HINTS AND PARTIAL SOLUTIONS TO SELECTED ODD-NUMBERED EXERCISES

SECTION 1.1

1. (a) false proposition (b) not a proposition (c) false proposition
   (d) not a proposition (e) not a proposition (f) true proposition

3. *Theorem.* madam
   Proof.
   S
   mSm
   maSam
   madam

5. Without further discussion, "hence" is not truth functional. Consider

   (a) *Spinach is a vegetable, hence there is at least one green vegetable.*

   (b) *Spinach is a vegetable, hence Mars is a planet.*

   Both are sentences of the form "*true sentence* hence *true sentence*." Most people would agree that (a) is true; but they would be reluctant to assign a truth value to (b), since "hence" suggests an evident linkage (missing in this example) between the components it connects.

SECTIONS 1.2 and 1.3

1. (a) $\neg P \wedge \neg Q$: Howard did not fall and Howard did not break his leg.
   $Q \vee \neg P$: Howard broke his leg or Howard did not fall.

L.J. Gerstein, *Introduction to Mathematical Structures and Proofs*, Undergraduate Texts in Mathematics,
DOI 10.1007/978-1-4614-4265-3, © Springer Science+Business Media, LLC 2012

3.

P	Q	$P \veebar Q$
F	F	F
F	T	T
T	F	T
T	T	F

5. The cards with the following markings need not be turned over: C, 2.

7. Let P denote the statement "I go to the movies in the afternoon," and let Q denote "It is rainy."
   (a) $Q \Rightarrow P$ (b) $P \Rightarrow Q$

9. (b) P: Napoleon is President of the United States.
   Q: Boston is a city.

11. (b)

P	$P \Rightarrow P$	$P \Rightarrow (P \Rightarrow P)$
F	T	T
T	T	T

13. 122

15. The new table has eight times as many rows as the old one.

17. $\neg(P \wedge \neg(\neg(Q \wedge \neg(\neg(R \wedge \neg S)))))$

19. The statements are equivalent since they have the same truth value (namely, *false*).

SECTION 1.4

1. Represent the statements "Dracula seizes power," "democracy is lost," "the use of food additives increases," and "mutations can be expected" by the letters P, Q, R, and S, respectively. Since we are given that P and $P \Rightarrow Q$ are both true, it follows by modus ponens that Q is true. From that and the given truth of $Q \Rightarrow R$, modus ponens yields the truth of R. Continue the argument in this way.

3. (c) I must prove that there exist a point P and a line L such that G consists of all the points that are equidistant from P and L.

5. Try a proof by contradiction. If 5 is *not* prime then (by definition of *prime*) 5 can be written in the form $a \cdot b$ for some integers a, b strictly between 1 and 5. Check that that is *not* the case.

7. Compare a side of the diamond to a radius of the circle, and use the fact that all radii of a given circle have the same length.

SECTION 1.5

1. Statement forms (a), (b), and (d) are tautologies, but (c) is not.

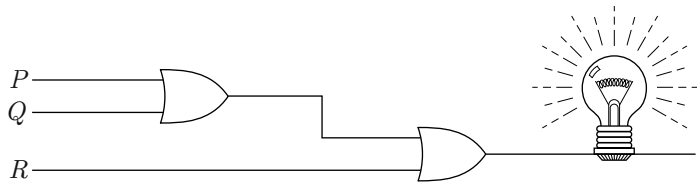
3. (a) $\neg((P \wedge Q) \wedge \neg R)$ (b) $\neg(\neg P \wedge \neg Q)$

5. $\neg P$: $P \downarrow P$; $P \vee Q$: $(P \downarrow Q) \downarrow (P \downarrow Q)$

7. Use the fact that $(R \vee \neg R)$ is a tautology to deduce that the given expression is logically equivalent to $(P \wedge Q) \vee (P \wedge \neg Q)$, then contract this further using a distributive law. Finally, conclude that the given expression is logically equivalent to P.

SECTION 1.6

1. (a)



SECTION 2.1

1. yes
3. (a) $\{2,3,5\}$ (b) $\{2,3,5,7\}$ (c) $\{1,2,3\}$ (d) $\{0,\pm1,\pm2,\pm3,\pm4\}$
5. (a) true (b) true (c) false (d) false (e) false (f) false (g) true (h) false (i) true (j) false
7. (a) $A = \emptyset$, $B = \{\emptyset\}$, $C = \{\{\emptyset\}\}$
   (b) $A = \emptyset$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$

SECTION 2.3

1. (a) $(\exists x)(x \in \mathbb{N} \wedge x^3 + 15 = 22)$ (c) $\neg(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y^2 = x)$
   (e) $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(y^3 = x)$
3. (a) With $A = \emptyset$, statements 2(a) and 2(b) are both true.
   (b) There is no set A for which the *hypothesis* of 2(b) is true, so for every set A, *statement* 2(b) is true.
5. (a) $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x')(|x' - x| < \delta \Longrightarrow |f(x') - f(x)| < \epsilon)$. More formally: $(\forall \epsilon)(\epsilon > 0 \Longrightarrow (\exists \delta)(\delta > 0 \wedge (\forall x')(|x' - x| < \delta \Longrightarrow |f(x') - f(x)| < \epsilon)))$
7. P: $x^2 = 5$; Q: $x = 17$
9. (a) true (c) true (e) false (g) true

SECTION 2.4

1. (b) The subsets are $\emptyset$, $\{1\}$, $\{\{2,3\}\}$, and B.
3. (a) $\{2,-1\}$ (c) $\{2,7\}$ (e) $S = \{-2,-1,0,1,2,\{1,2\}\}$
5. Suppose $\{a\} \subseteq S$. Then every member of $\{a\}$ is a member of S. But the only member of $\{a\}$ is a. This shows that $a \in S$, verifying "$\Rightarrow$".
7. (a) The left-hand set is equal to $\{1,2\}$, and $1 = 2^0$ and $2 = 2^1$.
9. Let $A = \{1\}$ and $B = \{1,\{1\}\}$.

SECTION 2.5

1. (a) $\{1, 2\}$ (c) $\{3, 4, 5\}$ (e) the set of all the even integers except for 4 and 6; in symbols: $\{0, \pm 2, -4, -6, \pm 8, \pm 10, \pm 12, \ldots\}$ (g) the members are 3, 5, and all the even integers: $\{3, 5, 0, \pm 2, \pm 4, \pm 6, \ldots\}$
   (i) $\emptyset$.
3. (a) false (b) true
5. (a) Observe that for any element x the statement $x \in A$ is equivalent to the statement
$$x \in A \ \vee \ x \in \emptyset.$$
   (c) First note that $A - A = \emptyset$, because no element satisfies "$x \in A \ \wedge \ x \notin A$." Then check that $\emptyset - A = \emptyset$.
   (e) For every element x, the statements "$x \in A \ \wedge \ x \in A$" and "$x \in A$" are equivalent.
7. (a) Every element of $A - B$ is an element of A, but no element of $B - A$ is. Therefore the given intersection contains no elements.
9. Let $z \in Z$. From the hypothesis we then have $z \in X$ and $z \in Y$, as desired.
11. (a) For every element x, the statements
$$x \in A \ \vee \ x \in B \quad \text{and} \quad x \in B \ \vee \ x \in A$$
   are equivalent. Therefore $A \cup B = B \cup A$. This proves that $A + B = B + A$. Proof of $AB = BA$ is similar, using intersection instead of union.
   (c) We have $AB = A \ \Leftrightarrow \ A \cap B = A \ \Leftrightarrow \ B \supseteq A$. Thus the sets B satisfying the given condition are the sets that contain A.
   (e) To prove "$\Leftarrow$", check that the statements "$x \in A$" and "$x \in A \vee x \in \emptyset$" have the same truth value. For "$\Rightarrow$", check that if X contains some element w, and A is a set not containing w, then $A + X \neq A$.

SECTION 2.6

1. (a) $\{1, 2, 3, 4, 0, -1\}$ (b) $\{1\}$ (c) $\mathbb{Z} - \{1\}$ (d) $\mathbb{Z} - \{-1, 0, 1, 2, 3, 4\}$
3. Let $x \in \cup A_i$. Then, by definition of union, $x \in A_i$ for some $i \in I$. But A_i is given to be a subset of A_n, so $x \in A_n$. This proves "$\subseteq$". Conversely, if $x \in A_n$ then since $n \in I$, we have that x is a member of $\cup A_i$, from the definition of union.
5. (a) $\mathbb{R}$ (b) $\{0\}$ (c) $[-5, 5]$
7. Proof of "$\subseteq$":
$$\begin{aligned} x \in A - \cup B_i \ &\Rightarrow \ x \in A \quad \text{and} \quad x \notin \cup B_i \\ &\Rightarrow \ x \in A \text{ and there is no } i \in I \text{ such that } x \in B_i \\ &\Rightarrow \ x \in A - B_i \text{ for each } i \in I \\ &\Rightarrow \ x \in \cap (A - B_i) \end{aligned}$$

9. (a) union: $\mathbb{R}$; intersection: $\emptyset$ (b) union: the set of all points in the plane whose horizontal coordinate x satisfies the inequality $-1 \leq x \leq 1$; intersection: $\emptyset$ (d) union: Π; intersection: $\emptyset$

SECTION 2.7

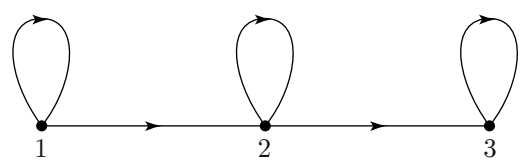
1. (a) $\{\emptyset, \{4\}\}$ (b) $\{\emptyset, \{5\}, \{6\}, \{5,6\}\}$
3. 127
5. $X \in P(A) \Rightarrow X \subseteq A \Rightarrow X \subseteq A \subset B \Rightarrow X \subset B \Rightarrow X \in P(B)$. This proves "$\subseteq$".
7. $A \cap B = \emptyset$

SECTION 2.8

1. (a) Associate the ordered pair (t, r) with an incoming call to room r on telephone line t.
   (b) The subset is a collection of four ordered pairs in $T \times R$, no two of which have the same first coordinate or same second coordinate.
3. "$\Rightarrow$". Let $A \times B = \emptyset$. If $A \neq \emptyset$ and $B \neq \emptyset$, let $a \in A$ and $b \in B$. Then $(a, b) \in A \times B$, contradicting the assumption that $A \times B = \emptyset$. Therefore, the assumption "$A \neq \emptyset$ and $B \neq \emptyset$" is false. That is, $A = \emptyset$ or $B = \emptyset$.
5. (a) Let $S = \{(1,2), (2,3)\}$. If $S = A \times B$, then $1, 2 \in A$ and $2, 3 \in B$. But then $(1,3) \in S$, a contradiction.
9. "$\subseteq$": Let $P \in (\cup A_i) \times S$. Then $P = (a, b)$ for some $a \in \cup A_i$ and some $b \in S$. Thus $a \in A_i$ for some $i \in I$, by definition of $\cup A_i$, hence $P \in A_i \times S$, and so $P \in \cup(A_i \times S)$.

SECTION 2.9

1. (a) {the set of negative real numbers, the set of nonnegative real numbers}; {the set of nonzero real numbers, $\{0\}$}; {the set of nonnegative rational numbers, the set of negative rational numbers, the set of irrational numbers}
3. (a) false (b) false (c) true (d) false
5. (a) $\Pi_1 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5,6\}\}$; $\Pi_2 = \{\{1,2\}, \{3,4\}, \{5,6\}\}$; $\Pi_3 = \{\{1,2,3,4\}, \{5,6\}\}$; $\Pi_4 = \{\{1,2,3,4,5,6\}\}$.
7. $R \cup \{(1,1), (2,2), (3,3), (4,4)\}$
9. $R \cup \{(1,1), (4,2), (4,3), (4,4)\}$
11. (a) Example 2.53(c), R_1:



(b) For each pair of vertices p, q for which there is an edge from p to q, there is also an edge from q to p.

(d) For each pair of vertices p, q for which there is a path from p to q formed by a succession of directed edges, there is also a single edge leading directly (that is, without passing through any other vertices along the way) from p to q.

15. (b) REFLEXIVE: for each $a \in \mathbb{N}$ we have $a \mid a$, because $a = a \cdot 1$.
ANTISYMMETRIC: if $a \mid b$ and $b \mid a$, then $b = ac$ and $a = bd$ for some $c, d \in \mathbb{N}$. Then $a = bd = acd$, therefore $cd = 1$, and so $c = d = 1$. This gives $a = b$.
TRANSITIVE: if $a \mid b$ and $b \mid c$ then $b = ad$ and $c = be$ for some $d, e \in \mathbb{N}$. Then $c = a(de)$, and so $a \mid c$.

17. (a) $(1,1), (2,2), (3,3)$ (b) impossible

19. We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. This gives

$$R \subseteq A \times B \subseteq (A \cup B) \times (A \cup B).$$

21. The relation $\{(1,2), (2,1), (1,1), (2,2)\}$ is symmetric and transitive, but not reflexive.

SECTION 2.10

1. When $n = 1$, both sides of the given expression are equal to 1. Having assumed the result to hold for $n = k$, observe that

$$1^2 + 2^2 + \cdots + (k+1)^2 = (1^2 + 2^2 + \cdots + k^2) + (k+1)^2. \tag{1}$$

Now apply the induction hypothesis to expression $(*)$, and use arithmetic to check that the result is equal to

$$\frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}.$$

3. If $n \leq 0$ the result follows from the fact that squares are nonnegative. So it is enough to prove the statement when $n \geq 1$, and for this we use induction. For $n = 1$ the statement is clearly correct. Now suppose we know that $k^2 \geq k$ for some positive integer k. Then we have

$$(k+1)^2 = k^2 + 2k + 1 \geq 3k + 1 \geq k + 1,$$

and this completes the proof.

5. Use the fact that $3k + 3 = 3(k + 1)$.

7. For $n = 10$: $1024 > 1000$. Now assume that $2^k > k^3$ for some $k \geq 10$; it must be shown that $2^{k+1} > (k+1)^3$. But $2^{k+1} = 2(2^k) > 2k^3$, so it will suffice to verify the inequality $2k^3 > k^3 + 3k^2 + 3k + 1$, or, equivalently, $k^3 - 3k^2 - 3k - 1 > 0$. This can be done in a variety of ways. For instance, the given inequality is equivalent to the inequality $k(k^2 - 3k - 3) > 1$, so it is enough to show that $k^2 - 3k - 3 > 0$ when $k > 10$.

9. Consider whether the argument is legitimate when $k = 1$.

11. Let A be as in the suggestion given with the problem, and let k be the smallest element of A. (Such a k exists by the well-ordering principle.) So k is the smallest positive integer for which $P(k)$ is false. Hence $P(k-1)$ is true. But then, by 2.67(b), statement $P(k)$ must be true. [Remember: $k = (k-1)+1$.] CONTRADICTION. Therefore we must have $A = \emptyset$. So $P(n)$ is true for all $n \in \mathbb{N}$.

13. If $n = 1$, then both sides of the equation being checked are equal to a^{b_1}. Now suppose $P(k)$ is true for some $k \geq 1$. That is, $\prod_{i=1}^{k} a^{b_i} = a^{\sum_{i=1}^{k} b_i}$. Then, by the laws of exponents and the induction hypothesis, we have

$$\prod_{i=1}^{k+1} a^{b_i} = \left(\prod_{i=1}^{k} a^{b_i}\right) \cdot a^{b_{k+1}} = a^{\left(\sum_{i=1}^{k} b_i\right)} \cdot a^{b_{k+1}} = a^{\left(\sum_{i=1}^{k} b_i\right)+b_{k+1}} = a^{\sum_{i=1}^{k+1} b_i}.$$

This completes the induction.

SECTION 3.1

1. Sets f and g are functions from A to B, but h is not a function; j is not a function from A to B, but j is a function from $\{2,3\}$ to B.

3. If $x \in X$ and $y_1, y_2 \in Y$, then $(x, y_1), (x, y_2) \in X \times Y$. Consider the consequences if $y_1 \neq y_2$.

5. 27

7. (a) $\chi_{\emptyset}(x) = 0 \quad \forall x \in A$

   (c) "$\Leftarrow$": Assume that $\chi_A = \chi_B$, and let $x \in A$. To show that $x \in B$, it suffices to check that $\chi_B(x) = 1$. But $\chi_B(x) = \chi_A(x) = 1$. (Here the first equality follows from the hypothesis $\chi_A = \chi_B$, and the second equality holds because $x \in A$.) This shows that $A \subseteq B$, and a similar argument shows that $A \supseteq B$.

9. (a) The answer is 0 if $\alpha \in \mathbb{Z}$, and otherwise the answer is -1.

   (b) If $m \leq \alpha < m+1$, with $m \in \mathbb{Z}$, then $n+m \leq n+\alpha < n+m+1$; so $[\alpha] = m$ and $[n+\alpha] = n+m = n+[\alpha]$.

   (c) Choose β to satisfy the inequality $(1 + 1/[\alpha]) < \beta < 2$.

   (d) First consider the case $m \geq n$, then the case $m < n$.

11. (a) $\{(1,1),(2,1)\}, \quad \{(1,1),(2,2)\}, \quad \{(1,2),(2,1)\}, \quad \{(1,2),(2,2)\}$

   (b) Let $f \in A^C$. That is, $f \subseteq C \times A$, and $\forall c \in C$ there is a unique $a \in A$ such that $f(c) = a$. But $A \subseteq B$, so $a \in B$ and $C \times A \subseteq C \times B$. Thus $f \subseteq C \times B$, and $\forall c \in C$ there is a unique $a \in B$ such that $f(c) = a$. That is, $f \in B^C$; this proves the result.

   (c) Each member of $\{1,2\}^{\{1,2,3\}}$ is a set of ordered pairs, and one of those pairs has 3 as its first coordinate. Consider whether the same can be said about the members of $\{1,2\}^{\{1,2\}}$.

SECTION 3.2

1. (a) $\operatorname{im} f = \operatorname{im} g = \{1\}$ (c) the set of nonnegative real numbers (e) $\operatorname{im} f$ is the set of nonnegative real numbers; $\operatorname{im} g$ is smaller, consisting of just the set of squares of rational numbers. Thus we can write

$$\operatorname{im} g = \{a^2/b^2 \mid a, b \in \mathbb{Z},\ b \neq 0\}.$$

3. (a) injective only (c) neither injective nor surjective
5. (a) $g^{-1} = \{(5,1), (-2,2), (6,3)\}$ (b) If $f\colon A \to B$ is not injective, then there exist distinct elements $a_1, a_2 \in A$ and an element $b \in B$ such that (a_1, b), $(a_2, b) \in f$. But then $(b, a_1), (b, a_2) \in f^{-1}$, and hence f^{-1} is not a function.
7. The only available representatives of $\{1\}$ and $\{6\}$ are 1 and 6, respectively. So the representative of $\{3,6\}$ must be 3, and therefore the representatives of $\{3,4\}$ and $\{1,2,3\}$ must be 4 and 2, respectively. This forces 5 and 7 to be the respective representatives of $\{2,4,5\}$ and $\{1,4,7\}$.
9. (a) The function f is a bijection if and only if $a \neq 0$. (b) The necessary and sufficient condition is that $a = \pm 1$.
13. Parts (a) and (b) follow from the fact that the absolute value of any number is nonnegative, and it is zero if and only if the number itself is zero.
15. Compare the current estimate with the preceding one or, if necessary, with the following one. If the current answer agrees with either of those through n decimal places, then it has the desired degree of accuracy.
17. The idea is to obtain a function h such that $h(1) = f(1)$, $h(2) = g(1)$, $h(3) = f(2)$, $h(4) = g(2)$, and so on. Such a function h will take the odd positive integers onto $\operatorname{im} f = A$ and the even positive integers onto $\operatorname{im} g = B$. Explicitly, define h by

$$h(x) = \begin{cases} f\left(\dfrac{x+1}{2}\right) & \text{if } x \text{ is odd,} \\ g\left(\dfrac{x}{2}\right) & \text{if } x \text{ is even.} \end{cases}$$

   Then check that this function works.
19. Go from $(1,1)$ to $(2,1)$ to $(2,2)$ to $(1,2)$, and continue.
21. Spiral out from the origin.

SECTION 3.3

1. (a) $\mathbb{R}$; (c) $\mathbb{R} - \{n\pi \mid n \in \mathbb{Z}\}$.
3. (a) If $f(x) = x + 1$ and $g(x) = x/2$, then $(f \circ g)(3) = \frac{5}{2} \neq (g \circ f)(3) = 2$.
7. (b) Let $c \in C$. Because $g \circ f$ is surjective, there exists $a \in A$ such that $(g \circ f)(a) = c$. That is, $g(f(a)) = c$. Thus we have found an element (namely, $f(a)$) of B that is taken to c by g.

9. Consider a function f defined by

$$f(x) = \begin{cases} x + 5 & \text{if } x \geq 0, \\ x & if\text{x } ¡\ 0. \end{cases}$$

This function "tears a hole" in $\mathbb{Z}$. Now try to patch the hole with a suitable function g.

11. The given function *is* a bijection.

13. (a) Suppose $A = \{1, 2\}$ and $C = \{3, 4\}$. Then, because $\operatorname{im} f^{-1} = A$ and $\operatorname{dom} g^{-1} = C$, it follows that $g^{-1} \circ f^{-1}$ is meaningless, though $(g \circ f)^{-1}$ has meaning.

15. To get the left cancellation law, notice that from the hypothesis we have

$$f^{-1} \circ (f \circ g) = f^{-1}(f \circ h),$$

and proceed from there.

SECTION 4.1

1. (b) We think of two collections as being the same "size" if each object in one can be paired with an object in the other so that when we're done there is nothing left over in either collection. (Demonstrate with a few pears and apples, say.) The statement says that if a first collection and a second collection have the same size in this sense, and also the second collection and a third collection have the same size, then the first and third collections have the same size. (Again demonstrate with actual objects, showing how the pairings for the first and second sets and the pairings for the second and third sets lead naturally to pairings of the elements of the first set with the elements of the third set.)

3. (a) Consider the mapping given by $n \mapsto 5n + 2$.

5. (a) Define $f\colon [0, 1] \to [2, 7]$ by $f(x) = 2 + 5x$.

(b) Define $f\colon \mathbb{Z} \to \{\text{positive evens}\}$ by

$$f(n) = \begin{cases} 4n & \text{if } n > 0 \\ 4\,|n| + 2 & \text{if } n \leq 0 \end{cases}$$

7. Use differentiation to show that the function is increasing, and hence one-to-one. To show surjectivity it suffices to show that the function is unbounded from above and therefore (why?) also from below.

11. For $n = 1$ there is really nothing to prove, since both sides are equal to $\#A_1$. Corollary 4.16 does the job for $n = 2$. Then rewrite $\cup_{i=1}^{k+1} A_i$ in the form $(\cup_{i=1}^{k} A_i) \cup A_{k+1}$, and use the case $n = 2$ to complete the induction step.

13. According to the definition of "+" in Exercise 10, the job here is to show that if $X \approx (B - A)$ and $Y \approx A$, where $X \cap Y = \emptyset$, then $X \cup Y \approx B$.

15. (a) Apply the pigeonhole principle (4.9) to a function from a given set of eight people to the set of days of the week.
17. (a) Use the product rule (4.22).
    (b) Once the activities have been chosen as in part (a), there remains the choice of *order* for the activities, and the number of possibilities for that is equal to the number of ways of listing the elements of the set {snack, movie, art}. Having determined that number, invoke the product rule.
19. Be careful not to count a given handshake more than once.
21. (a) Each of the n elements of B has m candidates in A that it might be paired with. In all, there are n choices to be made, with m possibilities for each. Invoke the product rule.
    (b) The hint gives the number of subsets of A that can serve as the function's image and, for each of those, part (a) gives the number of functions with domain B that have that image. Use the product rule.

SECTION 4.2

1. (a) "$\Leftarrow$": Assume $A = \emptyset$. The conditional statement

$$\text{If } x \in \emptyset \text{ then there exists } y \in \emptyset \text{ such that } \emptyset(y) = x.$$

   is true, because its hypothesis is false. Therefore $\emptyset$ is surjective.
3. This proof is a clone of the proof of the first part of the theorem.
5. (a) The mapping $x \mapsto x$ is an injection from $A - B$ to A.
   (c) Consider the consequences if $\#A > \#B$ and $A \cap B = \emptyset$.
7. A typical element of the square has the form

$$(.a_1a_2a_3\ldots, .b_1b_2b_3\ldots).$$

   Now follow the given suggestion.

SECTION 4.3

1. $\mathbb{Z} = \mathbb{N} \cup A \cup B$, where $A = \{n \in \mathbb{Z} \mid n \leq 0 \text{ and } n \text{ is even}\}$ and B is the set of odd negative integers
3. (a) By Corollary 4.15, at least one of the blocks must be infinite. Now apply Theorem 4.36(a).
5. Check that the mapping given by $n \mapsto 7n + 3$ is a bijection from $\mathbb{Z}$ to the given set. Then apply Example 4.34.
7. (a) Show that $\mathbb{N} \times \mathbb{N} = \cup_{i \in \mathbb{N}}(\mathbb{N} \times \{i\})$.
   (b) Theorem 4.39(a) will be useful.
9. First check that for each $q \in \mathbb{Q}$, the collection of all intervals having rational endpoints and right endpoint q is a countable set. Then use the result of Exercise 8.

11. INJECTIVITY: Suppose $g(m,n) = g(r,s)$. That is, $2^{m-1}(2n-1) = 2^{r-1}(2s-1)$. Since $2n-1$ and $2s-1$ are both odd it follows (from the Fundamental Theorem of Arithmetic) that $2^{m-1} = 2^{r-1}$ (and hence $m = r$), since both numbers represent the power of 2 in the standard factorization of $g(m,n)$. Therefore $2n-1 = 2s-1$, and so $n = s$. This proves that g is injective.
SURJECTIVITY: Let $k \in \mathbb{N}$; say $k = 2^t \cdot q$, with q odd and $t \geq 0$. Then $k = g(t+1, \frac{q+1}{2})$; so g is surjective.

13. Bet that neither coordinate will be rational.

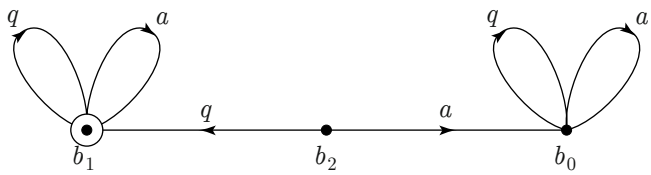
SECTION 4.5

1. View x and y as functions from $\mathbb{N}_m$ and $\mathbb{N}_n$ (respectively) into Σ. Then xy is the function from $\mathbb{N}_{m+n}$ into Σ given by

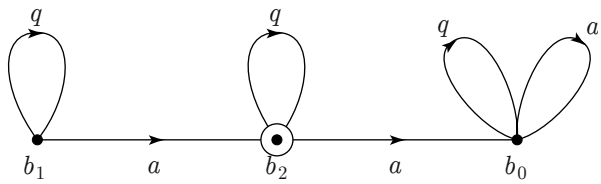
$$(xy)(i) = \begin{cases} x(i) & \text{if } 1 \leq i \leq m, \\ y(i-m) & \text{if } m < i \leq m+n. \end{cases}$$

3. (a) This consists of the set of all words of the form $w\,w$ with $w \in L$.
   (b) Any language containing the empty word ϵ will do.
5. The languages $L(M)$ and $L(N)$ are the same.
7. Let's say that the initial states of M_1 and M_2 are q_0 and q_0', respectively. Erase the arc starting at q_0 that is labelled b, and erase the arc starting at q_0' that is labelled a. Then drag the diagram for M_1 over to the diagram for M_2 in such a way that q_0 and q_0' become superimposed, but so that no other vertices of the two graphs come into contact. The result is the graph of the desired automaton. The point obtained from merging q_0 and q_0' represents the initial vertex of the new automaton, and the new collection of final states is the union of the final state sets of M_1 and M_2. It remains to check that this construction works.

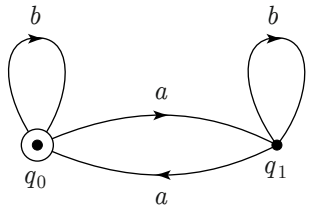
9. (a)



(c)



(e)



SECTION 5.2

3. (a) 360
   (b) 144 [First determine how many positive divisors *are* divisible by 10, then apply part (a).]
5. 1320

SECTION 5.3

1. (a) a permutation (c) not a permutation (e) a permutation
3. There are 24 members altogether.
5. (a) Given $\lambda \in S_m$, extend λ to an element $\lambda \in S_n$ by the formula

$$\lambda(x) = \begin{cases} \lambda(x) & \text{if } 1 \le x \le m, \\ x & \text{if } m+1 \le x \le n. \end{cases}$$

7. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$
9. Begin by noticing that the sets of elements moved by the given permutations are disjoint.
11. $8! = 40320$
13. (a) $10!$ (b) $2 \cdot 3! \cdot 7!$ (c) $8 \cdot 3! \cdot 7!$ (d) Arrange the 7 dogs in a line. There are 7! ways to do this. Each cat will go between two dogs or at an end location. So there are 8 positions in line available for cats. Pick a cat. For it there are 8 possible locations. Having put it in place, there are 7 possible locations for the next cat and 6 possibilities for the third. Final answer: $7! \cdot 8 \cdot 7 \cdot 6$. (e) $7 \cdot 6 \cdot 8!$
15. (a) If in a move the cup in position i goes to position j, associate $\sigma \in S_5$ such that $\sigma(i) = j$.
    (b) $\frac{5 \cdot 4}{2} = 10$ (c) $5!$
    (d) $4!$ (We know where the left-most cup goes. So it's just a matter of counting the repositions of the remaining 4 cups.)
17. $\frac{7!}{2}$

SECTION 5.4

1. Suppose the triangle's vertices are A, B, C, and let s be a symmetry. Show that if $s(A) = B$ then $s(B) = A$, and therefore $s(C) = C$. Then use this to obtain a contradiction.

3. (a) There are eight members.
   (b) Inscribe a regular triangle in a regular hexagon, and proceed from there.
5. (a) ten symmetries (c) one symmetry (e) four symmetries
   (g) one symmetry (i) the collection of symmetries is countably infinite
   (k) the collection of symmetries is uncountably infinite
7. (a) If $P = Q$ then $R_Q(P) = P$. If $P = (c, d) \neq Q$, then obtain the equation of the line through P and Q and the equation of the circle centered at Q with radius PQ. The intersection of the two geometric figures is obtained by solving these equations simultaneously, and the solution set is $\{P, R_Q(P)\}$.
   (b) Use the formula obtained in (a) to show that for any two points P_1, P_2, the distance between $R_Q(P_1)$ and $R_Q(P_2)$ is equal to the distance between P_1 and P_2.

SECTION 5.5

1. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$
3. (a) First notice that once σ^2 has been computed, only one more multiplication is required in order to compute σ^4.
5. They look the same, except that the edge directions are all reversed.
7. (a) The notation tells us that the listed elements are moved cyclically by σ, and that no other elements are moved by σ, but it does not tell us the whole domain.
   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 4 & 2 & 7 & 1 & 6 \end{pmatrix}$
9. (a) $\begin{pmatrix} 1 & 5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 6 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 4 & 3 \end{pmatrix}$
11. Consider the products $(\sigma\tau)(\tau^{-1}\sigma^{-1})$ and $(\tau^{-1}\sigma^{-1})(\sigma\tau)$.
13. If α satisfies the given equation, it follows that $\alpha = \sigma \begin{pmatrix} 2 & 1 & 4 \end{pmatrix} \lambda^{-1}$.
15. Notice that once *some* n is obtained for which S_n contains a non-cycle, then the same is true for all larger values of n.

SECTION 5.6

3. Use the method of Example 5.42.
5. Begin by factoring 2520 into primes, and then use this data to obtain a set of integers whose sum is 30 and whose least common multiple is 2520. Then apply Theorem 5.45.
7. (b) Every element of S_m has a natural extension to an element of S_n. See Exercise 5 in Section 5.3.
9. 18 (Use the method of Example 5.47.)

SECTION 5.7

1. (a) $(1\ \ 2)\,(1\ \ 5)\,(1\ \ 4)\,(1\ \ 6)\,(1\ \ 8)\,(1\ \ 7)\,(1\ \ 3)$ (c) $(1\ \ 2)\,(3\ \ 4)$
3. (a) two reversals (b) four reversals
5. Show that the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}$$

has the desired property.

7. Use Theorem 5.54.
9. Use Corollary 5.36, together with the fact that s moves exactly four elements.
11. (a) -1 (b) 1 (c) -1
13. (a) Observe that a shift of the puzzle pieces to the indicated configuration is associated with the permutation $\big(1\ \ 2\ \ 6\ \ 7\ \ 11\ \ 12\ \ 16\big)$, an even permutation.
15. Check each of the cases from $n = 1$ through $n = 5$ to get the idea.

SECTION 5.8

1. (a) 9863 (c) 1001 (e) 39711 (g) 1
5. (a) 336 (b) 231 (c) 2398
7. Use the binomial theorem.
11. Check the cases $n = 1$, 2, 3 individually. For $n \geq 4$, use Theorem 5.64.
13. (a) 1120 (b) 1883
15. $\frac{13 \cdot 48}{\binom{52}{5}} = \frac{1}{4165}$
17. (a) $\frac{35}{128}$ (b) $\frac{35}{64}$
19. If a set of n objects is to be assembled in sequence, with only two types of object in the set, consider the consequence of specifying the locations of one of the types.
21. 2,522,520
23. (a) 3,628,800 (b) 30,240 (c) 252
25. -280

SECTION 5.9

3. There are rs edges between vertices in V_1 and V_2. Et cetera.
5. (i) There is at least one cycle.
7. Find the smallest n such that $\binom{n}{2} \geq 50$ and explain why this is the answer.
9. Produce a spanning tree with a vertex of degree 3, and another without such a vertex.

11. (i) Consider a shortest path from v to x followed by such a path from x to w, and then think about the result.
13. If $e = xy$ is an edge then xey is a path from x to y.

SECTION 6.1

1. (a) The left side of the rule becomes $\cdot((a, +((b, c))))$.
3. n^3
5. The case $n = 3$: define a 3-ary (more commonly, *ternary*) operation $*_3 \colon S^3 \to S$ by
$$*_3((s_1, s_2, s_3)) = (s_1 * s_2) * s_3.$$
7. Suppose there *is* an identity element; call it e. Then for each $a \in \mathbb{R}$, the equation $a \div e = a$ holds. But this implies that $e = 1$. (Why?) Check that in fact 1 is *not* an identity element for $\div$.
9. The statement $(x * y) * z = z * (y * z)$ must be checked for each substitution of elements from the set $\{e, a\}$ in place of the symbols x, y, z. There are eight statements to check altogether.
11. First explain how to check the table to see if there is an identity element e for the operation. (If there is none, then no inverses exist.) Assuming that e exists, the element e must appear in every row of the table for inverses to exist. Having checked that, it remains to carry out an appropriate check of the *columns* of the table.
13. (a) yes, yes, no (c) yes, yes, yes (e) yes, no, no
15. If e is an identity for $*$, then
$$ra + se = a * e = e * a = re + sa, \quad \forall a \in \mathbb{Z}.$$
Consider the consequences when $a \neq e$.

SECTION 6.2

1.
$$\begin{aligned} (a+b)c &= c(a+b) && \text{since multiplication is commutative} \\ &= ca + cb && \text{by the left-hand rule in (6.7)} \\ &= ac + bc && \text{since multiplication is commutative.} \end{aligned}$$

3. (a) By using the distributive laws we obtain
$$(a+b)(c+d) = (a+b)c + (a+b)d = ac + bc + ad + bd.$$

5. Suppose that for some $a \in \mathbb{Z}$ we have $0 \cdot a = b \neq 0$. Then, from the properties discussed in the text, we deduce that
$$a = 1 \cdot a = (1+0) \cdot a = 1 \cdot a + 0 \cdot a = a + b.$$
Now add $-a$ to both sides of this equation to get a contradiction.

7. Use 6.11.
9. Define "$\leq$" in a way modelled after the definition of "$<$" in 6.11.

SECTION 6.3

1. From the hypothesis, we have $b = ax$ and $c = ay$ for some $x, y \in \mathbb{Z}$. This observation will lead to the result.
3. Notice that if $x \in \mathbb{R}$, then

$$[x] = x \quad \Leftrightarrow \quad x \in \mathbb{Z}.$$

7. After the sieving procedure is complete, thirty primes should remain on your list of integers.
9. It suffices to show that n is divisible by 2 and by 3. Achieve these goals one at a time. (The division algorithm (6.17) will be helpful for this purpose, with $k = a$.)
13. The expression on each side of the alleged equation represents a positive integer. It is enough to check that each of these is a divisor of the other.
17. Refer to the proof of Theorem 6.31 for the strategy.
19. Remember Pythagoras!
21. The least common multiple is 9,447,438.

SECTION 6.4

1. (a) true (c) false (e) true (g) false
3. Write n in the form $2k + 1$ for some integer k.
5. (b) Recall the formula $1 + 2 + 3 + \cdots + k = k(k + 1)/2$, and use the test (6.46) for divisibility by 9.
7. Refer to Example 6.40(a).
9. Use Theorem 6.45.
11. (b) As part of the solution, show that if $x, y \in \{0, k, 2k, \ldots, (m - 1)k\}$ and $x \equiv y \pmod{m}$, then $x = y$. Next apply part (a) to each element of the set $\{0, k, 2k, \ldots, (m - 1)k\}$.
15. Use a proof by contradiction. The equation $2^{ab} = (2^a)^b$ will be helpful.
17. (b) 131123031
19. Use Theorem 6.48 as a guide.

SECTION 6.5

1. (a) $\varphi(9) = 6, \quad \varphi(20) = 8, \quad \varphi(37) = 36.$
   (b) The numbers *not* relatively prime to p^k are $p, 2p, 3p, \ldots, p^{k-1}p$; therefore $\varphi(p^k) = p^k - p^{k-1}$.

3. (a) If $p = ab$, with $1 < a \leq p-1$, then $a \mid (p-1)!$; and also (from the hypothesis) we have $a \mid ((p-1)!+1)$. Therefore, $a \mid (((p-1)!+1) - (p-1)!)$; that is, $a \mid 1$, contradicting the fact that $a > 1$.

5. We have $2^{340} = 2^{256+64+16+4} = 2^{256} \cdot 2^{64} \cdot 2^{16} \cdot 2^4$. But

$$\begin{aligned} 2^4 &= 16 \\ 2^{16} &= 65536 \equiv 64 \pmod{341} \\ 2^{32} &\equiv 64^2 \equiv 4 \pmod{341} \\ 2^{64} &\equiv 4^2 \equiv 16 \pmod{341} \\ 2^{128} &\equiv 256 \pmod{341} \\ 2^{256} &\equiv 65536 \equiv 64 \pmod{341} \end{aligned}$$

Therefore, $2^{340} \equiv 64 \cdot 16 \cdot 64 \cdot 16 = 1048576 \equiv 1 \pmod{341}$.

7. (a) $683 = 512 + 128 + 32 + 8 + 2 + 1 \quad (= 2^9 + 2^7 + 2^5 + 2^3 + 2^1 + 2^0)$

(b) 683 in binary: 1010101011

9. (a) We can write $m_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $m_2 = p_{r+1}^{\alpha_{r+1}} \cdots p_n^{\alpha_n}$, with the p_i's distinct, since $(m_1, m_2) = 1$. The hypothesis gives that $m_1 \mid a - b$ and $m_2 \mid a - b$, hence $p_i^{\alpha_i} \mid a - b$ for all i. Therefore p_i appears to some power $\geq \alpha_i$ in the standard factorization of $a - b$. Thus $(\prod_{i=1}^n p_i^{\alpha_i}) \mid a - b$. That is, $a \equiv b \pmod{m_1 m_2}$.

(b) Let $x \in \mathbb{Z}$. By part (a) it suffices to check the congruences $x^5 \equiv x \pmod 2$ and
$x^5 \equiv x \pmod 5$.

SECTION 6.6

1. (a) $\varphi(360) = \varphi(2^3 \cdot 3^2 \cdot 5) = 360 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 96$.

3. Write $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n = mk = p_1^{\beta_1} \cdots p_r^{\beta_r} p_{r+1}^{\beta_{r+1}} \cdots p_t^{\beta_t}$, with $\alpha_i \leq \beta_i$ for $1 \leq\leq r$. Then

$$\varphi(m) = m \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \text{ and } \varphi(n) = n \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right) = \varphi(m) \cdot \left(k \prod_{i=r+1}^{t} \left(1 - \frac{1}{p_i}\right)\right)$$

5. $\varphi(8) \neq \varphi(4) \cdot \varphi(2)$

7. $\sum_{i=0}^{r} \varphi(p^i) = 1 + \sum_{i=1}^{r} \varphi(p^i) = 1 + \sum_{i=1}^{r} (p^i - p^{i-1}) = p^r$. (The sum "telescopes.")

9. (a) Since $n = qt + r$, there are q numbers in $\mathbb{N}_n$ that are divisible by t, namely $t, 2t, \ldots, qt$. But $[\frac{n}{t}] = [q + \frac{r}{t}] = q$, since $0 \leq r < q$.

(b) From part (a), we have $|A_5| = 200$, $|A_6| = 166$, and $|A_8| = 125$. Moreover $|A_5 \cap A_6| = |A_{30}| = 33$, $|A_5 \cap A_8| = |A_{40}| = 25$, $|A_6 \cap A_8| = |A_{24}| = 41$, and

$|A_5 \cap A_6 \cap A_8| = |A_{120}| = 8$. Then

$$\begin{aligned}|\,\overline{A_5} \cap \overline{A_6} \cap \overline{A_8}| &= 1000 - \big(|A_5| + |A_6| + |A_8|\big) \\ &\quad + \big(|A_5 \cap A_6| + |A_5 \cap A_8| + |A_6 \cap A_8|\big) - |A_5 \cap A_6 \cap A_8| \\ &= 1000 - (200 + 166 + 125) + (33 + 25 + 41) - 8 = 600.\end{aligned}$$

SECTION 6.7

1. Consider a prime divisor p of $(n-1)! - 1$. (Why does p exist?)
3. (i) and (ii). Note that p must be odd.
5. If $m \geq 2$ then $\dfrac{1}{1 - \frac{1}{m}} = \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{m}\right)^n$.

SECTION 6.8

1. If $a \in \mathbb{Z}_{23}^*$, then $\operatorname{ord}_{23} a \mid 22$.
3. $(ab)^{xy} = (a^x)^y (b^y)^x$
5. Consider $\varphi\big(\varphi(18)\big)$.
7. Use $r_1 \equiv r_2^{\log_{r_2} r_1} \pmod{\varphi(m)}$.
9. (i) Compute $\varphi\big(\varphi(18)\big)$.
11. Consider $\operatorname{ord}_{105} 2$.

SECTION 6.9

1. This is the Möbius function.
3. Use Corollary 6.90.
5. (ii) $\tau(n) = \sum_{d|n} 1$.
7. (i) $(f * I)(n) = \sum_{d|n} f(d)$.
   (ii) $g = f * I$, and get a similar formula for f.
9. Both functions are multiplicative. (Why?) Compare their values on prime powers.
11. The key is computing $(\mu * \mu)(p^\alpha)$ if p is prime.

SECTION 7.1

1. (a) $5 + 2i$ (c) $-i$ (e) 5 (g) $\frac{5}{7} - \frac{2}{7}i$
7. The complex number $\cos\frac{\pi}{9} + i \sin\frac{\pi}{9}$ is one such number.
9. (b) The exercise reduces to finding the points of intersection of two circles.
11. The average of the real parts of the given numbers is the real part of the midpoint, and similarly for the imaginary part.
13. (b) These points occupy the vertices of a regular octagon centered at the origin.

15. There *are* examples of the indicated kind.

17. (a) In $\mathbb{R}[x]$:

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1).$$

In $\mathbb{C}[x]$:

$$x^6 - 1 = (x - 1)\left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right) \times (x + 1)\left(x - \frac{1 + i\sqrt{3}}{2}\right)\left(x - \frac{1 - i\sqrt{3}}{2}\right)$$

SECTION 7.2

1. Any ring containing $\mathbb{Z}$ and i must contain products and sums of products of elements in $\mathbb{Z} \cup \{i\}$.

3. Consider how many ordered pairs (a, b) of integers satisfy $a^2 + b^2 \leq 20$.

5. $\dfrac{a + bi}{a - bi} = \dfrac{a^2 - b^2}{a^2 + b^2} + \dfrac{2ab}{a^2 + b^2}i$. When is this in $\mathbb{Z}[i]$?

7. (i) $\dfrac{18 + i}{5 + i} = \dfrac{91 - 13i}{26}$. Choose $q = 3$.

9. (i) Use the method of Example 7.26.

INDEX

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L.J. Gerstein, *Introduction to Mathematical Structures and Proofs*, Undergraduate Texts in Mathematics,
DOI 10.1007/978-1-4614-4265-3, © Springer Science+Business Media, LLC 2012