Appendix A
Classic differential operators

In this appendix, we very briefly revise several definitions and formulas relating to the basic differential operators, namely the gradient, divergence, curl, and Laplace operators. Throughout the text, we denote the scalar product of two vectors of $\mathbb{R}^d$ as $A \cdot B$ and the vector product of two vectors of $\mathbb{R}^3$ (an orientation being fixed) as $A \wedge B$.

1 The scalar and vector cases

1.1 Definitions

Definition A.1.1. *Let f be a scalar field and let g be a vector field, sufficiently regular and defined on a domain Ω of $\mathbb{R}^d$.*

- *The* gradient of f *refers to the vector field defined by its coordinates in any orthonormal basis of $\mathbb{R}^d$ by*

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{pmatrix}.$$

- *The* divergence of g *refers to the scalar field defined by*

$$\operatorname{div} g = \sum_{i=1}^{d} \frac{\partial g_i}{\partial x_i},$$

  where the coordinates of g are refered to as $g_1, \ldots, g_d$ in an othonormal basis of the space.
- *The* Laplacian of f *refers to the scalar field defined by*

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$$\Delta f = \operatorname{div}(\nabla f) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}.$$

- *In the case $d = 3$, the* curl *of g refers to the vector field denoted as* $\operatorname{curl} g$*, defined by its coordinates in the same orthonormal basis by*

$$\operatorname{curl} g = \begin{pmatrix} \dfrac{\partial g_3}{\partial x_2} - \dfrac{\partial g_2}{\partial x_3} \\ \dfrac{\partial g_1}{\partial x_3} - \dfrac{\partial g_3}{\partial x_1} \\ \dfrac{\partial g_2}{\partial x_1} - \dfrac{\partial g_1}{\partial x_2} \end{pmatrix}.$$

Moreover, if v is a vector field of $\mathbb{R}^d$, we denote the first order differential operator as $v \cdot \nabla$, defined by

$$(v \cdot \nabla) f = \sum_{i=1}^{d} v_i \frac{\partial f}{\partial x_i}.$$

The concepts defined above are actually intrinsic; that is, they do not depend on the choice of basis. Consequently, we can express these operators in non-Cartesian coordinates. We recall that the cylindrical coordinates in $\mathbb{R}^3$ are defined by

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,$$

and that a rotating othonormal reference frame is associated with the system of coordinates, denoted as (e_r, e_θ, e_z) and defined by

$$e_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix},\ e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix},\ e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Proposition A.1.2. *Let f be a scalar field and g be a vector field, sufficiently regular and defined in cylindrical coordinates on the cylindrical domain Ω of $\mathbb{R}^3$. We then have the following formulas*

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \theta} e_\theta + \frac{\partial f}{\partial z} e_z,$$

$$\operatorname{div} g = \frac{1}{r}\frac{\partial}{\partial r}(r g_r) + \frac{1}{r}\frac{\partial g_\theta}{\partial \theta} + \frac{\partial g_z}{\partial z}, \tag{A.1}$$

where $g = g_r e_r + g_\theta e_\theta + g_z e_z$, and

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}, \tag{A.2}$$

$$\operatorname{curl} g = \left(\frac{\partial g_z}{\partial \theta} - \frac{\partial g_\theta}{\partial z}\right) e_r + \left(\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r}\right) e_\theta + \left(\frac{1}{r}\frac{\partial (r g_\theta)}{\partial r} - \frac{1}{r}\frac{\partial g_r}{\partial \theta}\right) e_z.$$

1.2 Useful formulas

Let A, B and C be three vector fields and f a scalar field. Let us start this small formulary with the following algebraic formula known as the *vector triple product*

$$A \wedge (B \wedge C) = (A \cdot C)\, B - (A \cdot B)\, C. \tag{A.3}$$

We next have the following formulae which link the various differential operators introduced above

$$\operatorname{div}(fA) = f(\operatorname{div} A) + (A \cdot \nabla) f, \tag{A.4}$$
$$\operatorname{curl}(fA) = f(\operatorname{curl} A) + (\nabla f) \wedge A, \tag{A.5}$$
$$(A \cdot \nabla) A = \frac{1}{2}\nabla |A|^2 - A \wedge \operatorname{curl} A, \tag{A.6}$$
$$\operatorname{div}(A \wedge B) = B \cdot \operatorname{curl} A - A \cdot \operatorname{curl} B, \tag{A.7}$$
$$\operatorname{curl}(A \wedge B) = (\operatorname{div} B)A - (\operatorname{div} A)B + (B \cdot \nabla)A - (A \cdot \nabla)B, \tag{A.8}$$
$$\operatorname{curl}(\operatorname{curl} A) = \nabla \operatorname{div} A - \Delta A. \tag{A.9}$$

Of course, this list is a long way from being exhaustive and only contains the formulas which are most useful in fluid mechanics.

2 Extension to second-order tensors

Our goal is not to provide a detailed general study of tensors. In this text, we only need to consider in detail tensor fields of order 2, that is, the endomorphism fields of $\mathbb{R}^d$.

We therefore simply identify second-order tensors with square matrices via the canonical basis of $\mathbb{R}^d$.

Definition A.2.1. *Let Ω be a domain of $\mathbb{R}^d$.*

- *Lets v be a regular vector field on Ω; we call the* gradient *of v, the tensor field*
$$\nabla v = \left(\frac{\partial v_i}{\partial x_j}\right)_{1 \le i,j \le d}.$$
- *Let $\sigma = (\sigma_{ij})_{1 \le i,j \le d}$ be a regular tensor field on Ω; we call the* divergence *of σ, the vector field*

$$\operatorname{div}\sigma = \left(\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1\leq i\leq d}.$$

- *Let v be a regular vector field on Ω; we call the* Laplacian *of v, the vector field*

$$\Delta v = \operatorname{div}(\nabla v).$$

We can easily verify that Δv is the vector field whose components in a fixed othonormal basis of $\mathbb{R}^d$ are the Δv_i, where v_i are the components of the field v in this same basis.

Definition A.2.2 (Tensor product). *Let u, v be two vector fields; we refer to the* tensor product of u and v*, and we write $u \otimes v$, for the second-order tensor defined in an orthonormal basis by*

$$u \otimes v = \left(u_i v_j\right)_{1\leq i,j\leq d}.$$

If we apply $u \otimes v$ to a third vector field w, then we have

$$(u \otimes v).w = (v \cdot w)u.$$

Furthermore, we have the following formula for the divergence of a tensor product

$$\operatorname{div}(u \otimes v) = (\operatorname{div} v)u + (v \cdot \nabla)u. \tag{A.10}$$

Finally, we can define a scalar product on the set of second-order tensor fields, which is simply the Euclidean scalar product in the space of the matrices identified with $\mathbb{R}^{d\times d}$. In tensor language, this is called the contracted tensor product.

Definition A.2.3 (Contracted product). *Let σ, τ be two tensors of order 2; we define their* contracted product*, to be the real number defined by*

$$\sigma : \tau = \sum_{1\leq i,j\leq d} \sigma_{ij}\tau_{ij}.$$

The norm associated with this scalar product is simply denoted as $|\cdot|$, such that

$$|\sigma|^2 = \sigma : \sigma.$$

Appendix B
Thermodynamics supplement

The goal of this appendix is to recall, in an extremely succinct and therefore necessarily very incomplete way, the main concepts of the thermodynamics of fluids which are used in Chapter I. Our basic objective is to introduce the notion of entropy and the second law of thermodynamics (in particular the Clausius theorem). Among the various reference textbooks available on this topic, we refer, for example, to [94] for a much more detailed exposition.

Thermodynamics is the study of the evolution of material systems and of their energy taking into account their internal molecular structure. Such material systems are well described by a certain number of quantities, referred to as *state variables*, among which

- Volume, mass, and density which account for the size of the system and the quantity of matter in the system under study
- Temperature and energy which account for the excitation state of the elementary particles of the matter
- Pressure which is a mechanical measure of the random motion of the particles through the collisions of these particles on a (virtual) wall placed in the material.

Other state variables are often necessary when dealing with complex systems (such as multiphase systems, for instance).

Among all these quantities, there are some which do not vary when the size of the system is increased (temperature, pressure, etc.) and these are referred to as *intensive variables*. By contrast, some quantities increase with the size of the system (volume, energy, etc.) and are referred to as *extensive variables*.

In fact, the values of the state variables are not independent. It is admitted that each homogeneous thermodynamical system satisfies an *equation of state* which is a algebraic relationship between all the state variables. Depending on the system under study, it can take the form of an expression of the pressure as a function of the temperature and the density of the system, or of an

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expression of the temperature as a function of pressure and density, or any other combination of independent variables.

In the case of fluid dynamics, we need to apply the concepts of this appendix to fluid elements $(\Omega_t)_t$. They are assumed to be in instantaneous local thermodynamical equilibrium so that each elementary fluid particle can be considered as a closed homogeneous thermodynamical system. This means that the relaxation time towards the local molecular equilibrium at any point x and any time t in the fluid is much shorter than any characteristic time for the overall evolution of the fluid.

1 Heat capacity

Definition B.1.1. *We use the term* heat capacity at constant volume *(resp.* heat capacity at constant pressure*) denoted by C_V (or C_P, resp.) to refer to the quantity of heat required to increase the temperature of a system by $1K$ while maintaining the volume (resp. the pressure) constant. It follows that an increase of temperature ΔT requires us to furnish to the system an energy which is given by*

$$\Delta Q = C_V \Delta T, \ \textit{at constant } V,$$

$$\Delta Q = C_P \Delta T, \ \textit{at constant } P.$$

Of course C_V and C_P are extensive quantities and we often make use of the specific heat capacities (i.e., per unit mass) defined by

$$c_V = \frac{C_V}{M}, \quad c_P = \frac{C_P}{M},$$

where M is the mass of the system considered.

2 The first law of thermodynamics. Internal energy

Let us now consider a closed homogeneous thermodynamic system (i.e., one which does not exchange matter with the exterior but which, nevertheless, can exchange energy in the form of heat or work). The first law of thermodynamics tells us that there exists a state variable known as the *internal energy* and referred to as E, which depends on temperature, pressure, and density such that the change in energy of the system during a transformation must be exactly equal to the external energy provided to the system. This internal energy accounts for the kinetic energy of each individual molecule in the system even though the system is at rest at the macroscopic level.

For a given elementary transformation of our closed system, we have

$$\Delta E = \Delta Q + \Delta W,$$

where ΔQ is the heat provided and ΔW is the work done by external forces acting on the system during the transformation. If we assume that the system is submitted to no external forces other than the pressure, then the term for the work reduces to the work of the pressure forces and can be written as

$$\Delta W = -P\Delta V,$$

where ΔV is the volume change undergone by the system in the course of the transformation.

Remark B.2.1. In the energy equation above, ΔE represents the change in a state variable E when passing from an initial to a final state and, importantly, this value does not depend on the path by which the system passes from the initial state to the final state. However, the values ΔQ and ΔW do not represent changes of state variables and they do depend on the path followed by the system between the two states considered.

It is a standard and useful notation to write ΔE in the form of the exact differential dE of the function E (considered as a function of some set of independent variables) but we cannot do the same for ΔQ and ΔW.

The same "differential" notation is used for any state function that we encounter in this book but we do not need any precise element of the differential form theory.

Remark B.2.2. In the context of fluid mechanics, each fluid element contains internal energy so that the total energy of this fluid element is given by

$$E_{tot} = E_{int} + E_{kin} + E_{pot},$$

where E_{kin} is the macroscopic kinetic energy of the system (computed from the mean velocity field v in the flow) and E_{pot} is its potential energy. In this formula E_{int} is the total internal energy contained in the fluid element considered. In practice, this internal energy term can be expressed as an integral term of the density of internal energy.

3 The second law of thermodynamics

3.1 Entropy

The second law of thermodynamics asserts the existence of a new state variable S, called *entropy* such that for any quasi-static transformation of the closed system considered, we would have

$$\frac{\Delta Q}{T} \leq \Delta S, \tag{B.1}$$

the equality being true if, and only if, the transformation is *reversible*. In the case of an elementary reversible transformation we therefore have

$$\Delta Q = TdS.$$

Here, we used the fact that the entropy is a state function to replace the notation ΔS by the differential notation dS.

This statement, known as the Clausius theorem, is equivalent to saying that the transfer of heat from a cold system to a warm system is forbidden.

Now, by combining these two laws of thermodynamics, we can show that in an elementary reversible transformation the internal energy of a closed system changes according to the equation

$$dE = TdS - PdV. \tag{B.2}$$

Remark B.3.1. The preceding result only applies to reversible transformations. How should we handle irreversible transformations? We can use the fact that the change in energy depends only on the initial and final states and attempt to construct a reversible transformation which will take the system from the first state to the second. Along such a path, the above relation is valid and we can therefore integrate it.

3.2 Internal energy calculation

We can now give an expression for the internal energy as a function of the other variables T, P, V, and S. To do this, it is sufficient to use the intensive or extensive character of each variable.

Let us consider a system of size 1 and then consider the same system $(1+\varepsilon)$ times as large as before, with all its proportions being conserved. The temperature and pressure do not change in this system, whereas the energy, entropy, and volume are each multiplied by $(1+\varepsilon)$. By applying (B.2), and using that T and P are the same in each state, it follows that

$$\begin{aligned}\varepsilon E &= (1+\varepsilon)E - E = \Delta E = T(\Delta S) - P(\Delta V)\\ &= T((1+\varepsilon)S - S) - P((1+\varepsilon)V - V) = \varepsilon TS - \varepsilon PV;\end{aligned}$$

we then find the expression

$$E = TS - PV.$$

By taking the differential of this expression and by comparing it to (B.2), we obtain the *Gibbs–Duhem relation* which is written as

$$S dT - V dP = 0. \tag{B.3}$$

4 Specific variables

We now work with specific variables, that is, quantities per unit mass. We call e and s the specific energy and specific entropy of the system. Furthermore, we introduce the specific volume $v = 1/\rho$ (where ρ is the density of the system being studied) as a volume variable. The relations obtained in the previous section become

$$\begin{aligned} e &= Ts - Pv, \\ s\, dT - v\, dP &= 0. \end{aligned}$$

As we have seen above, for an elementary reversible transformation at constant specific volume, the quantity of heat acquired by the system is written as $\Delta Q = T ds$ but we also have $\Delta Q = c_v dT$, from which we can deduce that

$$\left(\frac{\partial s}{\partial T}\right)_v = \frac{c_v}{T},$$

where we use here the notation $(\partial F/\partial x)_y$ for the partial derivative of the quantity F with respect to x, at a fixed value of the variable y. This notation is useful to make precise the choice of independent state variables which is done in each computation. Hence, we can write the differential of the specific entropy (considered as a function of T and v) in the form

$$ds = \frac{c_v}{T} dT + \left(\frac{\partial s}{\partial v}\right)_T dv.$$

From the Gibbs–Duhem relation (B.3), we have:

$$d(Pv) = P dv + v dP = P dv + s dT;$$

we can therefore use Schwarz's theorem on crossed second-order derivatives for any (smooth) function, which gives us one of the so-called *Maxwell relations*

$$\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial P}{\partial T}\right)_v.$$

It follows that

$$ds = \frac{c_v}{T} dT + \left(\frac{\partial P}{\partial T}\right)_v dv,$$

and then returning to the specific internal energy, we have

$$de = Tds - Pdv = c_v dT + \left(T \left(\frac{\partial P}{\partial T} \right)_v - P \right) dv. \tag{B.4}$$

Therefore, if we know the equation of state for the fluid under study (i.e., the expression of P as a function of T and v), we can calculate all of the coefficients of these last two relations.

Some classic equations of state for the dynamics of a gas are, for example:

- The ideal gas law
$$P = k \frac{T}{v},$$
where k is a constant depending only on the molar mass of the fluid study.
- The Van der Waals equation of state
$$\left(P + \frac{a}{v^2} \right) \left(1 - \frac{b}{v} \right) = k \frac{T}{v},$$
where a and b are two new constants which depend on the fluid studied. Unlike the very simple equation of state for an ideal gas, this law takes account of the diameter of the molecules (assumed to be hard spheres) and the attractive forces between molecules.

Liquids also satisfy equations of state but in that case, as explained in Section 6 of Chapter I, the flow can often be considered to be incompressible so that the particular expression of the equation of state is not useful for the description of the flow.

We observe finally that to obtain Equation (B.4), we have not used the explicit expression for the entropy and, furthermore, the entropy did not appear in the formula obtained. The simple fact of postulating the existence of this state variable, S, allows us to apply the Schwarz theorem and to deduce the fundamental thermodynamic relations from it. In the incompressible framework which is the main topic of this book, the entropy does not explicitly appear in the final system of equations describing the flow. However, it is a fundamental quantity in compressible flow modelling.

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