Appendix A
Zeta Functions in Number Theory

In this appendix we collect some basic facts about zeta functions in number theory to which our theory of explicit formulas can be applied. We refer to [Lan, Chapters VIII and XII–XV], [ParsSh1, Chapter 4] and [ParsSh2, Chapter 1, §6, Chapter 2, §1.13] for more complete information and proofs.

A.1 The Dedekind Zeta Function

Let K be an algebraic number field of degree d over $\mathbb{Q}$, and let $\mathcal{O}$ be the ring of integers of K. The *norm* of an ideal $\mathfrak{a}$ of $\mathcal{O}$ is defined as the number of elements of the ring $\mathcal{O}/\mathfrak{a}$:

$$N\mathfrak{a} = \#\mathcal{O}/\mathfrak{a}. \tag{A.1}$$

The norm is multiplicative: $N(\mathfrak{a}\mathfrak{b}) = N\mathfrak{a} \cdot N\mathfrak{b}$. Furthermore, an ideal has a unique factorization into prime ideals,

$$\mathfrak{a} = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_k^{a_k}, \tag{A.2}$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are prime ideals.

We denote by r_1 the number of real embeddings $K \to \mathbb{R}$ and by r_2 the number of pairs of complex conjugate embeddings $K \to \mathbb{C}$. Thus we have that $r_1 + 2r_2 = d$. Further, w stands for the number of roots of unity contained in K. Associated with K, we need the *discriminant* $\operatorname{disc}(K)$, the *class number* h, and the *regulator* $\mathfrak{R}$. We refer to [Lan, p. 64 and p. 109] for the definition of these notions.

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The *Dedekind zeta function* of K is defined for $\operatorname{Re} s > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s} = \sum_{n=1}^{\infty} A_n n^{-s}. \tag{A.3}$$

Here, $\mathfrak{a}$ runs over the ideals of $\mathcal{O}$, and, in the second expression, A_n denotes the number of ideals of $\mathcal{O}$ of norm n. This function has a meromorphic continuation to the whole complex plane, with a unique (simple) pole at $s = 1$, with residue

$$\frac{2^{r_1}(2\pi)^{r_2} h\mathfrak{R}}{w\sqrt{\operatorname{disc}(K)}}. \tag{A.4}$$

(See [Lan, Theorem 5 and Corollary, p. 161].)

From unique factorization and the multiplicativity of the norm, we deduce the Euler product for $\zeta_K(s)$: for $\operatorname{Re} s > 1$,

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - (N\mathfrak{p})^{-s}}, \tag{A.5}$$

where $\mathfrak{p}$ runs over the prime ideals of $\mathcal{O}$.

Example A.1 (The case $K = \mathbb{Q}$). Ideals of $\mathbb{Z}$ are generated by positive integers, and the norm of the ideal $n\mathbb{Z}$ is n, for $n \geq 1$. Hence,

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s), \tag{A.6}$$

so that the Dedekind zeta function of $\mathbb{Q}$ is the Riemann zeta function. The Euler product for $\zeta_{\mathbb{Q}}(s)$ is given by

$$\zeta_{\mathbb{Q}}(s) = \prod_{p} \frac{1}{1 - p^{-s}}, \tag{A.7}$$

where p runs over the rational prime numbers.

A.2 Characters and Hecke L-series

Let χ be an ideal-character of $\mathcal{O}$, belonging to the cycle $\mathfrak{c}$ (see [Lan, Chapter VIII, §3 and Chapter VI, §1] for a complete explanation of the terms). The *L-series* associated with χ is defined by

$$L_{\mathfrak{c}}(s, \chi) = \sum_{(\mathfrak{a},\mathfrak{c})=1} \chi(\mathfrak{a}) (N\mathfrak{a})^{-s}. \tag{A.8}$$

By the multiplicativity of the norm, this function has an Euler product

$$L_{\mathfrak{c}}(s,\chi) = \prod_{\mathfrak{p} \not| \mathfrak{c}} \frac{1}{1-\chi(\mathfrak{p})\,(N\mathfrak{p})^{-s}}. \tag{A.9}$$

This zeta function can be completed with factors corresponding to the divisors of $\mathfrak{c}$, to obtain a function $L(s,\chi)$, called the *Hecke L-series* associated with χ. It is related to the Dedekind zeta function as follows: Let L be the class field associated with an ideal class group of $\mathcal{O}$, and let ζ_L be the Dedekind zeta function of L. Let χ run over the characters of the ideal class group. Then,

$$\zeta_L(s) = \prod_{\chi} L(s,\chi). \tag{A.10}$$

Example A.2 (The case $K = \mathbb{Q}$). A multiplicative function χ on the positive integers gives rise to a *Dirichlet L-series*

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}. \tag{A.11}$$

The value $\chi(n)$ only depends on the class of n modulo a certain positive integer c. The minimal such c is called the *conductor* of χ.

A.3 Completion of L-Series, Functional Equation

The fundamental property of the Dedekind zeta function and of the L-series is that it can be completed to a function that is symmetric about $s = \frac{1}{2}$. Let $\Gamma(s)$ be the gamma function. Let

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) \quad \text{and} \quad \zeta_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)\,.$$

Denote by $\operatorname{disc}(K)$ the discriminant of K. Then the function

$$\xi_K(s) = \operatorname{disc}(K)^{s/2}\big(\zeta_{\mathbb{R}}(s)\big)^{r_1}\big(\zeta_{\mathbb{C}}(s)\big)^{r_2}\zeta_K(s) \tag{A.12}$$

has a meromorphic continuation to the whole complex plane, with simple poles located only at $s = 1$ and $s = 0$, and it satisfies the functional equation

$$\xi_K(1-s) = \xi_K(s). \tag{A.13}$$

We deduce from this key fact that the function

$$\psi_K(\sigma) := \limsup_{t\to\infty} \frac{\log|\zeta_K(\sigma+it)|}{\log t}, \tag{A.14}$$

defined for $\sigma \in \mathbb{R}$, is given by the following simple formula for $\sigma \notin (0,1)$:

$$\psi_K(\sigma) = \begin{cases} 0, & \text{for } \sigma \geq 1, \\ \frac{d}{2}\left(1 - 2\sigma\right), & \text{for } \sigma \leq 0. \end{cases}$$

It is known that ψ_K is convex on the real line. (The Lindelöf hypothesis says that $\psi_K(1/2) = 0$.) We deduce the following property (see [Lan, Chapter XIII, §5]): For every real number σ and $\varepsilon > 0$, there exists a constant C, depending on σ and ε, such that for all real numbers t with $|t| > 1$,

$$|\zeta_K(\sigma + it)| \leq C|t|^{\psi_K(\sigma)+\varepsilon}. \tag{A.15}$$

Thus $\zeta_K(s)$ satisfies the hypotheses **L1** and **L2** of Chapter 5. (See also Remark A.5 below.)

The formalism required to prove for general L-series the functional equation (A.13) and the estimate (A.15) about the growth along vertical lines was developed in Tate's thesis [Ta]. We refer to [Lan, Chapter XIV] for the corresponding results.

Remark A.3. Our theory also applies to the more general L-series associated with nonabelian representations of $\mathbb{Q}$, such as those considered in [RudSar]. (The abelian case corresponds to the Hecke L-series discussed above.) These L-series can also be completed at infinity, and they satisfy a functional equation, relating the zeta function associated to a given representation with the zeta function associated to the contragredient representation. Moreover, they have an Euler product representation, much like that of $L(s,\chi)$, except that the p-th Euler factor may be a polynomial in p^{-s} of degree larger than one.

These zeta functions are called primitive L-series in [RudSar]. According to the Langlands Conjectures, they are the building blocks of the most general L-series occurring in number theory. See, for example, [Gel, KatSar, RudSar].

A.4 Epstein Zeta Functions

A natural generalization of the Riemann zeta function is provided by the Epstein zeta functions [Ep].[1] Let $q = q(\mathbf{x})$ be a positive definite quadratic form of $\mathbf{x} \in \mathbb{R}^d$, with $d \geq 1$. Then the associated *Epstein zeta function* is

[1] See, for example, [Ter, Section 1.4] for detailed information about these functions. However, we use the convention of [Lap2, §4], which is different from the traditional one used in [Ter].

defined by

$$\zeta_q(s) = \sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\{0\}} q(\mathbf{n})^{-s/2}. \tag{A.16}$$

It can be shown that $\zeta_q(s)$ has a meromorphic continuation to all of $\mathbb{C}$, with a simple pole at $s = d$ having residue $\pi^{d/2}(\det q)^{-1/2}\Gamma(d/2)^{-1}$. Further, $\zeta_q(s)$ satisfies a functional equation analogous to that satisfied by $\zeta(s)$; namely, for the completed zeta function

$$\xi_q(s) = \pi^{-s/2}\Gamma(s/2)\,\zeta_q(s),$$

we have

$$\xi_q(s) = (\det q)^{-1/2}\xi_{q^{-1}}(d - s), \tag{A.17}$$

where q^{-1} is the positive definite quadratic form associated with the inverse of the matrix of q. (See [Ter, Theorem 1, p. 59].)

In the important special case when $q(x) = x_1^2 + \cdots + x_d^2$, the corresponding Epstein zeta function is denoted $\zeta_d(s)$ in Section 1.4, Equation (1.47), and can be viewed as a natural higher-dimensional analogue of the Riemann zeta function $\zeta(s)$. Indeed, we have $\zeta_1(s) = 2\zeta(s)$. (See, for example, [Lap2, §4] and the end of Section 1.4.) Since in this case, q is associated with the d-dimensional identity matrix, we have $q^{-1} = q$ and $\det q = 1$, so that (A.17) takes the form of a true functional equation relating $\zeta_d(s)$ and $\zeta_d(d - s)$. Moreover, $\zeta_d(s)$ has an Euler product like that of $\zeta(s)$ if and only if $d = 1$, 2, 4 or 8, corresponding to the real numbers, the complex numbers, the quaternions and the octaves, respectively. (See, e.g., [Bae].)

Remark A.4. More generally, one can also consider the Epstein-like zeta functions considered in [Ess1]. Such Dirichlet series are associated with suitable homogeneous polynomials of degree greater than or equal to one.

Remark A.5. For all the zeta functions in Sections A.1–A.4, hypotheses **L1** and **L2′** (Equations (5.19) and (5.21), page 147) are satisfied with a window W equal to all of $\mathbb{C}$. Moreover, for the example from Section A.6 below, one can take W to be a right half-plane of the form $\operatorname{Re} s \geq \sigma_0$, for a suitable $\sigma_0 > 0$.

A.5 Two-Variable Zeta Functions

Let C be an algebraic curve over a finite field, as in Section 11.6. In that section, we introduced the zeta function of C, $\zeta_C(s)$. In the paper [Pel], Pellikaan introduced a zeta function $\zeta_C(s, t)$ that specializes to $\zeta_C(s)$ for $t = 1$. Later, Schoof and van der Geer introduced the analogue for the integers, inspired by their work on positivity for Arakelov divisors [SchoG]. We present here a brief summary of their results.

A.5.1 The Zeta Function of Pellikaan

We consider a complete nonsingular curve C over the finite field with q elements $\mathbb{F}_q$. A divisor of C is a formal sum of valuations of $k = \mathbb{F}_q(C)$, the field of functions on C. In particular, the divisor of a function f is

$$(f) = \sum_v \operatorname{ord}(f, v)v,$$

where the sum is over all valuations of k. The degree of a divisor

$$\mathfrak{D} = \sum_v D_v v$$

is given by

$$\deg \mathfrak{D} = \sum_v D_v \deg v,$$

where $\deg v$ is the dimension of the residue class field at v over $\mathbb{F}_q$. Two divisors are said to be (linearly) equivalent if their difference is the divisor of a function. Let $\mathrm{Cl} = \mathrm{Cl}(C)$ be the group of divisor classes of C. This group has a grading by the degree, since $\deg(f) = 0$ for any nonzero function f on C, and we write Cl_n for the subset[2] of classes of degree n. A divisor is said to be *positive* if $D_v \geq 0$ for all v. Let $l(\mathfrak{D})$ denote the dimension over $\mathbb{F}_q$ of the vector space of functions f such that $\mathfrak{D} + (f) \geq 0$. We will need the theorem of Riemann–Roch,

$$l(\mathfrak{D}) = \deg \mathfrak{D} + 1 - g + l(\mathfrak{K} - \mathfrak{D}),$$

where $\mathfrak{K}$ is the canonical divisor of C, and $g = l(\mathfrak{K})$ is the genus of C. It follows that $l(\mathfrak{D}) = 0$ for $\deg \mathfrak{D} < 0$, and $l(\mathfrak{D}) = \deg \mathfrak{D} + 1 - g$ for $\deg \mathfrak{D} > \deg \mathfrak{K} = 2g - 2$.

Define, for $\operatorname{Re} t < \operatorname{Re} s < 0$, the function of two complex variables

$$\zeta_C(s, t) = \frac{1}{q^t - 1} \sum_{\mathfrak{D} \in \mathrm{Cl}} q^{tl(\mathfrak{D})} q^{-s(\deg \mathfrak{D} + 1 - g)}. \tag{A.18}$$

We first derive an expression for this function that converges for all s and t. Note that for $g = 0$, that is, when $C = \mathbb{P}^1$ is the projective line, as discussed in Example 11.34 for the case of the one-variable zeta functions, we have $l(\mathfrak{D}) = 0$ for $\deg \mathfrak{D} < 0$ and $l(\mathfrak{D}) = \deg \mathfrak{D} + 1$ for $\deg \mathfrak{D} \geq 0$. Also, there is only one divisor class of degree 0; i.e., $h = 1$. Hence the sum over the divisor classes in (A.18) becomes, for $\operatorname{Re} t < \operatorname{Re} s < 0$,

$$\sum_{n=-\infty}^{-1} q^{-s(n+1)} + \sum_{n=0}^{\infty} q^{t(n+1)} q^{-s(n+1)}.$$

[2]Thus Cl is isomorphic to the product $\mathrm{Cl}_0 \times \mathbb{Z}$. It is known that Cl_0 is a finite group. We denote its order by h.

Thus we find

$$\zeta_{\mathbb{P}^1}(s,t) = \frac{1}{(1-q^{t-s})(q^s-1)}.$$

In general, we write $h = \#\mathrm{Cl}_0 = \#\mathrm{Cl}_n$ for the class number of C (see also Section 11.6). Then

$$\zeta_C(s,t) = \sum_{n=-\infty}^{\infty} \sum_{\mathfrak{D}\in\mathrm{Cl}_n} \frac{q^{tl(\mathfrak{D})} - q^{t\max\{0,n+1-g\}}}{q^t-1} q^{-s(n+1-g)}$$
$$+ \frac{h}{q^t-1} \sum_{n=-\infty}^{\infty} q^{t\max\{0,n+1-g\}} q^{-s(n+1-g)}.$$

The first sum is finite, and the second sum equals $h\zeta_{\mathbb{P}^1}(s,t)$. Hence we find

$$\zeta_C(s,t) = \sum_{n=-\infty}^{\infty} \sum_{\mathfrak{D}\in\mathrm{Cl}_n} \frac{q^{tl(\mathfrak{D})} - q^{t\max\{0,n+1-g\}}}{q^t-1} q^{-s(n+1-g)} + h\zeta_{\mathbb{P}^1}(s,t). \tag{A.19}$$

We see that $\zeta_C(s,t)$ has poles at $q^s = 1$ and at $q^s = q^t$. Using the Riemann–Roch formula, we can continue to verify that $\zeta_C(s,t) = \zeta_C(t-s,t)$. Moreover, we see from (A.19) that $q^{-sg}\zeta_C/\zeta_{\mathbb{P}^1}$ is a polynomial in q^{-s} and q^t of degree $2g$ in q^{-s}.

Remark A.6. In [Na], Naumann proves that this polynomial is irreducible. Thus the zeros of $\zeta_C(s,t)$ lie in an irreducible family. It is well known that for $t = 1$, $\zeta_C(s,t)$ satisfies the Riemann hypothesis; i.e., all zeros in s of $\zeta_C(s,1)$ have real part 1/2.

To relate this function to the divisors of C (and not to the classes of divisors), and hence derive the Euler product for $\zeta_C(s,1)$, we use that

$$\frac{q^{l(\mathfrak{D})}-1}{q-1} \tag{A.20}$$

equals the number of positive divisors in the divisor class $\mathfrak{D}$. Note that

$$\sum_{n=0}^{\infty} \frac{q^{t\max\{0,n+1-g\}}-1}{q^t-1} q^{-s(n+1-g)} = \zeta_{\mathbb{P}^1}(s,t).$$

Hence we find for $\mathrm{Re}\, s > \max\{0, \mathrm{Re}\, t\}$ that

$$\zeta_C(s,t) = \sum_{n=0}^{\infty} \sum_{\mathfrak{D}\in\mathrm{Cl}_n} \frac{q^{tl(\mathfrak{D})}-1}{q^t-1} q^{-s(n+1-g)}. \tag{A.21}$$

Hence by (A.20), for $t = 1$, we find the Euler product

$$\zeta_C(s,1) = q^{s(g-1)} \prod_v \frac{1}{1 - q^{-s \deg v}}.$$

The value $t = 0$ is also interesting. Taking the limit as $t \to 0$ in Equation (A.21), we find that

$$\zeta_C(s,0) = \sum_{n=0}^{\infty} \sum_{\mathfrak{D} \in \mathrm{Cl}_n} l(\mathfrak{D}) q^{-s(n+1-g)}.$$

A.5.2 The Zeta Function of Schoof and van der Geer

An Arakelov divisor is a formal linear combination of valuations of $\mathbb{Q}$, where the archimedean valuation has a real coefficient and the p-adic valuations have an integer coefficient. For example, the divisor of a number f is

$$(f) = -(\log |f|)v_\infty + \sum_p \mathrm{ord}(f,p) v_p,$$

where the sum is over all prime numbers. The degree of a divisor

$$\mathfrak{D} = \sum_v D_v v$$

is given by

$$\deg \mathfrak{D} = D_\infty + \sum_p D_p \log p,$$

which is a real number, not necessarily an integer. Clearly, the group of divisor classes of $\mathbb{Q}$ is isomorphic to $\mathbb{R}$. Thus, there is only the grading by the degree, and $h = 1$. Let

$$\theta(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u^2}.$$

We use this function to measure positivity of a divisor at the archimedean valuation. A divisor is said to be *positive* if $D_p \geq 0$ for all p-adic valuations, and $\log \theta\left(e^{-2D_\infty}\right)$ is large.[3] We have the theorem of Riemann–Roch,

$$\log \theta(1/u) = \log u + \log \theta(u), \tag{A.22}$$

for $u > 0$. This is proved using the Poisson Summation Formula; see [Tit, Section 2.3], [Pat, Theorem 2.2] or [Schw1, Eq. (VII.7.5)]. It follows that

$$\theta(u) = 1 + O\left(e^{-\pi u^2}\right) \quad \text{for } u \to \infty,$$

[3] Positivity is not a definite notion: the larger $\log \theta\left(e^{-2D_\infty}\right)$, the more positive the divisor is said to be.

and

$$\theta(u) = u^{-1} + O\left(u^{-1}e^{-\pi/u^2}\right) \quad \text{for } u \to 0.$$

Define, for $\operatorname{Re} t < \operatorname{Re} s < 0$,

$$\zeta_{\mathbb{Z}}(s,t) = \frac{1}{t}\int_0^\infty \theta(u)^t u^s \,\frac{du}{u}. \tag{A.23}$$

We first derive an expression that converges for all s and t:

$$\zeta_{\mathbb{Z}}(s,t) = \frac{1}{t}\int_0^\infty \left(\theta(u)^t - \max\{1/u,1\}^t\right) u^s \,\frac{du}{u} \\ + \frac{1}{t}\left(\int_0^1 u^{s-t}\,\frac{du}{u} + \int_1^\infty u^s\,\frac{du}{u}\right).$$

The first term converges for every s and t, and the second is defined for $\operatorname{Re} t < \operatorname{Re} s < 0$, but we can easily compute it to find

$$\zeta_{\mathbb{Z}}(s,t) = \frac{1}{t}\int_0^\infty \left(\theta(u)^t - \max\{1/u,1\}^t\right) u^s \frac{du}{u} + \frac{1}{s(s-t)}, \tag{A.24}$$

for every $s, t \in \mathbb{C}$. We see that $\zeta_{\mathbb{Z}}(s,t)$ has poles at $s = 0$ and at $s = t$. Using the Riemann–Roch formula, we can continue to verify the functional equation $\zeta_{\mathbb{Z}}(s,t) = \zeta_{\mathbb{Z}}(t-s,t)$.

To relate this function to the divisors of $\mathbb{Q}$, and hence derive the Euler product for $\zeta_{\mathbb{Z}}(s,1)$, we use that $\theta(u) - 1$ 'equals' the number of positive divisors in the same divisor class as u. Note that

$$\int_0^\infty \frac{\max\{1/u,1\}^t - 1}{t} u^s \,\frac{du}{u} = \frac{1}{s(s-t)}.$$

Hence we find for $\operatorname{Re} s > \max\{0, \operatorname{Re} t\}$ that

$$\zeta_{\mathbb{Z}}(s,t) = \int_0^\infty \frac{\theta(u)^t - 1}{t} u^s \,\frac{du}{u}$$

For $t = 1$, we find the Euler product

$$\zeta_{\mathbb{Z}}(s,1) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\prod_p \frac{1}{1-p^{-s}} = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Remark A.7. Considering the zeros of $\zeta_{\mathbb{Z}}(s,t)$ for varying t, we see that the zeros of the Riemann zeta function lie in an analytic family. In analogy with the geometric case (see Remark A.6), we might conjecture that this family is irreducible.

There is also an Euler product for $t = 2$, 4 and 8, corresponding to the Gaussian integers, the quaternions and the octonions, respectively. Surprisingly, there is also an Euler product for $t = 0$, as we verify directly (see also [LagR]). Taking the limit as $t \to 0$, we find that

$$\zeta_{\mathbb{Z}}(s,0) = \int_0^\infty (\log \theta(u))\, u^s \,\frac{du}{u},$$

for $\operatorname{Re} s > 0$. Using the Jacobi triple product identity [HardW, Theorem 352, p. 282],

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} (1 + q^{2n-1})^2 (1 - q^{2n}),$$

we can compute

$$\log \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right) = 2 \sum_{n=1}^{\infty} (-1)^{n-1} q^n \sigma_{-1}(n|n|_2),$$

where $|n|_2 = 2^{-\operatorname{ord}(n,2)}$ denotes the 2-adic valuation (so that $n|n|_2$ is the largest odd factor of n), and $\sigma_{-1}(n) = \sum_{d \mid n} \frac{1}{d}$. We thus obtain

$$\zeta_{\mathbb{Z}}(s,0) = \pi^{-s/2} \Gamma(s/2) \zeta(s/2) \zeta(1 + s/2)(1 - 2^{1-s/2})(1 - 2^{-1-s/2}),$$

where $\zeta(s)$ is the Riemann zeta function. Substituting the Euler product for the Riemann zeta function, we obtain the Euler product for $\zeta_{\mathbb{Z}}(s,0)$, valid for $\operatorname{Re} s > 2$. Using the functional equation for the Riemann zeta function,

$$\Gamma(s)\Gamma(-s) = \frac{\pi}{-s \sin \pi s},$$

the doubling formula

$$2^{2s-1}\Gamma(s)\Gamma(s+1/2) = \Gamma(2s)\Gamma(1/2),$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$, we obtain the alternative expression

$$\zeta_{\mathbb{Z}}(s,0) = \zeta(s/2)\zeta(-s/2)(1 - 2^{-1-s/2})(1 - 2^{-1+s/2}) \frac{4\pi}{s \sin(\pi s/4)},$$

which shows clearly that $\zeta_{\mathbb{Z}}(-s,0) = \zeta_{\mathbb{Z}}(s,0)$.

Remark A.8. It would be interesting to consider two-variable dynamical zeta functions in the context of Chapter 7, in the spirit of [Lag2]. We hope to do so in some future work.

A.6 Other Zeta Functions in Number Theory

The flexibility of our theory of explicit formulas with an error term allows us to apply it to other zeta functions that do not necessarily satisfy a functional equation.

As an example, we mention the zeta function

$$\mathcal{P}(s) = \sum_{p} (\log p)\, p^{-s},$$

which was studied by M. van Frankenhuijsen in [vF2, §3.9] in connection with the ABC conjecture. To obtain information about this function, we consider the logarithmic derivative of the Euler product of the Riemann zeta function,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} (\log p)\, p^{-ms},$$

where p ranges over the rational primes (as above), and m over the positive integers. This function has simple poles at $s = 1$ and at each zero of $\zeta(s)$. By Möbius inversion,

$$\begin{aligned} \mathcal{P}(s) &= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(2s)}{\zeta(2s)} + \frac{\zeta'(3s)}{\zeta(3s)} + \frac{\zeta'(5s)}{\zeta(5s)} - \frac{\zeta'(6s)}{\zeta(6s)} + \dots \\ &= \sum_{n=1}^{\infty} \mu(n) \left(-\frac{\zeta'(ns)}{\zeta(ns)} \right), \end{aligned}$$

where $\mu(n)$ is the Möbius function, defined on the positive integers by

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ is not square-free}, \\ (-1)^k, & \text{if } n = p_1 \dots p_k \text{ is square-free}. \end{cases}$$

Thus the poles of $\mathcal{P}(s)$ are contained in the set

$$\{s/n\colon s \text{ is a zero of } \zeta, n = 1, 2, \dots\}.$$

If the Riemann hypothesis holds, then each of these points is a pole of $\mathcal{P}$, and in general, the number of possible cancelations is small. It follows that the poles of $\mathcal{P}(s)$ accumulate on the line $\operatorname{Re} s = 0$. Hence this line is a natural boundary for the analytic continuation of $\mathcal{P}(s)$. But on a line $\operatorname{Re} s = \sigma > 0$ that does not meet any of the poles of $\mathcal{P}(s)$, this function is bounded by a constant times $\log |t|$, for $t = \operatorname{Im} s > 2$. Hence this function satisfies hypotheses **L1** and **L2** (Equations (5.19) and (5.20), page 147) with $W = \{s\colon \operatorname{Re} s \geq \sigma_0\}$, where $\sigma_0 > 0$ is suitably chosen. For example, if the Riemann hypothesis is true, one can take any positive value for σ_0 other than $1/n$, for $n = 1, 2, \dots$.

Appendix B

Zeta Functions of Laplacians and Spectral Asymptotics

In this appendix, we provide a brief overview of some of the results from spectral geometry that are relevant to the study of the spectral zeta function associated with a Laplacian Δ on a smooth compact Riemannian manifold M. For the simplicity of exposition, we focus on the case when M is a closed manifold (i.e., is without boundary). However, as is briefly explained at the end of the appendix, all the results stated for closed manifolds are known to have a suitable counterpart for the case of a compact manifold with boundary. An important special case of the latter situation is that when M is a smooth bounded open set in Euclidean space $\mathbb{R}^d$ and $\Delta = -\sum_{k=1}^{d} \partial^2/\partial x_k^2$ is the associated Dirichlet or Neumann Laplacian.

By necessity of concision, our presentation is somewhat sketchy and imprecise. For a much more detailed treatment of these matters, we refer the interested reader to some of the articles and books cited below, including [Min1–2, MinPl, Kac, McKSin, Se1, BergGM, AtPSin, Gil, Hö1–3, Gru, AndLap1–2], along with the relevant references therein.

B.1 Weyl's Asymptotic Formula

Let M be a closed, d-dimensional, smooth, compact and connected Riemannian manifold. We assume throughout that the closed manifold M is equipped with a fixed Riemannian metric g. Let Δ be the (positive) La-

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placian (or Laplace–Beltrami operator) on M associated with g.[1] It is well known that Δ has a discrete (frequency) spectrum, written in increasing order according to multiplicity:

$$0 < f_1 \leq f_2 \leq \cdots \leq f_j \leq \ldots,$$

where $f_j \to +\infty$ as $j \to \infty$. Here, by convention, the frequencies of Δ are defined as the square root of its eigenvalues λ_j.[2,3]

Next, let $N_\nu(x) = N_{\nu,M}(x)$ be the associated *spectral counting function* (or counting function of the frequencies):

$$N_\nu(x) = \#\{j \geq 1\colon f_j \leq x\}, \quad \text{for } x > 0. \tag{B.1}$$

Then Weyl's classical asymptotic formula [Weyl1–2] states that

$$N_\nu(x) = c_d \operatorname{vol}(M) x^d + o\big(x^d\big), \tag{B.2}$$

as $x \to \infty$, where $c_d = (2\pi)^{-d}\mathcal{B}_d$ and $\mathcal{B}_d$ denotes the volume of the unit ball in $\mathbb{R}^d$. Recall that $\mathcal{B}_d = \pi^{d/2}/\Gamma(d/2+1)$, where $\Gamma = \Gamma(s)$ is the usual gamma function. Further, $\operatorname{vol}(M)$ denotes the Riemannian volume of M.

The leading term in (B.2),

$$W(x) = W_M(x) = (2\pi)^{-d}\mathcal{B}_d \operatorname{vol}(M) x^d, \tag{B.3}$$

is often referred to as the Weyl term in the literature.

Remark B.1. Weyl's original result has been improved in various ways. One extension consists in giving a (sharp) remainder estimate for Weyl's asymptotic law, of the form

$$N_\nu(x) = c_d \operatorname{vol}(M) x^d + O\big(x^{d-1}\big), \tag{B.4}$$

as $x \to \infty$. This result is due to Hörmander [Hö1] in the case of closed manifolds, and to Seeley [Se4–5] (for $d \leq 3$) or to Pham The Lai [Ph] (for $d \geq 4$) in the case of manifolds with boundary (for example, for the Dirichlet or Neumann Laplacian on a smooth bounded open set in $\mathbb{R}^d$). We refer the interested reader to [Hö2–3] for a detailed exposition of these results.

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[1] Using Einstein's summation convention, Δ is given in local coordinates by

$$\Delta = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_\alpha}\, g^{\alpha\beta}\sqrt{\det g}\,\frac{\partial}{\partial x_\beta},$$

where $g = \big(g_{\alpha\beta}\big)_{\alpha,\beta=1}^d$ and $g^{-1} = \big(g^{\alpha\beta}\big)_{\alpha,\beta=1}^d$.

[2] We have used a slightly different normalization in the rest of this book; see, for example, Section 1.3 and footnote 1 of the introduction.

[3] Throughout this discussion, we ignore the zero eigenvalue of the Neumann Laplacian. Alternatively, we can replace Δ by $\Delta + \alpha$, for some positive constant α.

B.2 Heat Asymptotic Expansion

We denote by $z_\nu(t) = z_{\nu,M}(t)$ the trace of the heat semigroup $\{e^{-t\Delta}\colon t \geq 0\}$ generated by Δ.[4] Thus, $z_\nu(t)$ is given by

$$z_\nu(t) = \text{Trace}\left(e^{t\Delta}\right) = \sum_{j=1}^{\infty} e^{-t\lambda_j}, \tag{B.5}$$

for every $t > 0$.

A well-known Tauberian argument shows that Weyl's formula (B.2) is equivalent to the following asymptotic formula for $z_\nu(t)$ (see, for example, [Kac] or [Sim]):

$$z_\nu(t) = e_d \operatorname{vol}(M) t^{-d/2} + o\big(t^{-d/2}\big), \tag{B.6}$$

as $t \to 0^+$, where $e_d = \Gamma(d/2+1)c_d$. Using the above expression for c_d and $\mathcal{B}_d$, one finds $e_d = (4\pi)^{-d/2}$.

Remark B.2. The fact that (B.2) implies (B.6) is immediate and follows from a simple Abelian argument; see, e.g., [Sim, Theorem 10.2, p. 107] or [Lap1, Appendix A, pp. 521–522]. However, the converse relies on Karamata's Tauberian Theorem [Sim, Theorem 10.3, p. 108] (which is closely related to the Wiener–Ikehara Tauberian Theorem [Pos, Section 27, pp. 109–112; Shu, Theorem 14.1, p. 115]). In addition, even the existence of an error term in (B.6) does not imply the corresponding Weyl formula with error term in (B.2).[5]

More generally, a key result due in its original form to Minakshisundaram and Pleijel [MinPl] (building, in particular, on work of the first of these authors [Min1–2] in a closely related context) states that $z_\nu(t)$ has the following asymptotic expansion (in the sense of Poincaré):[6]

$$z_\nu(t) \sim \sum_{k\geq 0} \alpha_k t^{-(d-k)/2}, \tag{B.7}$$

[4]For convenience, we use here for $z_\nu(t)$ the standard convention encountered in the literature on spectral geometry; that is, we work with the trace of $e^{-t\Delta}$ rather than of $e^{-t\sqrt{\Delta}}$. The latter choice would correspond to the spectral partition function

$$\theta_\nu(t) = \text{Trace}\left(e^{-t\sqrt{\Delta}}\right) = \sum_{j=1}^{\infty} e^{-tf_j} = \int_0^\infty e^{-tx}\, dN_\nu(x),$$

as we defined it in Section 6.2.3 above; see, for example, Equation (6.29).

[5]We stress that in contrast to much of the rest of this book, all the asymptotic formulas in this appendix are interpreted pointwise.

[6]This asymptotic formula can be interpreted as follows: For each fixed integer $k_0 \geq 0$,

$$z_\nu(t) = \sum_{k\leq k_0} \alpha_k t^{-(d-k)/2} + O\big(t^{(k_0+1-d)/2}\big),$$

as $t \to 0^+$.

as $t \to 0^+$, where the coefficients $\alpha_k = \alpha_k(M)$ are integrals with respect to the Riemannian volume measure of M of suitable local geometric invariants of M. Namely, for each $k = 0, 1, 2, \ldots,$

$$\alpha_k(M) = \int_M \alpha_k(y, M)\, d\,\mathrm{vol}_M(y), \tag{B.8}$$

where the function $\alpha_k(\cdot, M)$ can be expressed as a (locally invariant) polynomial of (suitable contractions of) the Riemann curvature tensor of M and of its covariant derivatives. In this sense, it is a local invariant of M.

Moreover, in the present situation, one can show that

$$\alpha_k(M) = 0, \quad \text{if } k \text{ is odd.} \tag{B.9}$$

Remark B.3. For example, α_0 is equal to $e_d \operatorname{vol}(M)$, while α_2 is proportional to the integral over M of the scalar curvature of M.[7] In general, the explicit computation of the coefficients α_k is difficult but a large amount of information is now available, particularly in the present case of closed manifolds. We refer to Gilkey's book [Gil, Sections 1.7, 1.10, 4.8, and 4.9] for a detailed treatment of this matter.

B.3 The Spectral Zeta Function and its Poles

Let us next introduce the *spectral zeta function* of the Laplacian on M (or simply, the zeta function of Δ) $\zeta_\nu(s) = \zeta_{\nu,M}(s)$:[8]

$$\zeta_\nu(s) = \operatorname{Trace}\left(\Delta^{-s/2}\right) = \sum_{j=1}^{\infty} f_j^{-s}. \tag{B.10}$$

In view of (B.5), we have the following relation between $\zeta_\nu(s)$ and $z_\nu(t)$,

$$\zeta_\nu(2s) = \frac{1}{\Gamma(s)} \int_0^\infty z_\nu(t) t^{s-1}\, dt. \tag{B.11}$$

Hence, by Weyl's asymptotic formula (B.2) (or equivalently, by (B.6)), $\zeta_\nu(s)$ extends holomorphically to the open right half-plane $\operatorname{Re} s > d$. Further,

[7] By application of the Gauss–Bonnet formula (as extended by S.-S. Chern [Chern1–2] to every dimension d), it follows from the latter statement that the Euler characteristic of M is audible (i.e., can be recovered from the spectrum of M); see, e.g., [McKSin, pp. 44–45]. (Recall that the Euler characteristic of M vanishes when d is odd.)

[8] In the usual terminology, $\zeta_\nu(s)$ is the zeta function of $\sqrt{\Delta}$, because, according to our present conventions, the frequencies f_j of Δ are given by $f_j = \sqrt{\lambda_j}$, where the λ_j's are the eigenvalues of Δ, written in nondecreasing order. The reader should keep this in mind when comparing our formulas with those in [Gil], for example.

according to (B.7), $\zeta_\nu(s)$ has a simple pole at $s = d$. It follows that the abscissa of convergence of the Dirichlet series $\zeta_\nu(s) = \sum_{j=1}^{\infty} f_j^s$ is equal to d, the dimension of the manifold M.

More generally, the asymptotic expansion (B.7) combined with relation (B.11) above yields the following key theorem (see [MinPl] and, for instance, [Gil, Section 1.10, especially Lemma 1.10.1, p. 79]):

Theorem B.4. *The spectral zeta function $\zeta_\nu(s)$ of a closed Riemannian manifold M has a meromorphic extension to the whole complex plane, with simple poles located at d and at a (subset of) the points $d-2, d-4, \ldots$. Further, for $k = 0, 1, 2, \ldots,$ the residue at $s = d-k$ is equal to*

$$\frac{2\alpha_k(M)}{\Gamma((d-k)/2)},$$

where $\alpha_k = \alpha_k(M)$ is the k-th coefficient in the heat asymptotic expansion (B.7).

More precisely, $\zeta_\nu(s)$ is holomorphic except for simple poles located at

$$\begin{cases} s = d - 2q, \ \ q = 0, 1, 2, \ldots, & \text{if } d \text{ is odd,} \\ s = d, d-2, d-4, \ldots, 4, 2, & \text{if } d \text{ is even.} \end{cases}$$

Remark B.5. From our present point of view, the first part of Theorem B.4 is the most important one. It implies that all the poles of the spectral zeta function of a smooth manifold are located on the real axis, in contrast to what happens for fractal manifolds, as illustrated in the main body of this book.

Remark B.6. As was explained above, $s = d$ is always a (simple) pole of $\zeta_\nu(s)$. On the other hand, the other points mentioned in Theorem B.4 may not be poles of $\zeta_\nu(s)$, because the associated residue may vanish. For example, if $M = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the standard flat d-dimensional torus (i.e., the unit cube $[0,1]^d$ with its faces identified, as at the end of Section 1.4), then $\zeta_\nu(s) = \zeta_{\nu,M}(s)$ is the normalized Epstein zeta function associated with the standard quadratic form $q_d(x) = x_1^2 + \cdots + x_d^2$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Namely,

$$\zeta_\nu(s) = \zeta_d(s) = \sum_{(n_1,\ldots,n_d)\in\mathbb{Z}^d\setminus\{0\}} \left(n_1^2 + \cdots + n_d^2\right)^{-s/2}$$

as in Equation (1.47) above.[9] Therefore, for any $d \geq 1$, $s = d$ is the only pole of $\zeta_\nu(s) = \zeta_d(s)$, and it is simple. (See Appendix A, Section A.4, or [Ter, Section 1.4].) In order to reconcile this fact with the statement of

[9] For convenience, we are using here the normalized eigenvalues of Δ on $\mathbb{T}^d$.

Theorem B.4, it suffices to note that $M = \mathbb{T}^d$ has zero Euler characteristic and vanishing curvature. An entirely analogous comment can be made about a general Epstein zeta function $\zeta_q(s)$ considered in Section A.4 of Appendix A (or in [Ter, Section 1.4]), which can be viewed as the spectral zeta function of the Laplacian on a flat torus $M = \mathbb{R}^d/\Lambda$, where Λ is a lattice of $\mathbb{R}^d$ with associated positive definite quadratic form $q = q(x)$. (See Equation (A.16).)

B.4 Extensions

Various extensions of the above results are known in spectral geometry. We mention only a few, which are most relevant to our situation or that may help clarify certain issues:

(i) Formulas (B.2), (B.7) and Theorem B.4 apply to the more general situation of a (positive) elliptic differential operator $\mathcal{P}$ (instead of the Laplacian Δ). If $\mathcal{P}$ is of order $m > 0$, then we define the j-th frequency of $\mathcal{P}$ by $f_j = \lambda_j^{1/m}$, where λ_j is the j-th eigenvalue of $\mathcal{P}$, written in nondecreasing order according to multiplicity. With this convention, the exponent of x in (B.2) remains equal to d, while the exponent of t^{-1} in (B.6) and (B.7) is now equal to d/m and $(d-k)/m$, respectively. Further, the poles of $\zeta_\nu(s)$ also remain the same as in Theorem B.4. On the other hand, in (B.2) and in (B.3), the constant $(2\pi)^{-d}\mathcal{B}_d$ will be replaced by $(2\pi)^{-d}$ times a volume in phase space (i.e., in the cotangent bundle of M) determined by the principal symbol of $\mathcal{P}$. Moreover, with the obvious change in notation, in the analogue of (B.7) and (B.8), the local invariants $\alpha_k(\cdot, \mathcal{P})$ are now expressed as (locally invariant) polynomials of the total symbol of $\mathcal{P}$ and of its covariant derivatives.

(ii) Let us now assume that M is a (smooth, compact) manifold with boundary. For elliptic boundary value problems on M (and, in particular, for the prototypical cases of the Dirichlet and Neumann Laplacians on a smooth bounded open set of d-dimensional Euclidean space $\mathbb{R}^d$), the analogue of Weyl's asymptotic formula (B.2) and of the Minakshisundaram–Pleijel heat asymptotic expansion (B.7) still holds. It takes the same form as above, except that in the counterpart of (B.7), the coefficients α_k (or the corresponding local invariants) are more complicated to compute.[10] In addition, a suitable counterpart of Theorem B.4 also holds; see [MinPl] and [McKSin]. In particular, the poles of ζ_ν are all simple and located on the real axis. Perhaps the most complete treatment of these questions in the case of manifolds with boundary can be found in Grubb's book [Gru], which

[10] In fact, to our knowledge, no explicit algorithm is known to calculate every α_k in this case, although a great deal of information is available.

also deals with the more general case of elliptic pseudodifferential boundary value problems on M.[11] Besides the earlier papers [Min1–2, MinPl] (which study slightly different notions of spectral zeta functions of Laplacians, motivated by the work of Carleman [Car]), other useful references in this setting include the aforementioned paper by McKean and Singer [McKSin], along with the classical paper by Mark Kac [Kac] entitled *Can one hear the shape of a drum?,* which gives some related results on certain planar domains.

In a seminal paper, entitled *Complex powers of elliptic operators,* Seeley [Se1] has used modern analytical tools to study spectral zeta functions. In turn, Seeley's paper (along with its sequel for boundary value problems [Se2–3]) has stimulated a number of further developments related to the zeta functions of elliptic pseudodifferential operators. (See, for example, [Shu, Chapter II] and [Gru].)

B.4.1 Monotonic Second Term

Under the assumptions of Remark B.1 for a manifold with smooth boundary, it need not be the case that $N_\nu(x)$ admits (pointwise) an asymptotic second term as $x \to \infty$. (Contrast this statement with the fact that $z_\nu(t)$ has an asymptotic expansion of every order as $t \to 0^+$; see formula (B.7).) Knowing when $N_\nu(x)$ admits a monotonic asymptotic second term (i.e., of the form a nonzero constant times x^{d-1}) is the object of Hermann Weyl's conjecture [Wey1–2]. In a beautiful work, Ivrii [Ivr1–2] has partially solved this conjecture. More specifically, for example for the Dirichlet or Neumann Laplacian, respectively, he shows that on a manifold M with boundary ∂M, we have (with the obvious notation for the volume of M and ∂M),

$$N_\nu(x) = c_d \operatorname{vol}_d(M)x^d \mp g_{d-1} \operatorname{vol}_{d-1}(\partial M)x^{d-1} + o\big(x^{d-1}\big), \tag{B.12}$$

as $x \to \infty$, provided a suitable condition is satisfied.[12] (Here, the positive constant g_{d-1} is explicitly known in terms of $d-1$, the dimension of the smooth boundary ∂M.) Positive results toward Weyl's Conjecture were also obtained by Melrose [Mel1–2] for manifolds with concave boundary. We refer the interested reader to volumes III and IV of Hörmander's treatise [Hö3] as well as to Ivrii's recent book [Ivr3] for further information about this subject.

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[11] For a general pseudodifferential operator $\mathcal{P}$ on M, the heat asymptotic expansion may contain logarithmic terms, corresponding to the singularities of the symbol of $\mathcal{P}$. This is not the case, however, for an elliptic differential operator, and hence for a Laplacian on M. (See Corollary 4.27, page 388 and the comment on page 390 in [Gru].)

[12] Roughly speaking, this condition says that the set of multiply reflected periodic geodesics of M forms a set of measure zero, with respect to Liouville measure in phase space (i.e., in the cotangent bundle of M). This condition (which is sufficient but not necessary) is known to be generic among smooth Euclidean domains, but is very difficult to verify in any concrete example.

Finally, we note that situations where $N_\nu(x)$ has an oscillatory behavior (beyond the Weyl term) have been analyzed, in particular, by Duistermaat and Guillemin [DuiGu] in terms of the concentration of periodic geodesics (or, more generally, of bicharacteristics). See also the beginning of Section 12.5.3 for a sample of related mathematical and physical works, including the papers by Colin de Verdière [Col] and Chazarain [Chaz].

B.5 Notes

We note that Weyl's formula plays an important role in mathematical physics and can be given interesting physical interpretations; see, for example, [CouHi, Kac, ReSi3, Sim], along with [BaltHi].

Further information about heat asymptotic expansions and related issues can be found in the papers by McKean and Singer [McKSin] or Atiyah, Patodi and Singer [AtPSin], and in [AndLap1–2] or in the first unnumbered subsection of [JohLap, Section 20.2.B], along with the relevant references therein. See also [BergGM] for many interesting examples of spectra of Laplacians on Riemannian manifolds.

Additional information regarding spectral zeta functions and some of their connections with dynamical or with arithmetic zeta functions can be found in the book [Lap-vF8].

Appendix C
An Application of Nevanlinna Theory

In this appendix, we briefly discuss aspects of Nevanlinna theory and give in Section C.2, Theorem C.1, an application of that theory to the complex zeros of Dirichlet polynomials, as defined and studied in Chapter 3. This theorem is used in the proof of Equation (2.37) of Theorem 2.16 (see Sections 2.5 and 2.6). Note however, that in Chapter 3 we obtain a better asymptotic density estimate (with the $O(\sqrt{\varrho}\,)$ of Equation (C.8) below replaced by $O(1)$ in Theorem 3.6, Equation (3.10)).

Nevanlinna theory was developed in the 1930s to study the distribution of solutions in z of the equation $f(z) = a$ (so-called a-points) of a meromorphic function f. Recall that a meromorphic function is an analytic function

$$f\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\},$$

where $f(x)$ is defined to be ∞ if f has a pole at x. In this case, the function $g(z) = 1/f(z)$ is defined in a neighborhood of x and g is holomorphic at $z = x$ if we set $g(x) = 0$. In other words, we can view f as a holomorphic function to the Riemann sphere

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}.$$

The starting point of Nevanlinna theory is the Poisson–Jensen formula. It was Nevanlinna's insight that this formula can be interpreted as saying that the number of a-points of f in a disc plus the average closeness of f to a on the boundary of this disc (measured in a suitable way) equals the number of poles of f plus the average size of f on this boundary. We exploit this

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fact for a holomorphic function, by counting the number of a-points of f in a disc by computing the average size of f on the boundary of this disc, provided one can bound the average closeness of f to a on the boundary.

We refer the reader to the monographs [Hay] and [LanCh] for an exposition of Nevanlinna theory.

C.1 The Nevanlinna Height

Recall that $\mathbb{P}^1(\mathbb{C})$ denotes the complex projective line; i.e., the complex line $\mathbb{C}$, completed by a point at infinity, denoted ∞. Alternatively, $\mathbb{P}^1(\mathbb{C})$ could be realized as the Riemann sphere. The *distance* between two points a and a' in $\mathbb{P}^1(\mathbb{C})$ is defined as

$$\|a, a'\| = \frac{|a - a'|}{\sqrt{1+|a|^2}\sqrt{1+|a'|^2}}, \quad \text{if } a, a' \neq \infty, \tag{C.1a}$$

and the distance of a point to the point at infinity is given by

$$\|a, \infty\| = \frac{1}{\sqrt{1+|a|^2}}, \quad \text{if } a \neq \infty. \tag{C.1b}$$

Here, $|z|$ denotes the ordinary absolute value of the complex number z. (See, e.g., [Bea, §2.1].) When one views $\mathbb{P}^1(\mathbb{C})$ as a sphere of diameter 1 in three-dimensional Euclidean space, the distance is simply the chordal distance between the inverse images of a and a' under stereographic projection.

Let f be a nonconstant meromorphic function and let $a \in \mathbb{P}^1(\mathbb{C})$. The *mean proximity* function of f is the function of the positive real variable ϱ given by

$$m_f(a, \varrho) = \int_{|z|=\varrho} -\log \|f(z), a\| \, \frac{dz}{2\pi i z}. \tag{C.2}$$

The *counting function* of f is defined as[1]

$$n_f(a, \varrho) = \#\left\{z \in \mathbb{C} \colon |z| \leq \varrho,\ f(z) = a\right\}, \tag{C.3}$$

and, for $a \neq f(0)$, we set

$$N_f(a, \varrho) = \int_0^{\varrho} n_f(a, t) \, \frac{dt}{t}. \tag{C.4}$$

Finally, the *Nevanlinna height* of f is defined, for $a \neq f(0)$, by

$$T_f(\varrho) = N_f(a, \varrho) + m_f(a, \varrho) + \log \|f(0), a\|, \tag{C.5}$$

which is independent of a (cf. [LanCh, Theorem 1.6, p. 19]). It is this independence that we will exploit, for $a = \infty$ and for $a = 0$.

[1] This function takes finite values since the zeros of a nonconstant meromorphic function form a discrete subset of $\mathbb{C}$.

C.2 Complex Zeros of Dirichlet Polynomials

We investigate the distribution of the roots of the equation

$$\sum_{j=0}^{M} m_j r_j^s = \sum_{j=0}^{M} m_j e^{-w_j s} = 0,$$

where $r_0 = 1 > r_1 > r_2 > \ldots > r_M > 0$. The weights w_j are defined by $r_j = e^{-w_j}$, so that $w_0 = 0 < w_1 < w_2 < \cdots < w_M$.

Our analysis is partly similar to that of Jorgenson and Lang [JorLan3]. In particular, the result that the zeros of a Dirichlet polynomial lie in a bounded strip can be found in [JorLan3, p. 58]. On the other hand, in the present situation, we obtain more precise results than in [JorLan3]. Similar results were also obtained by B. Jessen. (See [Bohr, Appendix II].)

Let σ_l and σ_r be defined by the equations

$$e^{(w_M - w_{M-1})\sigma_l} \sum_{j=0}^{M} |m_j| = \frac{1}{2}\,|m_M|\,, \tag{C.6a}$$

and

$$e^{-w_1 \sigma_r} \sum_{j=0}^{M} |m_j| = \frac{1}{2}\,|m_0|\,. \tag{C.6b}$$

In other words, writing $\sum |m_j| = \sum_{j=0}^{M} |m_j|$, we have

$$\sigma_l = -\frac{\log\left(2\sum |m_j|/|m_M|\right)}{w_M - w_{M-1}} \quad \text{and} \quad \sigma_r = \frac{\log\left(2\sum |m_j|/|m_0|\right)}{w_1}. \tag{C.7}$$

Theorem C.1. *Let $w_0 = 0 < w_1 < \ldots < w_M$ and let $m_0, \ldots, m_M$ be arbitrary nonzero complex numbers. Define*

$$f(s) = \sum_{j=0}^{M} m_j e^{-w_j s}$$

and assume that $f(0) = \sum_{j=0}^{M} m_j \neq 0$. Then the number of complex zeros of $f(s)$, counted according to multiplicity in the closed disc of radius ϱ, equals[2]

$$\frac{\log r_M^{-1}}{\pi}\varrho + O(\sqrt{\varrho}), \quad \textit{as } \varrho \to \infty. \tag{C.8}$$

[2] In Theorem 3.6, Equation (3.10), the density is given as $\frac{\log r_M^{-1}}{2\pi}\varrho + O(1)$, as $\varrho \to \infty$; i.e., half of the density as given here. The reason is that in Theorem 3.6, we count zeros in the upper half of a vertical strip, $\{s\colon 0 \le \operatorname{Im} s \le \varrho\}$.

Moreover, the zeros lie in the horizontally bounded strip $\sigma_l \leq \operatorname{Re} s \leq \sigma_r$, *where* σ_l *and* σ_r *are given respectively by Equations* (C.6a) *and* (C.6b) *and formula* (C.7) *above.*

Proof. To obtain information about the zeros of $f(s)$, we estimate the counting function $n_f(0, \varrho)$. To accomplish this, we first compute the height

$$T_f(\varrho) = m_f(\infty, \varrho) + \log \|f(0), \infty\|$$

and then combine the relation

$$T_f(\varrho) = N_f(0, \varrho) + m_f(0, \varrho) + \log \|f(0), 0\|$$

with estimates for $m_f(0, \varrho)$ to obtain an estimate for $N_f(0, \varrho)$. Finally, we use Lemma C.2 below to deduce the estimate for $n_f(0, \varrho)$.

The height of f is

$$m_f(\infty, \varrho) = \int_{|s|=\varrho} \log \sqrt{1 + |f(s)|^2}\, \frac{ds}{2\pi i s} + \log \|f(0), \infty\|.$$

We need to estimate $|f(s)|$ in the above integral. For $\sigma = \operatorname{Re} s \geq \sigma_l$, the function $|f(s)|$ is bounded by a constant depending on w_j and m_j ($j = 0, \ldots, M$). On the other hand, for $\sigma \leq \sigma_l$, we have

$$\begin{aligned} |f(s)| = \left| \sum\nolimits_{j=0}^{M} m_j r_j^s \right| &\geq |m_M| r_M^\sigma - \sum\nolimits_{j=0}^{M-1} |m_j| r_j^\sigma \\ &\geq r_M^\sigma \left(|m_M| - \sum\nolimits_{j=0}^{M-1} |m_j| e^{(w_M - w_j)\sigma} \right) \\ &\geq r_M^\sigma \left(|m_M| - e^{(w_M - w_{M-1})\sigma_l} \sum\nolimits_{j=0}^{M} |m_j| \right) \\ &= \frac{|m_M|}{2} r_M^\sigma. \end{aligned} \tag{C.9}$$

Putting these estimates together, we find that

$$\log \sqrt{1 + |f(s)|^2} = \begin{cases} w_M |\sigma| + O(1) & (\sigma \leq \sigma_l), \\ O(1) & (\sigma \geq \sigma_l), \end{cases}$$

as $|s| = \varrho \to \infty$. On the circle with radius ϱ, the real part of $s = \varrho e^{i\theta}$ equals $\varrho \cos \theta$. For the height, we thus find that

$$T_f(\varrho) = -w_M \int_{\pi/2}^{3\pi/2} \varrho \cos \theta \, \frac{d\theta}{2\pi} + O(1) = \frac{w_M}{\pi} \varrho + O(1),$$

as $|s| = \varrho \to \infty$, where the implied constant depends only on M and the numbers r_j and m_j.

Now, clearly, $m_f(0,\varrho)$ is positive. Hence $N_f(0,\varrho) \le \frac{w_M}{\pi}\varrho + O(1)$. To show that this is the correct asymptotic order for $N_f(0,\varrho)$, we have to bound the function $m_f(0,\varrho)$ from above. In view of (C.1), inequality (C.9) shows that

$$\|f(s),0\|^{-1} = \frac{\sqrt{1+|f(s)|^2}}{|f(s)|} = \sqrt{1+|f(s)|^{-2}}$$

is uniformly bounded for $\sigma \le \sigma_l$. For $\sigma \ge \sigma_r$, on the other hand, we have

$$|f(s)| \ge |m_0| - \sum_{j=1}^{M} |m_j| r_j^\sigma \ge |m_0| - r_1^\sigma \sum_{j=0}^{M} |m_j| \ge \frac{|m_0|}{2}.$$

Thus $\log\|f(s),0\|^{-1}$ is uniformly bounded for $\operatorname{Re} s \ge \sigma_r$ and $\operatorname{Re} s \le \sigma_l$. Observe that this shows that the complex zeros of f lie in the horizontally bounded strip $\sigma_l \le \operatorname{Re} s \le \sigma_r$. The integral for $m_f(0,\varrho)$ over the parts of the circle $|s| = \varrho$ between $\operatorname{Re} s = \sigma_l$ and $\operatorname{Re} s = \sigma_r$ is bounded since $x \mapsto \log|x|$ is an integrable function around $x = 0$. This shows that

$$N_f(0,\varrho) = \frac{w_M}{\pi}\varrho + O(1),$$

as $\varrho \to \infty$. Finally, in view of (C.3) and (C.4), the statement for $n_f(0,\varrho)$ is a consequence of the following general calculus lemma, applied to the functions $n(t) := n_f(0,t)$ and $N(\varrho) := N_f(0,\varrho)$. □

Lemma C.2. *Let $n(t)$ be a nondecreasing, nonnegative function on $[0,\infty)$ for which there exists $t_0 > 0$ such that $n(t) = 0$ for $t \le t_0$. Let*

$$N(\varrho) = \int_0^\varrho n(t)\,\frac{dt}{t}$$

and suppose that there exist positive constants c and C such that

$$|N(\varrho) - c\varrho| \le C \qquad \text{for all } \varrho > 0.$$

Then

$$|n(\varrho) - c\varrho| \le \sqrt{8Cc\varrho}$$

for all sufficiently large positive values of ϱ.

Proof. Consider a value of ϱ for which $n(\varrho) > c\varrho$. For this value, we have

$$\begin{aligned} n(\varrho) + C &\ge N(n(\varrho)/c) \\ &= N(\varrho) + \int_\varrho^{n(\varrho)/c} n(t)\,\frac{dt}{t} \ge c\varrho - C + n(\varrho)\log\frac{n(\varrho)}{c\varrho}. \end{aligned}$$

Hence, writing $x = \frac{n(\varrho)}{c\varrho}$, we deduce that

$$2C \ge c\varrho - n(\varrho) + n(\varrho)\log\frac{n(\varrho)}{c\varrho} = c\varrho\,(1 - x + x\log x)\,.$$

Consider now a value of ϱ for which $n(\varrho) < c\varrho$. In the same way as above, we find

$$c\varrho - C \le N(\varrho) = N(n(\varrho)/c) + \int_{n(\varrho)/c}^{\varrho} n(t)\,\frac{dt}{t} \le n(\varrho) + C + n(\varrho)\log\frac{c\varrho}{n(\varrho)}.$$

Again, we deduce that $c\varrho\,(1 - x + x\log x) \le 2C$, with $x = \frac{n(\varrho)}{c\varrho}$.

The function

$$1 - x + x\log x = \frac{(x-1)^2}{2} + O\left((x-1)^3\right) \qquad (\text{as } x \to 1)$$

is nonnegative and vanishes at $x = 1$. It follows that for ϱ sufficiently large, x is close to 1. Around $x = 1$, this function takes values larger than $(x-1)^2/4$. Hence for large positive ϱ, $(x-1)^2 \le \frac{8C}{c\varrho}$. This is equivalent to $|n(\varrho) - c\varrho| \le \sqrt{8Cc\varrho}$, as was to be proved. □

Remark C.3. In Chapter 3, we define numbers d_l and d_r such that the Dirichlet polynomial $f(s)$ does not vanish for $\operatorname{Re} s < d_l$ and $\operatorname{Re} s > d_r$ (see formula (3.18)). The numbers σ_l and σ_r defined above give weaker bounds, because they are defined to satisfy the stronger property that $|f(s)|$ (respectively, $|f(s)|r_M^{-s}$) is bounded away from 0 by a fixed distance for $\operatorname{Re} s \ge \sigma_r$ (respectively, $\operatorname{Re} s \le \sigma_l$).

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It was also presented by the first author in invited talks at the CBMS-NSF Conference on “Spectral Problems in Geometry and Arithmetic" (Iowa City, August 1997) and at the Conference on “Recent Progress in Noncommutative Geometry" (Lisbon, Portugal, September 1997), as well as in the Workshops and Programs on “Spectral Geometry” (June–July 1998) and on “Number Theory and Physics” (September 1998), both held at the Erwin Schroedinger International Institute for Mathematical Physics in Vienna, Austria.

In addition, the continuation of this work was presented by one or both of the authors at numerous conferences, workshops, summer schools and meetings of professional societies. A list of the corresponding talks prior to the publication of the first edition of this book in 2006 can be found on pages 440 and 441 of [Lap-vF11]. We mention just a few of the most important ones. It was presented by the authors at the Special Session on “Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot” held during the Annual Meeting of the American Mathematical Society (San Diego, California, January 2002), and at the International Conference on “Analysis, Fractal Geometry, Dynamical Systems and Economics” held at the University of Messina on the occasion of the Award (to MLL) of the 2011 Anassilaos International Prize in Mathematics (Messina, Sicily, and Reggio Calabria, Italy, November 2011), and at the Special Session on “Fractal Geometry in Pure and Applied Mathematics: In Memory of Benoît Man-

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Conventions

$f(x) = O(g(x))$	$f(x)/g(x)$ is bounded.
$f(x) = o(g(x))$	$f(x)/g(x)$ tends to 0.
$f(x) \sim g(x)$	$f(x)/g(x)$ tends to 1.
$f(x) \ll g(x)$	same meaning as $f(x) = O(g(x))$.
$\approx$	approximately equal to.
$d \mid n$	d divides n.
$A \backslash B$	the set of points in A that do not lie in B.
$\#A$	the cardinality of the finite set A.
$\mathbb{N}$	the set of nonnegative integers $0, 1, 2, 3, \ldots$
$\mathbb{N}^* = \mathbb{N} \backslash \{0\}$	the set of positive integers $1, 2, 3, \ldots$
$\mathbb{Z}$ and $\mathbb{Q}$	the sets of integers and rational numbers, respectively.
$\mathbb{R}$ and $\mathbb{C}$	the sets of real and complex numbers, respectively.
$\mathbb{R}_+^*$	the multiplicative group of positive real numbers.
i	the square root of -1; $e^{\pi i/2}$.
$s = \sigma + it$	s is a complex number with $\sigma = \operatorname{Re} s$ and $t = \operatorname{Im} s$.
$\log x$	the natural logarithm of x.
$\log_a x$	$\log x / \log a$, the logarithm of x with base a.
$[x]$	the greatest integer less than or equal to x.
$\{x\} = x - [x]$	the fractional part of x.

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