291, North-Holland, Amsterdam (1982)
31. Kallenberg, O.: Foundations of Modern Probability, Probability and Its Applications. Springer, New York (1997)
32. Kingman, J.F.C.: Poisson Processes. Oxford University Press, New York (1993)
33. Ledoux, M.: The concentration of measure phenomenon. In: Mathematical Surveys and Monographs, vol. 89. American Mathematical Society, Providence (2001)
34. Lovász, L., Szegedy, B.: Limits of dense graph sequences. J. Combin. Theory Ser. B **96**(6), 933–957 (2006)
35. Lovász, L., Szegedy, B.: Szemerédi's lemma for the analyst. Geom. Funct. Anal. **17**(1), 252–270 (2007)
36. Lovász, L., Szegedy, B.: Limits of compact decorated graphs. arXiv:1010.5155 (2010)
37. Mézard, M., Parisi, G., Sourlas, N., Toulouse, G., Virasoro, M.A.: On the nature of the spin-glass phase. Phys. Rev. Lett. **52**, 1156 (1984)
38. Mézard, M., Parisi, G., Sourlas, N., Toulouse, G., Virasoro, M.A.: Replica symmetry breaking and the nature of the spin-glass phase. J. de Physique **45**, 843 (1984)
39. Mézard, M., Parisi, G., Virasoro, M.: Random free energies in spin glasses. J. Phys. Lett. **46**, L217–L222 (1985)

40. Mézard, M., Parisi, G., Virasoro, M.A.: Spin glass theory and beyond. In: World Scientific Lecture Notes in Physics, vol. 9. World Scientific, Teaneck (1987)
41. Panchenko, D.: A note on Talagrand's positivity principle. Electron. Comm. Probab. **12**, 401–410 (2007)
42. Panchenko, D.: On differentiability of the Parisi formula. Elect. Comm. in Probab. **13**, 241–247 (2008)
43. Panchenko, D.: A connection between Ghirlanda-Guerra identities and ultrametricity. Ann. Probab. **38**(1), 327–347 (2010)
44. Panchenko, D.: On the Dovbysh-Sudakov representation result. Electron. Comm. Probab. **15**, 330–338 (2010)
45. Panchenko, D.: The Ghirlanda-Guerra identities for mixed p-spin model. C. R. Acad. Sci. Paris Ser. I **348**, 189–192 (2010)
46. Panchenko, D.: Spin glass models from the point of view of spin distributions. To appear in Ann. of Probab. arXiv: 1005.2720 (2010)
47. Panchenko, D.: Ghirlanda-Guerra identities and ultrametricity: An elementary proof in the discrete case. C. R. Acad. Sci. Paris Ser. I **349**, 813–816 (2011)
48. Panchenko, D.: The Parisi formula for mixed p-spin models. To appear in Ann. of Probab. arXiv:1112.4409 (2011)
49. Panchenko, D.: A unified stability property in spin glasses. Comm. Math. Phys. **313**(3), 781–790 (2012)
50. Panchenko, D.: The Parisi ultrametricity conjecture. Ann. of Math. (2) **177**(1), 383–393 (2013)
51. Parisi, G.: Infinite number of order parameters for spin-glasses. Phys. Rev. Lett. **43**, 1754–1756 (1979)
52. Parisi, G.: A sequence of approximate solutions to the S-K model for spin glasses. J. Phys. A **13**, L-115 (1980)
53. Parisi, G.: Order parameter for spin glasses. Phys. Rev. Lett. **50**, 1946 (1983)
54. Parisi, G., Talagrand, M.: On the distribution of the overlaps at given disorder. C. R. Math. Acad. Sci. Paris **339**(4), 303–306 (2004)
55. Pastur, L.A., Shcherbina, M.V.: Absence of self-averaging of the order parameter in the Sherrington-Kirkpatrick model. J. Statist. Phys. **62**(1–2), 1–19 (1991)
56. Ruelle, D.: A mathematical reformulation of Derrida's REM and GREM. Comm. Math. Phys. **108**(2), 225–239 (1987)
57. Schechter, M.: Principles of Functional Analysis. Academic, New York (1971)
58. Sherrington, D., Kirkpatrick, S.: Solvable model of a spin glass. Phys. Rev. Lett. **35**, 1792–1796 (1975)
59. Talagrand, M.: Gaussian averages, Bernoulli averages, and Gibbs' measures. Random Struct. Algorithms **21**(3–4), 197–204 (2002)
60. Talagrand, M.: On Guerra's broken replica-symmetry bound. C. R. Math. Acad. Sci. Paris **337**(7), 477–480 (2003)
61. Talagrand, M.: On the meaning of Parisi's functional order parameter. C. R. Math. Acad. Sci. Paris **337**(9), 625–628 (2003)
62. Talagrand, M.: Spin Glasses: A Challenge for Mathematicians. Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge A Series of Modern Surveys in Mathematics, vol. 43. Springer (2003)
63. Talagrand, M.: Parisi measures. J. Funct. Anal. **231**(2), 269–286 (2006)
64. Talagrand, M.: The Parisi formula. Ann. Math. (2) **163**(1), 221–263 (2006)
65. Talagrand, M.: Construction of pure states in mean-field models for spin glasses. Probab. Theory Relat. Fields **148**(3–4), 601–643 (2010)
66. Talagrand, M.: Mean-Field Models for Spin Glasses. Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge A Series of Modern Surveys in Mathematics, vol. 54, 55. Springer (2011)

Index

D. Panchenko, *The Sherrington-Kirkpatrick Model*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-6289-7, © Springer Science+Business Media New York 2013

R

S

T

U