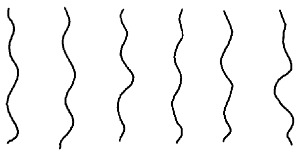
Additional Exercises

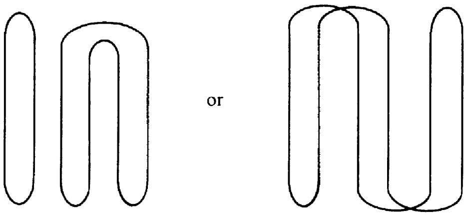
Note: Exercises 1 to 16 are relative to Chapters 1, 2, and 3. Chapters 4 and 5 are needed for Exercises 17 to 28.

Exercise 1 (A Sequence of Liars). Consider a sequence of n "liars" $L_1, \ldots, L_n$. The first liar L_1 receives information about the occurrence of some event ("yes" or "no") and transmits it to L_2, who transmits it to L_3, etc .... Each liar transmits what he hears with probability $p (0 < p < 1)$, and the contrary with probability $q = 1 - p$. The decision of lying or not lying is made independently by each liar. What is the probability p_n to obtain the correct information from L_n? What happens when n increases to infinity?

Exercise 2 (The Golden Ring). There are $2n$ bits of thread:



Two people, operating independently of each other, make knots. The first one makes knots on the upper extremities, and the other one on the lower extremities. Each lower (resp. upper) extremity is involved in one and only one knot. For instance with $2n = 6$, you can obtain, among other configurations:



In the second situation, you have just one piece of thread forming a ring. What is the probability P_{2n} for this event to occur in the general case?
Hint: Find P_2 and a recurrence relation between P_{2n} and P_{2n-2}.

Exercise 3 (Shall I Get This Book?). You are looking for a book in the campus libraries. Each library has it with probability 0.60, but the book may have been borrowed by some other patron with probability 0.25. If there are 3 libraries, what are your chances of obtaining the book?

Exercise 4 (Winning a Game of Heads and Tails). Two players A and B with respective initial fortunes \$$a$ and \$$b$ (a, b strictly positive integers) play a game of heads and tails, betting an amount of \$1 at each toss. The outcome "heads" has probability $p(0 < p < 1)$. Player A wins on "heads." Compute the probability for A to win (the game ends when one of the players is broke).
Hint: You must compute the probability $u(a)$ that A starting with a fortune of $x = a$ reaches fortune $c = a + b$ without getting broke in the meantime. Of course $x = a$ is of interest, but you will compute the probability $u(x)$ for all integers $x \in [0, c]$. To do this derive a recurrence relation for $u(x)$, and solve it.

Exercise 5 (Heads and Tails Again). A person, named A, throws an *unbiased* coin N times and obtains T_A "tails." Another person, B, throws his own unbiased coin $N + 1$ times and has T_B "tails." What is the probability that $T_A \geqslant T_B$?
Hint: Introduce H_A and H_B the number of "heads" obtained by A and B respectively, and use a symmetry argument.

Exercise 6 (Boys and Girls Are Independent). Let X be a Poisson random variable with mean $\lambda > 0$, independent of $(Y_n, n \geqslant 0)$, a sequence of $\{0, 1\}$ valued iid random variables with $P(Y_n = 1) = p$. Show that $U = \sum_{n=1}^{X} Y_n$ and $V = X - U$ are independent Poisson random variables of mean λp and $\lambda(1 - p)$ respectively.

Remark 1. This is generally considered a "paradoxical" result since it would be obviously false if X were replaced by a fixed number k.

Remark 2. The title refers to a model where the total progeny of a couple consists of X boys and girls, with U boys and V girls.

Exercise 7 (Group Testing). N individuals must undergo a medical test which is somewhat expensive. Instead of analyzing a blood sample for each of them (a procedure which would require N separate analyses), the blood samples are mixed and the test is performed on this "group sample." If it is positive (i.e., if at least one of the patients has a positive reaction), then all N patients undergo an individual test. The probability of a positive reaction for a given patient is p, and it is assumed that, with respect to this test, the patients react independently of one another. What is the average number of tests required in the group testing procedure? Compare it to N when $N = 4$, $p = 1/10$.

Exercise 8 (Decoding a Characteristic Function). Find the distribution of a random variable X with cf $\phi_X(u) = 1/(2e^{-iu} - 1)$.

Exercise 9 (The Covariance Matrix of a Multinomial Vector). Let $X = (X_1, \ldots, X_k)$ be a k-dimensional random vector where the X_i's take their values in $\mathbb{N}$. Let $g(s_1, \ldots, s_k)$ be the generating function of this vector. Show that $E[X_i X_j - \delta_{ij}] = (\partial^2 g/\partial s_i \partial s_j)(1, \ldots, 1)$. Apply this to vector $X \sim \mathcal{M}(n, k, p_i)$ (Eq. (11) of Chapter 2), for which it is known (E21 of Chapter 2) that $g(s_1, \ldots, s_k) = (p_1 s_1 + \cdots + p_k s_k)^n$, to compute its covariance matrix.

Exercise 10 (The Random Pen Club). You write n personal letters to be sent to n of your friends and you write the addresses at random on the envelopes. What is the probability that at least one of the envelopes has the correct adress? What if n is very large?

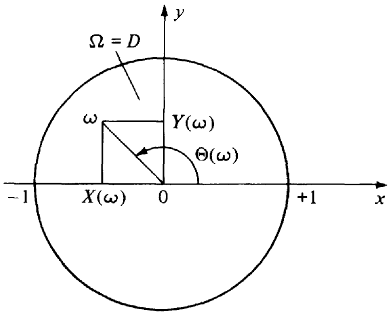
Exercise 11 (The Matchbox (Banach's Problem)). A smoker has one matchbox with n matches in each pocket. He reaches at random for one box or the other. What is the probability that, having eventually found an empty matchbox, there will be k matches left in the other box?

Exercise 12 (Infinite Expectation in the Coin Tossing Game). Two players toss an unbiased coin in turn until both get the same number of heads. Let $2N$ be the total number of coin tosses needed for equalization. Find $P(2N = 2n)$ and give the expectation of $2N$.

Exercise 13 (Lottery Tickets). Lottery tickets have numbers going from 0 0 0 0 0 0 to 9 9 9 9 9 9. Find the probability of purchasing a ticket with the sum of the first three digits equal to the sum of the last three digits.
Hint: Use generating functions.

Exercise 14 (The Uniform Distribution on a Disk). The following model of a point selected at random uniformly in the closed unit disk D of R^2 centered at the origin is proposed: Take $\Omega = D$ and let $\mathscr{F}$ be the family of subsets of D for which the area can be defined, and for each set of A of $\mathscr{F}$ with area

$S(A)$ define $P(A) = S(A)/S(\Omega) = S(A)/\pi$. The figure below introduces a few notations.



Define $Z = (X^2 + Y^2)^{1/2}$. Thus Z is a random variable taking its values in $[0, 1]$ and Θ takes its values in $[0, 2\pi)$. Compute $P(0 \leqslant a \leqslant Z \leqslant b \leqslant 1, 0 \leqslant \theta_1 \leqslant \Theta < \theta_2 < 2\pi)$ and show that Z and Θ are independent.

Exercise 15 (Equation with Random Coefficients). The numbers a and b are selected independently and uniformly on the segment $[-1, +1]$. Find the probability that the roots of the equation $x^2 + 2ax + b$ are real.

Exercise 16 (Infinum of Exponential Random Variables). Let $X_1, \ldots, X_n$ be independent exponential random variables, with $E[X_i] = 1/\lambda_i$ $(1 \leqslant i \leqslant n)$. Find the pdf of $Y = \inf(X_1, \ldots, X_n)$.

Exercise 17 (Gaussian Vector). Let X and Y be two independent standard Gaussian random variables. Show that the two-dimensional random vector $((X + Y)/\sqrt{2}, (X - Y)/\sqrt{2})$ is a standard Gaussian vector, *without using the formula of change of variables.*

Exercise 18 (Spherical Coordinates). The random variables $X_1, \ldots, X_n$ are iid $\mathcal{N}(0, \sigma^2)$. Find the pdf of the vector $R, \Phi_1, \ldots, \Phi_{n-1}$ where $R \geqslant 0$, $\Phi_i \in [0, 2\pi)$ $(1 \leqslant i \leqslant n - 1)$, and

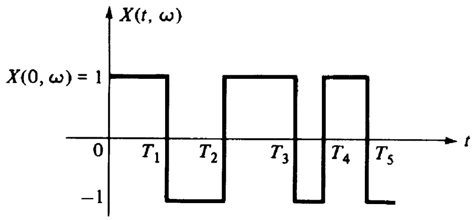
$$\begin{aligned}
X_1 &= R \sin \Phi_1 \\
X_2 &= R \sin \Phi_2 \quad \cos \Phi_1 \\
X_3 &= R \sin \Phi_3 \quad \cos \Phi_2 \quad \cos \Phi_1 \\
&\vdots \\
X_{n-1} &= R \sin \Phi_{n-1} \quad \cos \Phi_{n-2} \quad \cdots\cdots\cdots \quad \cos \Phi_1 \\
X_n &= R \cos \Phi_{n-1} \quad \cos \Phi_{n-2} \quad \cdots \quad \cdots \quad \cos \Phi_1 .
\end{aligned}$$

Exercise 19 (Integral Powers of Cauchy Random Variables). Let X be a random variable with the pdf $(1/\pi)(a/a^2 + x^2)$. Find the pdf of $Y = X^n$ where n is a positive integer.

Exercise 20 (Change of Variables with Exponential Random Variables). Let $U_1, \ldots, U_{n+1}$ be $n + 1$ iid random variables with pdf $f(u) = \lambda e^{-\lambda u}$ if $u > 0$, $f(u) = 0$ if $u \leqslant 0$. Define $Y_i = U_i/(U_1 + \cdots + U_{n+1})$ for $1 \leqslant i \leqslant n$ and $S_n = U_1 + \cdots + U_{n+1}$. Compute the pdf's of the vectors $(Y_1, \ldots, Y_n, S_n)$ and $(Y_1, \ldots, Y_n)$, and of the random variable nY_i.

Exercise 21 (Poisson Process and Binomial Random Variable). Let $(N(t), t \geqslant 0)$ be the counting process of a homogeneous Poisson process $(T_n, n \geqslant 1)$ with intensity $\lambda > 0$. What is the conditional distribution of $N(s)$ given $N(t) = n$ $(s \leqslant t)$? [i.e., compute $P(N(s) = k \mid N(t) = n)$].

Exercise 22 (The Flip-Flop Stochastic Process). Let $(N(t), t \geqslant 0)$ be the counting process of a homogeneous Poisson process $(T_n, n \geqslant 1)$ with intensity $\lambda > 0$. Define for each t the random variable $X(t)$ by $X(t) = X(0)(-1)^{N(t)}$ where $X(0)$ is a random variable taking the values -1 and $+1$, independent of $(N(t), t \geqslant 0)$. Therefore for each $\omega \in \Omega$, $t \to X(t, \omega)$ is a function flipping from -1 to $+1$ and flopping from $+1$ to -1:



When $P(X(0) = 1) = p$, compute $P(X(t) = 1)$ for each $t \geqslant 0$. Find $\lim_{t \uparrow \infty} P(X(t) = 1)$.

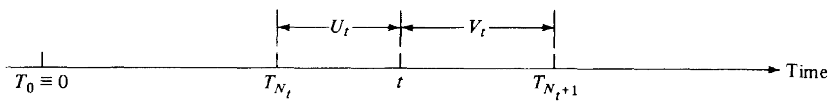
Exercise 23 (A Poisson Series). Use the central limit theorem to prove that

$$\lim_{n \to \infty} \sum_{k=0}^{n} e^{-n} \frac{n^k}{k!} = \frac{1}{2}.$$

Exercise 24 (Standardized Poisson Counting Process). Let $(N_t, t \geqslant 0)$ be the counting process of a homogeneous Poisson process with intensity $\lambda > 0$. Compute the cf of $(N_t - \lambda t)/\sqrt{\lambda t}$ and show that the latter rv converges in distribution to $\mathcal{N}(0, 1)$ as t goes to ∞.

Exercise 25 (Asymptotic Estimate of the Intensity of a Homogeneous Poisson Process). Use the strong law of large numbers to show that if $(N_t, t \geqslant 0)$ is the counting process of a homogeneous Poisson process with intensity $\lambda > 0$, then $\lim_{t\to\infty} N_t = \infty$, P – as. Show that $\lim_{t\to\infty} N_t/t = \lambda$, P – as.

Exercise 26 (Feller's Paradox). Let $(T_n, n \geqslant 1)$ be a homogeneous Poisson process over $\mathbb{R}_+$, with intensity $\lambda > 0$. Let $T_0 = 0$. For a *fixed* $t > 0$, define $U_t = t - T_{N_t}$, $V_t = T_{N_{t-1}} - t$ (see figure below).



Note that T_{N_t} is the first random point T_n strictly to the left of t and $T_{N_{t+1}}$ is the first random point T_n after t. Find the joint distribution of U_t and V_t. Find the distribution of $T_{N_{t+1}} - T_{N_t}$. Examine the case $t \to \infty$.

Remark. Feller's paradox consists in observing that although the random variables $T_{n+1} - T_n$ $(n \geqslant 0)$ have the same exponential distribution, $T_{N_{t+1}} - T_{N_t}$ is not an exponential random variable.

Exercise 27 (Sum of Sums of Gaussian Random Variables). Let $(\varepsilon_n, n \geqslant 1)$ be a sequence of iid standard Gaussian random variables and define the sequence $(X_n, n \geqslant 1)$ by: $X_n = \varepsilon_1 + \cdots + \varepsilon_n$. Show that $(X_1 + \cdots + X_n)/n\sqrt{n} \xrightarrow{\mathscr{L}} N(0, \sigma^2)$ for some σ to be computed.

Exercise 28 (Gaussian Random Variables Converging in Law). Let $(U_n, n \geqslant 0)$ be a sequence of iid random variables, Gaussian, mean 0 and variance 1. Define $(X_n, n \geqslant 0)$ by $X_0 = U_0$, $X_{n+1} = aX_n + U_{n+1}$ $(n \geqslant 0)$.

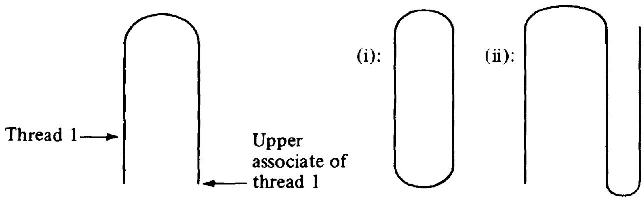
(i) Show that X_n is a Gaussian random variable, and that if $a < 1$, $X_n \xrightarrow{\mathscr{L}} \mathscr{N}(0, \sigma^2)$ for some σ^2 to be identified.

(ii) If $a > 1$ show that X_n/a^n converges in quadratic mean. Does it converge in distribution?

Solutions to Additional Exercises

1. $p_n = (1 - (p - q)^n)/2, \quad \lim_{n\uparrow\infty} p_n = \frac{1}{2}$.

2. $P_2 = 1, P_{2n} = P_{2n-2}(2n - 2)/(2n - 1)$. To see this, consider thread 1 and its "upper associate"



Then either (i) or (ii) occurs. (ii) occurs with probability $(2n - 2)/(2n - 1)$. In situation (ii), the piece has become one bit of thread, to be linked to the $2n - 3$ remaining pieces. Finally $P_{2n} = 2 \cdot 4 \cdot 6 \ldots (2n - 2)/1 \cdot 3 \cdot 5 \ldots (2n - 1)$.

3. $1 - [(0.60)(0.25) + 0.40]^3 = 1 - (0.55)^3$.

4. If $x = 0$, then clearly $u(0) = 0$. If $x = c$, then clearly $u(c) = 1$. For $0 < x < c$, after a toss of the coin, player A has fortune $x + 1$ with probability p, and fortune $x - 1$ with probability q. From $x + 1$ (resp. $x - 1$), the probability of reaching c without getting broke in the meantime is $u(x + 1)$ (resp. $u(x - 1)$), and therefore (exclusive and exhaustive causes)

$$u(x) = pu(x + 1) + qu(x - 1).$$

The solution of this recurrence equation with boundary conditions $u(0) = 0$, $u(c) = 1$ is (computations) for $x = a$:

$$u(a) = \begin{cases} a/(a+b) & \text{if } p = q = 1/2 \\ [1-(q/p)^a]/[1-(q/p)^{a+b}] & \text{if } p \neq q. \end{cases}$$

5. By symmetry (since the coins are *unbiased*), $P(T_A \geqslant T_B) = P(H_A \geqslant H_B)$. But $H_A = N - T_A$ and $H_B = N + 1 - T_B$ so that $H_A \geqslant H_B \Leftrightarrow T_A < T_B$. Therefore $P(H_A \geqslant H_B) = P(T_A < T_B) = 1 - P(T_A \geqslant T_B)$ and, using the first equality, $P(T_A \geqslant T_B) = 1 - P(T_A \geqslant T_B)$, i.e., $P(T_A \geqslant T_B) = 1/2$.

6. $P(U = k, V = l) = P(U = k, X = k + l) = P(X = k + l, Y_1 + \cdots + Y_{k+l} = k)$

$$= P(X = k + l)P(Y_1 + \cdots + Y_{k+l} = k)$$

$$= e^{-\lambda}\frac{\lambda^{k+l}}{(k+l)!} \times \frac{(k+l)!}{k!\,l!}p^k q^l$$

$$= e^{-\lambda p}\frac{(\lambda p)^k}{k!} \times e^{-\lambda q}\frac{(\lambda q)^l}{l!}.$$

7. $1 + (1 - (1 - p)^N)N$. Numerical application: 2.3756.

8. $P(X = k) = 2^{-k}$ (k integer $\geqslant 1$).

9. We do the case $i = 1$, $j = 2$. The generating function of (X_1, X_2) is $f(s_1, s_2) = g(s_1, s_2, 1, \ldots, 1)$. Also $f(s_1, s_2) = \sum_{k_1 \in \mathbb{N}} \sum_{k_2 \in \mathbb{N}} P(X_1 = k_1, X_2 = k_2) s_1^{k_1} s_2^{k_2}$, so that $(\partial^2 f/\partial s_1 \partial s_2)(1, 1) = \sum_{k_1 \in \mathbb{N}} \sum_{k_2 \in \mathbb{N}} k_1 k_2 P(X_1 = k_1, X_2 = k_2) = E[X_1 X_2]$. The multinomial case: $E[X_i X_j] = n(n-1)p_i p_j$ if $i \neq j$ and $\mathrm{Var}(X_i^2) = np_i(1 - p_i)$ (Why?). Also $E[X_i] = np_i$. Therefore $\Gamma_x = \{\gamma_{ij}\}$ with $\gamma_{ij} = -np_i p_j$ if $i \neq j$, $\gamma_{ii} = np_i(1 - p_i)$.

10. A_k = (k^{th} envelope has the correct address). To be computed $P(\bigcup_{k=1}^n A_k)$. But

$$P(A_k) = \frac{1}{n} = \frac{(n-1)!}{n!}, \quad P(A_k \cap A_j) = P(A_k)P(A_j | A_k) = \frac{(n-2)!}{n!},$$

$$P(A_k \cap A_j \cap A_l) = \frac{(n-3)!}{n!}, \ldots, P\left(\bigcap_{k=1}^n A_k\right) = \frac{1}{n!}$$

Therefore the result is (inclusion–exclusion formula)

$$\binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} - \cdots + (-1)^{n-1}\frac{1}{n!}.$$

That is to say

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!}.$$

If n is very large, this is close to $1 - e^{-1}$.

11. $\binom{2n-k}{n} \Big/ 2^{2n-k}$.

12. $P(2N = 2n) = \dfrac{(2n-2)!}{n!(n-1)!} \cdot \dfrac{1}{2^{2n-1}}$; $E[2N] = +\infty$

(use the equality $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n \frac{(2n)!}{(n!)^2} = (1 - x)^{-1/2}$).

13. Call $X_1X_2X_3X_4X_5X_6$ the random number. The X_i's are independent and identically distributed random variables, with $P(X_i = k) = p = 1/10$ for $k = 0, \ldots, 9$. The generating functions of the X_i's are identical, equal to

$$\frac{1}{10}(1 + s + \cdots + s^9) = \frac{1}{10} \cdot \frac{1 - s^{10}}{1 - s}.$$

The generating function $g(s)$ of $X_1 + X_2 + X_3$ is

$$\frac{1}{10^3} \cdot \frac{(1 - s^{10})^3}{(1 - s)^3}$$

and is equal to the gf of $X_4 + X_5 + X_6$. The coefficient of s^r in $g(s)$ is the probability that the sum of the first 3 digits is r, and the coefficient of s^{-r} in $g(s^{-1})$ is the probability that the sum of the last 3 digits is r. Therefore the coefficient of s^0 in $g(s)g(s^{-1})$ is the probability that we are looking for. We have

$$g(s)g(s^{-1}) = \frac{1}{10^6} \cdot \frac{1}{s^{27}} \left(\frac{1 - s^{10}}{1 - s} \right)^6.$$

But

$$(1 - s^{10})^6 = 1 - \binom{6}{1} s^{10} + \binom{6}{2} s^{20} + \cdots$$

and

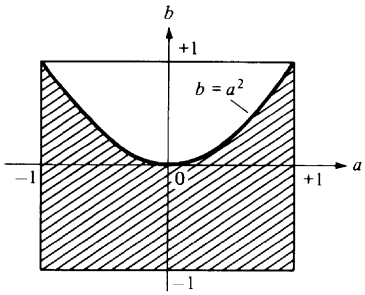
$$(1 - s)^{-6} = 1 + \binom{6}{5} s + \binom{7}{5} s^2 + \cdots$$

and therefore the answer is

$$\frac{1}{10^6}\left(\binom{32}{5} - \binom{6}{1}\binom{22}{5} + \binom{6}{2}\binom{12}{5}\right) = 0.05525.$$

14. $P(a \leqslant Z \leqslant b, \theta_1 \leqslant \Theta < \theta_2) = (b^2 - a^2)(\theta_2 - \theta_1)/2\pi$. Therefore $P(a \leqslant Z \leqslant b) = b^2 - a^2$ (take $\theta_1 = 0, \theta_2 = 2\pi$), and $P(\theta_1 \leqslant \Theta \leqslant \theta_2) = (\theta_2 - \theta_1)/2\pi$ (take $a = 0, b = 1$).

15. $P(a^2 - b \geqslant 0) = \frac{1}{2} + \frac{1}{4} \cdot 2 \int_0^1 x^2\, dx = \frac{2}{3}$.



16. If $y \geqslant 0$, $1 - P(Y > y) = 1 - P(X_1 > y, \ldots, X_n > y) = 1 - e^{-\lambda_1 y} \ldots e^{-\lambda_n y}$. Therefore $f_Y(y) = (\lambda_1 + \cdots + \lambda_n)e^{-(\lambda_1 + \cdots + \lambda_n)y}$ $(y \geqslant 0)$.

17. $((X + Y)/\sqrt{2}, (X - Y)/\sqrt{2})$ is Gaussian, because it is a linear function of the Gaussian vector (X, Y). (By the way, why is (X, Y) Gaussian?) $(X + Y)/\sqrt{2}$ and $(X - Y)/\sqrt{2}$ are independent because they are uncorrelated, and their variance is 1(compute).

18. $f(r, \phi_1, \ldots, \phi_{n-1}) = \dfrac{r^{n-1}\cos^{n-2}\phi_1 \cos^{n-3}\phi_2 \ldots \cos\phi_{n-2}}{(2\pi)^{n/2}\sigma^n} e^{-r^2/2\sigma^2}.$

19. If n is odd: $f_Y(y) = \dfrac{ay^{-(n-1)/n}}{\pi n(a^2 + y^{2/n})}$.

$$\text{If } n \text{ is even: } f_Y(y) = \begin{cases} \dfrac{2ay^{-(n-1)/n}}{\pi n(a^2 + y^{2/n})} & \text{if } y \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

20. $f_{Y_1,\ldots,Y_n,S_n}(y_1, \ldots, y_n, s)$

$$= \begin{cases} \lambda^{n+1}s^n e^{-\lambda s} & \text{if } (y_1, \ldots, y_n, s) \in \mathbb{R}_+^{n+1} \text{ and } 0 \leqslant y_1 + \cdots + y_n \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \begin{cases} n! & \text{if } (y_1, \ldots, y_n) \in \mathbb{R}_+^n \text{ and } 0 \leqslant y_1 + \cdots + y_n \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{nY_i}(z) = \begin{cases} (1 - z/n)^{n-1} & \text{if } 0 \leqslant z \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

21. $(s/t)^k(1 - s/t)^{n-k} n!/(k!(n-k)!) \qquad (k = 0, \ldots, n)$.

22. $1/2 + (p - 1/2)e^{-2\lambda t} \to 1/2$.

23. If $(X_n, n \geqslant 1)$ are iid Poisson random variables with mean 1 and variance 1,

$$\lim_{n \uparrow \infty} P\left(\frac{(X_1 + \cdots + X_n) - n}{\sqrt{n}} \geqslant 0\right) = \frac{1}{2} \qquad \text{(central limit theorem).}$$

Observe that the sum $X_1 + \cdots + X_n$ is Poisson with mean n, and the conclusion follows.

24. The cf of $(N_t - \lambda t)/\sqrt{\lambda t}$ is $\exp\{\lambda t(e^{iu/\sqrt{\lambda t}} - 1 - iu/\sqrt{\lambda t})\} \to \exp\{-\frac{1}{2}u^2\}$ as $t \to +\infty$.

25. From the law of large numbers:

$$\frac{N_n}{n} = \frac{1}{n}\sum_{k=0}^{n-1}[N_{k+1} - N_k] \xrightarrow[n\to\infty]{\text{as}} E[N_1] = \lambda > 0,$$

therefore

$$\lim_{t\to\infty} N_t = \lim_{n\to\infty} N_n = +\infty, \qquad P-\text{as}.$$

Observe that

$$\frac{N_t}{S_1 + \cdots + S_{N_t} + S_{N_t+1}} \leqslant \frac{N_t}{t} \leqslant \frac{N_t}{S_1 + \cdots + S_{N_t}}.$$

Also

$$\lim_{t\to\infty} \frac{S_1 + \cdots + S_{N_t}}{N_t} = \lim_{n\to\infty} \frac{S_1 + \cdots + S_n}{n}$$

since $N_t \to \infty$ as. The latter limit is $E[S_1] = 1/\lambda$ (law of large numbers). The conclusion follows from the above remarks.

26. $\{U_t \leqslant u, V_t \leqslant v\} = \{N_t - N_{t-u} \geqslant 1, N_{t+v} - N_t \geqslant 1\} \qquad (0 \leqslant u < t, v \geqslant 0).$

Therefore, for $0 \leqslant u < t$, $v \geqslant 0$,

$$P(U_t \leqslant u, V_t \leqslant v) = P(N_t - N_{t-u} \geqslant 1)P(N_{t+v} - N_t \geqslant 1) = (1 - e^{-\lambda u})(1 - e^{-\lambda v}).$$

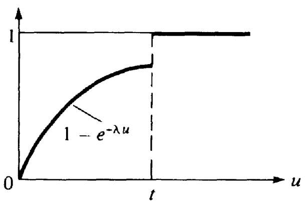
Also for $u = t$, $v \geqslant 0$:

$$P(U_t = t, V_t \leqslant v) = P(N_t = 0, N_{t+v} - N_t \geqslant 1) = e^{-\lambda t}(1 - e^{-\lambda v}).$$

Finally for $u \geqslant 0$, $v \geqslant 0$:

$$P(U_t \leqslant u, V_t \leqslant v) = (1_{\{u \geqslant t\}} + 1_{\{u<t\}}(1 - e^{-\lambda u}))(1 - e^{-\lambda v}).$$

Therefore U_t and V_t are independent, V_t is an exponential rv of mean $1/\lambda$, whereas U_t as the cdf given by the figure below.



If $t \to \infty$, U_t converges in law to an exponential rv with mean $1/\lambda$. In the limit $t \to \infty$, $T_{N_{t+1}} - T_{N_t}$ is the sum of two independent exponential rv's with mean $1/\lambda$.

27. The cf of the vector $(X_1, \ldots, X_n)$ is $\phi_{X_1 \ldots X_n}(u_1, \ldots, u_n) = \exp\{-\frac{1}{2}[u_n^2 + (u_n + u_{n-1})^2 + \cdots + (u_1 + \cdots + u_n)^2]\}$. The cf of $(X_1 + \cdots + X_n)/n\sqrt{n}$ is $\exp\{-\frac{1}{2}u^2[1/n^3(1^2 + 2^2 + \cdots + n^2)]\}$ and tends to $\exp\{-\frac{1}{2}\cdot(u^2/3)\}$ as n goes to ∞; $\sigma^2 = 1/3$.

28. (i) X_n is a linear combination of $U_0, \ldots, U_n$ and $(U_0, \ldots, U_n)$ is a Gaussian vector since $U_0, \ldots, U_n$ are independent Gaussian vectors. Therefore X_n is Gaussian. $EX_n = 0$ and therefore

$$\phi_{X_n}(u) = e^{-(1/2)E[X_n^2]u^2}.$$

But $E[X_{n+1}^2] = a^2E[X_n^2] + E[U_{n+1}^2] = a^2E[X_n^2] + 1$. Therefore since $E[X_0^2] = E[U_0^2] = 1$, $\lim_{n\to\infty} E[X_n^2] = 1/(1 - a^2)$, and

$$\lim \phi_{X_n}(u) = e^{-(1/2)(1/1-a^2)u^2} \qquad \text{i.e., } X_n \xrightarrow{\mathcal{L}} N\left(0, \frac{1}{1-a^2}\right).$$

(ii)
$$\frac{X_{n+1}}{a^{n+1}} = \frac{X_n}{a^n} + \frac{U_{n+1}}{a^{n+1}}.$$

Therefore

$$\frac{X_n}{a_n} = U_0 + \frac{U_1}{a} + \cdots + \frac{U_n}{a^n}.$$

Letting

$$Z_n = \frac{X_n}{a^n},$$

we have

$$E[|Z_{m+n} - Z_n|^2] = E\left[\left|\frac{U_{n+1}}{a^{n+1}} + \cdots + \frac{U_{n+m}}{a^{n+m}}\right|\right]$$

$$= \frac{1}{a^{n+1}}\left(1 + \cdots + \frac{1}{a^{m-1}}\right) = \frac{1}{a^{n+1}} \frac{1 - \dfrac{1}{a^m}}{1 - \dfrac{1}{a}} \to 0 \text{ as } n, m \to \infty.$$

Thus $(Z_n, n \geqslant 0)$ converges in quadratic mean (Theorem T13). Convergence in qm implies convergence in law.

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