

Bibliography

1. Aoki, M. *Optimization of Stochastic Systems*. New York, London: Academic Press, 1967.
2. Astrem, K. J. *Introduction to Stochastic Control Theory*. New York: Academic Press, 1970.
3. Bellman, R. A Markov decision process. *J. Math. Mech.*, **6**, 679–684, 1957.
4. Bellman, R. *Dynamic Programming*. Princeton: Princeton University Press, 1957.
5. Benes, V. E. Existence of optimal strategies based on specified information, for a class of stochastic decision problems. *SIAM J. Control*, **2**, 179–188, 1970.
6. Benes, V. E. Existence of optimal stochastic control laws. *SIAM J. Control*, **9**, 446–472, 1971.
7. Blackwell, D. Discrete dynamic programming. *Ann. Math. Stat.*, **33**, 719–726, 1962.
8. Blackwell, D. Positive dynamic programming. Proc. of the 5th Berkeley Symp., vol. 3, 415–428, 1965.
9. Blackwell, D. On stationary policies, Proc. of the 37th Session of the Intern. Statist. Institute, London, Sept. 1969. (Russian translation in *Matematika* **14**, No. 2, 155–159, 1970.)
10. Davis, M. H. A. On the existence of optimal policies in stochastic control. *SIAM J. Control*, **11**, 587–994, 1973.
11. Davis, M. H. A., Varaiya, P. P. Dynamic programming conditions for partially observable stochastic integrals. *SIAM J. Control*, **11**, 226–261, 1973.
12. Duncan, T. E., Varaiya, P. P. On the solution of a stochastic control system. *SIAM J. Control*, **9**, 354–371, 1971.
13. Dunford, N., Schwartz, J. T. *Linear Operators, General Theory*. New York: Interscience Publishers, 1958.
14. Dynkin, E. B., Yushkevich, A. A. *Controlled Markov Processes and their Applications*. Moscow: Nauka, 1975 (in Russian).
15. Fleming, W. Some Markovian optimization problems. *J. Math. and Mech.*, **12**, 131–140, 1963.
16. Fleming, W. Duality and a priori estimates in Markovian optimization problems. *J. Math. Anal. Appl.*, **16**, 254–279, 1966. Erratum, *ibid.* **19** (1966) p. 204.
17. Fleming, W. Optimal continuous parameter stochastic control. *SIAM Rev.*, **11**, 470–509, 1969.

18. Gihman, I. I. On a weak compactness of a set of measures corresponding to solutions of stochastic differential equations. In: *Mathematical Physics 7*, Kiev: Naukova Dumka, 49–65, 1970 (in Russian).
19. Gihman, I. I., Skorohod, A. V. *Stochastic Differential Equations*. Berlin, Heidelberg, New York: Springer-Verlag, 1972.
20. Gihman, I. I. and Skorohod, A. V. *The Theory of Stochastic Processes*, Volumes I, II, III. Berlin, Heidelberg, New York: Springer-Verlag, 1974, 1975, 1979.
21. Girsanov, I. V. On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory Probab. and Applic.* **5**, 285–301, 1960.
22. Gubenko, L. G. Controlled Markov and semi-Markov models and some specific problems of optimization of the stochastic system. Abstract of candidate's thesis, Kiev State University, Kiev, 1972 (in Russian).
23. Gubenko, L. G. and Shtatland, E. S. On controlled discrete time Markov decision processes. *Teor. Veroyatn. i Matemat. Statist.*, **7**, 51–64, 1972. English translation **7**, 47–62, 1975.
24. Hausdorff, F. *Grundzüge der Mengenlehre*, Leipzig: Teubner 1914.
25. Hinderer, K. *Foundation of Nonstationary Dynamic Programming with Discrete Time Parameter*. Berlin, Heidelberg, New York: Springer-Verlag, 1970.
26. Howard, R. A. *Dynamic Programming and Markov Processes*. Cambridge: Technology Press, and New York: J. Wiley, 1960.
27. Howard, R. A. *Dynamic Probabilistic Systems, V.2. Semi-Markov and Decision Processes*. New York: J. Wiley, 1971.
28. Krylov, N. W. On the existence of ε -optimal homogeneous Markov strategies for controlled chains. *Dokl. Akad. Nauk SSSR*, **155**, 747–750, 1964 (in Russian).
29. Krylov, N. W. The construction of an optimal strategy for a finite controlled chain. *Theory Probab. and Applic.*, **10**, 45–54, 1965.
30. Krylov, N. W. Control of a solution of a stochastic integral equation. *Theory Probab. and Applic.*, **17**, 114–131, 1972.
31. Krylov, N. W. On controlling a solution of a stochastic integral equation in the presence of degeneracy. *Izvest. Akad. Nauk SSSR, Ser. Matemat.* **36**, 248–261 (in Russian) 1972.
32. Krylov, N. W. On Bellman's equation. In the volume: *Trudy Shkoly-Seminara po Teorii Sluchainykh Protsessov (Proc. of the Seminar on the Theory of Stochastic Processes)*. Izdat. Akad. Nauk Lit SSSR, Vil'nyus, 1975.
33. Kushner, H. J. *Stochastic Stability and Control*. New York, London: Academic Press, 1967.
34. Kushner, H. J. *Introduction to Stochastic Control*. New York: Holt, 1971.
35. Lipcer, R. Sh. and Shirayev, A. N. *Stochastics of Random Processes*. Berlin, Heidelberg, New York: Springer-Verlag, 1976.
36. Dellacherie, C. and Meyer, P. A. *Probabilities and Potential*. New York: Elsevier, 1977.
37. Portenko, N. I. and Skorohod, A. V. On the existence of ε -optimal Markov strategies for controlled diffusion processes. In: *Voprosy Statistiki i Upravleniya Sluchainymi Protsessami (Problems of Statistics and Control of Random Processes)*. Izdat. IM Akad. Nauk USSR, Kiev, 204–227, 1974.
38. Pragarauskas, G. On a control theory for discontinuous random processes. In: *Trudy Shkoly-Seminara po Teorii Sluchainykh Protsessov (Proc. of the Seminar on the theory of stochastic processes)*, Izdat. Akad. Nauk LitSSR, Vil'nyus, 1975.
39. Rishel, R. Necessary and sufficient dynamic programming conditions for continuous-time stochastic optimal control. *SIAM J. Control*, **8**, 559–571, 1970.
40. Shiryaev, A. N. *Statistical Sequential Analysis*. Providence: Amer. Math. Soc., 1973 (Russian edition, Nauka 1969).
41. Skorohod, A. V. *Studies in the Theory of Random Processes*. Reading: Addison Wesley, 1965.

42. Skorohod, A. V. and Slobodenyuk, N. P. *Limit Theorems for Random Walks*. Kiev: Naukova Dumka, 1970.
43. Wald, A. *Statistical Decision Function*. New York: J. Wiley, 1950.
44. Wonham, W. M. Stochastic problems in optimal control. Tech. report, Baltimore, 63–74, 1963.

Additional References for the English Edition

45. Wonham, W. M. Random differential equations in control theory. In A. T. Bharucha-Reid ed., *Probabilistic Methods in Applied Math.*, vol 2, New York, London: Academic Press, 131–212, 1970.
46. Chow, Y. S., Robbins, H., Siegmund, D. *Great Expectations: The Theory of Optimal Stopping*. Boston: Houghton Mifflin, 1971.
47. Fleming, W. H., Richel, R. W. *Deterministic and Stochastic Control*. Berlin, Heidelberg, New York: Springer-Verlag, 1975.
48. Gatun, A. P., Gihman, I. I. Controlled stochastic differential equations without an after-effect. *Theory of stoch. processes*, **5**, 14–21, Kiev: Naukova Dumka, 1977 (in Russian).
49. Krylov, N. W. *Controlled Processes of the Diffusion Type*. Moscow: Nauka 1977 (in Russian).
50. Skorohod, A. W. *On One General Scheme of Controlled Stochastic Processes. Controlled Stochastic Processes and Systems*. Kiev: Naukova Dumka, 1972 (in Russian).
51. Snell, J. L. Application of martingale system theorems. *Trans. Amer. Math. Soc.*, **73**, 293–312, 1952.
52. Stone, L. D. Necessary and sufficient conditions for optimal control of semi-Markov jump process. *SIAM J. Control*, **11**, 187–201, 1973.
53. Strauch, R. E. Negative dynamic programming. *Ann. Math. Stat.* **37**, 871–890, 1966.

Historical and Bibliographical Remarks

Chapter 1

The basic ideas of optimal control theory are linked to sequential analysis and the theory of statistical decision functions which originated and were developed by A. Wald (see, e.g. [43]). R. Bellman developed this theory for more specific cases under the name of dynamic programming (see [3], [4]). Further development of multi-staged Markov processes was stimulated by Howard [26], [27] who also studied the problem of the controlled Markov chain. The general results on optimal controlled Markov chains with arbitrary sets of states were obtained by D. Blackwell [7], [8], [9], R. E. Strauch [53], and N. W. Krylov [28], [29]. A detailed exposition of the theory of controlled Markov chains is contained in Kushner's monograph [34] and a more general theory may be found in E. B. Dynkin and A. A. Yushkevich's memoir [14].

The general description of controlled stochastic objects presented in the first chapter of the present book and the constructions of optimal and ε -optimal controls are based on A. V. Skorohod's paper [50].

The bibliography of works devoted to the problem of optimal stopping is quite substantial. A general solution of this problem is due to Snell [51]. A more detailed exposition of optimal stopping theory is presented in A. N. Shiryaev's [40] and in Y. S. Chow *et al.*'s monographs [46].

Chapter 2

The definition and construction of arbitrary controlled (in general non-Markov) objects with continuous time given in the second chapter is along the lines of A. V. Skorohod's paper [50] cited above as well as N. I. Portenko and A. V. Skorohod's work [37]; however, our definition is wider in scope and more far reaching. This construction of the representation of a controlled object appears in the literature presumably for the first time.

Optimal controlled semi-Markov processes with continuous time are discussed by L. D. Stone [52].

Chapter 3

The theory of stochastic differential equations is studied in more detail in the following books: [19] and [20] vol 3, for example.

Initially the development of the theory of optimal control for solutions of stochastic differential equations was concerned solely with linear systems with a square cost of control and with problems related to the theory of "Bellman's equation".

Section 5 deals with the problem of control for a solution of a linear system of stochastic differential equations with complete observations of a somewhat more general type than usually considered in the literature, namely the random perturbations affecting the system may be general continuous processes with independent increments and finite moments of the second order.

A review of the results dealing with the theory of linear systems is presented for instance in A. W. Wonham's paper [45].

In connection with Bellman's equation and its justification for stochastic processes with continuous time, the reader is referred to the works of H. J. Kushner [33], W. Fleming [16], [17], and N. W. Krylov [30–32], where the results of the theory of parabolic equations with partial derivatives are substantially utilized. The most far-reaching study of controlled diffusion processes based on a direct probabilistic approach to Bellman's equation is presented in N. W. Krylov's monograph [49]; see also W. H. Fleming and R. W. Rishel's memoir [47].

The convergence of finite-difference approximations in the problem of optimal control for solutions of stochastic differential equations was considered in the papers of A. V. Skorohod [50], A. P. Gatun and I. I. Gihman [48].

The material in Sections 6 and 7 is based on M. H. A. Davis and P. P. Varaiya's [11] and M. H. A. Davis' contributions [10] (see also V. E. Benes [5–6].)

In this monograph only the case of complete observations is studied. A more general case of controlling incomplete observations is discussed in [11]. The construction of ε -optimal Markov controls is based on N. I. Portenko and A. V. Skorohod's exposition [37].

A

additive noise 191

B

Baire function of the first class 68
 basic process 7 (*see also* controlled process)
 value of 80
 Bellman equation 111, 178, 192
 analog of 211
 approximate solution of 181
 generalized 180, 187
 uniqueness of solution of 113
 Bellman's principle 172 (*see also* optimality principle)

C

control 79, 153
 admissible 37, 109, 153
 -non-randomized 37
 continuous time 79
 generalized 86
 non-randomized 80
 step 81
 cost 8, 153

discrete time
 ε -optimal 74, 77
 Markovian 52, 57
 ε -optimal 8, 74, 77, 157
 feedback 153
 generalized 86
 non-randomized 7, 11, 80
 optimal 8
 phase space of 1
 sequence of 6
 step 81
 controlled Markov chain 50, 56, 59
 homogeneous 61, 75
 controlled Markov object 105–106
 (*see also* Markovian controlled object)
 controlled object 1, 79, 85
 Gaussian 93, 94, 95
 Markovian 50, 106, 114
 representation of 85, 87
 stochastic equivalence of 90
 with independent increments 93, 94
 controlled process 7, 86
 controlled random sequence 7
 cost function with discounting 62
 cost functional 8
 cost of controlling 8, 154, 200 (*see also* control cost)
 ε -optimal 154
 optimal 154
 regular 34
 current of σ -algebras 116

D

density of measures 148
 diffusion solution 176
 Doob's inequality 122, 132, 139

E

essential supremum 44–46
 excessive function 75, 76

F

Fatou's theorem 33, 136
 feedback control with incomplete data 153
 finite-difference approximation 161
 function
 Baire, of the first class 68
 Borel 16, 20, 27, 69, 112
 excessive 75, 76
 loss 154
 lower semi-continuous 11, 12
 non-anticipative 127
 functional
 of evolutionary type 51, 106, 177
 regularization of 35

G

generalized control 86, 98, 154

H

Hamiltonian 214
 homogeneous process 117
 with independent increments 118

I

infinitely divisible distribution 162
 Ito's formula 138, 150, 196

J

jump Markovian object 107–109

K

Kolmogorov's theorem 33, 56

L

Lebesgue's theorem 33, 50
 Levy's theorem 124, 202
 Lipschitz condition
 local 137, 156, 223
 uniform 131, 134, 136, 159
 with respect to semi-norm 162
 loss function 154

M

Markov process 176, 216, 220
 Markov time 77
 Markovian control 177, 198, 216
 ϵ -optimal 216
 non-randomized 52, 57, 109, 115
 optimal 224
 Markovian controlled object 106, 114
 martingale 122
 characteristic of 123, 130, 206
 mutual 123
 continuous local 150
 local square integrable 123
 square integrable 122
 Meyer's theorem 210

N

norm of a vector 122

O

optimality principle 172

P

phase space of controls 172
 phase space of the basic process 1, 79
 Poisson measure 94, 117
 procedure for correcting strategies 64, 65
 process with independent increments 116

R

random interference 191
 random perturbations 191
 random time 123
 regularization of a functional 35

S

semi-norm 162
 sequence of controls 6 (*see also*
 strategy)
 square variation of a process 124
 stationary Markov strategy 63–65, 67
 optimal 63
 step control 81
 stochastic differential equation 153
 stochastic functional-differential
 equation 128
 linear 187
 linearly bounded 129
 of continuous type 138
 of diffusion type 143
 solution of 128
 with a lag 129
 without an after-effect 129
 stochastic integral 119
 basic result on 125

stochastic process 116
 adapted to a current 116
 stochastically continuous process 117
 stopping rule 43
 stopping time 43, 75
 ε -optimal 49
 optimal 48
 truncated 71
 supermartingale 76, 151, 203, 209
 relation to martingale 151
 strategy 6–7, 71, 153
 admissible 153

V

value 71, 78
 value of the basic process 80

W

weak compactness of measures 144, 205
 Wiener process 94, 117, 126, 152, 202,
 210
 definition of 117
 relation to local martingale 124